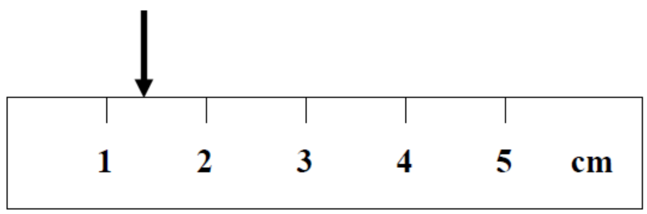
Basic Concepts of Error Analysis

SIGNIFICANT FIGURES

The laboratory usually involves measurements of several physical quantities such as length, mass, time, voltage and current. The values of these quantities should be presented in terms of **Significant Figures**.

For example, the location of the arrow is to be determined in the figure below.



It is obvious that the location is between 1 cm and 2 cm. The correct way to express this location is to make one more estimate based on your intuition. That is, in this case, a reading of 1.3 cm is estimated. This measurement is said to contain two significant figures. Note that there should only be one estimated place in any measurement. If data are to contain, say, three significant figures, two must be known, and the third estimated. Do not try to locate the position of the arrow in fig. 1 as 1.351 cm.

The following rules dictate the handling of significant figures.

a Specify the measured value to the same accuracy as the error. For example, we report that a physical quantity is $x = 3.45 \pm 0.05$, not 3.4 ± 0.05 and not 3.452 ± 0.05.

b When adding or subtracting numbers, the answer is only good to the least accurate number present. For example, $50.3 + 2.555 = 53.9$ and not 52.855.

c When multiplying or dividing, keep the same number of significant figures as the factor with the fewest number of significant figures. For example, $5.0 \cdot 1.2345 = 6.2$ and not 6.1725.

TYPES OF ERRORS

Every measurement has its error. In general, there are three types of errors that will be explained below.

Random Errors

This type of error is usually referred to as a statistical error. This class of error is produced by unpredictable or unknown variations in the measuring process. It always exists even though one does the experiment as carefully as is humanly possible. One example of these uncontrollable variations is an observer's inability to estimate the last significant digit for a given measurement the same way every time.

Systematic Errors

This class of error is commonly caused by a flaw in the experimental apparatus. They tend to produce values either consistently above the true value, or consistently below the true value. One example of such a flaw is a bad calibration in the instrumentation.

Personal Errors

This class of error is also called "mistakes." It is fundamentally different from either the systematic or random errors stated above, and can be completely eliminated if the experimenter is careful enough. One example of this type of error is to misread the scale of an instrument.

MEAN AND STATISTICAL DEVIATION

Let's assume that both the systematic and personal errors can be eliminated by careful experimental procedures, then we can conclude that the experimental errors are governed by random statistical errors. If there are a total number of N measurements made of some physical quantity, say, x, and the i-th value is denoted by x_i, the statistical theory says that the "mean" of the above N measurements is the best approximation to the true value; i.e., the mean $\overline{x}$ is given by

$$\overline{x} = \left(\frac{1}{N}\right) \sum_{i=1}^{N} \equiv \left[\frac{1}{N}\right] (x_1 + x_2 + x_3 + - - - + x_N), \tag{1}$$

where Σ means summation.

The statistical theory states that the precision of the measurement can be determined by the calculation of the quantity called "standard deviation" from the mean of the measurements, which is defined by the following equation:

$$\sigma = \sqrt{\left(\frac{1}{N-1}\right) \sum_{i=1}^{N} (x_i - \overline{x})^2} = \sqrt{\left(\frac{1}{N-1}\right) [(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + ... + (x_N - \overline{x})^2]}. \tag{2}$$

The statistical theory states that approximately 68% of all the repeated measurements should fall within a range of plus or minus σ from the mean, and about 95% of all the repeated measurements should fall within a range of 2σ around the mean. In other words, if one of your measurements is 2σ or farther from the mean, it is very likely that it is due to either systematic or personal error.

REPORTING OF RESULTS

Typically, in the laboratory, one will be asked to make a number of repeated measurements on a given physical quantity, say x. The measured value is customarily expressed in the laboratory report as

$$x = \overline{x} \pm \sigma, \tag{3}$$

where $\overline{x}$ is the mean and σ is the standard deviation. Equivalently, we can also write

$$x = \overline{x} \pm \left(\frac{\sigma}{\overline{x}} \cdot 100\% \right), \tag{4}$$

where $\left(\frac{\sigma}{x} \right) \cdot 100\%$ is called percentage error.

PROPAGATION OF ERRORS

In carrying out an experiment, a specific physical quantity of interest is usually obtained by an indirect measurement and a simple manipulation of other physical quantities. For instance, in the laboratory, speed is determined indirectly by the division of the distance traveled and the time taken to travel that distance.

It is clear from our previous discussion that these measurements of distance and time inevitably have errors associated with them. In evaluating the speed, these errors on distance and time will pass on to the speed.

Consider two independent physical quantities x and y with their associated errors Δx and Δy, respectively. We are interested in knowing how the errors propagate to another physical quantity z formed under the following specific operations.

Addition

$$z = x + y \tag{5}$$

The error associated with z is therefore given by

$$\Delta z = \Delta x + \Delta y. \tag{6}$$

For example, $(3.0 \pm 0.1) + (4.0 \pm 0.2) = 7.0 \pm 0.3$.

Subtraction

$$z = x - y \tag{7}$$

The error associated with z is therefore given by

$$\Delta z = |\Delta x| + |-\Delta y| \tag{8}$$

or

$$\Delta z = \Delta x + \Delta y. \tag{9}$$

For example, $(5.0 \pm 0.1) - (1.0 \pm 0.3) = 4.0 \pm 0.4$.

Multiplication

$$z = x \cdot y \tag{10}$$

The error associated with z is therefore given by

$$\Delta z = \Delta x \cdot y + x \cdot \Delta y \tag{11}$$

or

$$\frac{\Delta z}{z} = \frac{\Delta x}{x} + \frac{\Delta y}{y}. \tag{12}$$

For example, $(1.0 \pm 0.1) \cdot (3.0 \pm 0.3) = (1.0 \pm 10\%) \cdot (3.0 \pm 10\%) = 3.0 \pm 20\% = 3.0 \pm 0.6$.

In other words, one forms the percentage error for each of the physical quantities x and y, and then adds the two percentage errors to obtain the final error for z.

Division

$$z = \frac{x}{y} \tag{13}$$

The error associated with z is therefore given by

$$\Delta z = \left|\frac{\Delta x}{y}\right| + \left|x \cdot \frac{-\Delta y}{y^2}\right|, \tag{14}$$

or

$$\frac{\Delta z}{z} = \frac{\Delta x}{x} + \frac{\Delta y}{y}. \tag{15}$$

For example, $\dfrac{(3.0 \pm 0.3)}{(1.0 \pm 0.1)} = \dfrac{(3.0 \pm 10\%)}{(1.0 \pm 10\%)} = 3.0 \pm 20\% = 3.0 \pm 0.6$.

In other words, one forms the percentage error for each of the physical quantities x and y first, and then adds the two percentage errors to obtain the final error for z.

Powers or Exponential

$$z = x^k \tag{16}$$

The error associated with z is therefore given by

$$\Delta z = k \cdot x^{k-1} \cdot \Delta x \tag{17}$$

or

$$\frac{\Delta z}{z} = k \cdot \frac{\Delta x}{x}. \tag{18}$$

For example, $(3.0 \pm 0.3)^2 = (3.0 \pm 10\%)^2 = 9.0 \pm 2 \cdot 10\% = 9.0 \pm 20\% = 9.0 \pm 1.8$.

In other words, one multiplies the power (k) by the percentage error of x to obtain the error associated with z.

