Chapter 10

Exponential Functions

If we start with a single yeast cell under favorable growth conditions, then it will divide in one hour to form two identical "daughter cells". In turn, after another hour, each of these daughter cells will divide to produce two identical cells; we now have four identical "granddaughter cells" of the original parent cell. Under ideal conditions, we can imagine how this "doubling effect" will continue:

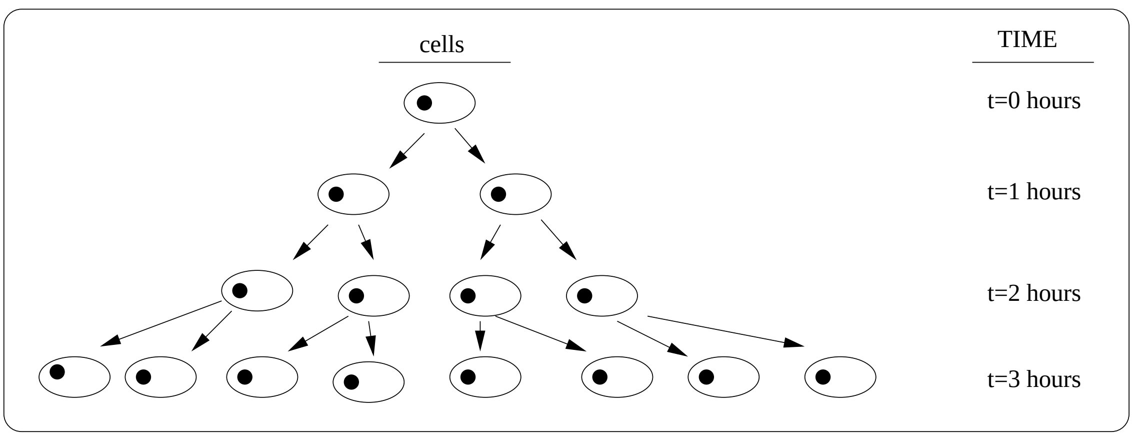


Figure 10.1: Observing cell growth.

The question is this: *Can we find a function of* t *that will predict (i.e. model) the number of yeast cells after* t *hours?* If we tabulate some data (as at right), the conclusion is that the formula

$$N(t) = 2^t$$

predicts the number of yeast cells after t hours. Now, let's make a very slight change. Suppose that instead of starting with a single cell, we begin with a population of 3×10^6 cells; a more realistic situation. If we assume

Total hours	**Number of yeast cells**
0	$1=2^0$
1	$2=2^1$
2	$4=2^2$
3	$8=2^3$
4	$16=2^4$
5	$32=2^5$
6	$64=2^6$

Table 10.1: Cell growth data.

that the population of cells will double every hour, then reasoning as above will lead us to conclude that the formula

$$N(t) = (3 \times 10^6)2^t$$

gives the population of cells after t hours. Now, as long as t represents a non-negative integer, we know how to calculate $N(t)$. For example, if $t = 6$, then

$$\begin{aligned} N(t) &= (3 \times 10^6)2^6 \\ &= (3 \times 10^6)(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) \\ &= (3 \times 10^6)64 \\ &= 192 \times 10^6. \end{aligned}$$

The key point is that computing $N(t)$ only involves simple arithmetic. But what happens if we want to know the population of cells after 6.37 hours? That would require that we work with the formula

$$N(t) = (3 \times 10^6)2^{6.37}$$

and the rules of arithmetic do not suffice to calculate $N(t)$. We are stuck, since we must understand the meaning of an expression like $2^{6.37}$. In order to proceed, we will need to review the algebra required to make sense of raising a number (such as 2) to a non-integer power. We need to understand the precise meaning of expressions like: $2^{6.37}$, $2^{\sqrt{5}}$, $2^{-\pi}$, etc.

10.1 Functions of Exponential Type

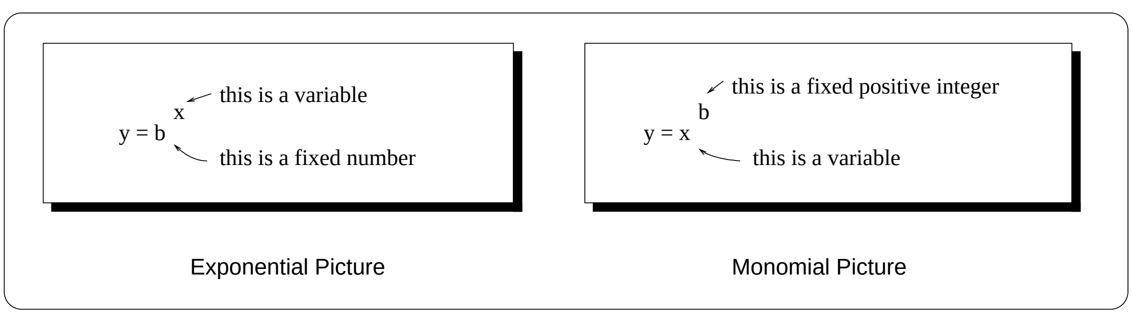


Figure 10.2: Viewing the difference between exponential and monomial functions.

On a symbolic level, the class of functions we are trying to motivate is easily introduced. We have already studied the monomials $y = x^b$, where x was our input variable and b was a fixed positive integer exponent. What happens if we turn this around, interchanging x and b, defining a new rule:

$$y = f(x) = b^x. \tag{10.1}$$

We refer to x as the *power* and b the *base*. An expression of this sort is called a function of *exponential type*. Actually, if your algebra is a bit rusty, it is easy to initially confuse functions of exponential type and monomials (see Figure 10.2).

10.1.1 Reviewing the Rules of Exponents

To be completely honest, making sense of the expression $y = b^x$ for *all* numbers x requires the tools of *Calculus*, but it is possible to establish a reasonable comfort level by handling the case when x is a rational number. If $b \geq 0$ and n is a positive integer (i.e. $n = 1, 2, 3, 4, \ldots$), then we can try to solve the equation

$$t^n = b. \tag{10.2}$$

A solution t to this equation is called an n^{th} root of b. This leads to complications, depending on whether n is even or odd. In the odd case, for any real number b, notice that the graph of $y = b$ will always cross the graph of $y = t^n$ exactly once, leading to one solution of (10.2).

On the other hand, if n is even and $b < 0$, then the graph of $y = t^n$ will miss the graph of $y = b$, implying there are no solutions to the equation in (10.2). (There will be *complex* solutions to equations such as $t^2 = -1$, involving the imaginary complex numbers $\pm i = \pm\sqrt{-1}$, but we are only working with real numbers in this course.) Also, again in the case when n is even, it can happen that there are two solutions to (10.2). We do not want to constantly worry about this even/odd distinction, so we will henceforth **assume** $b > 0$. To eliminate possible ambiguity, we will single out a particular n^{th}-root; we define the symbols:

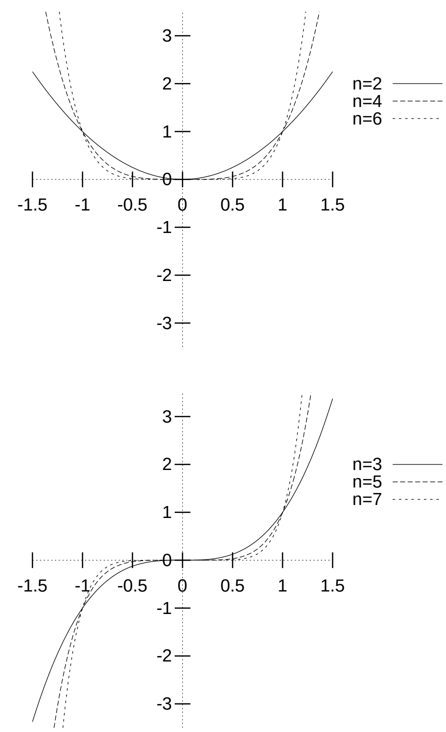


Figure 10.3: Even and odd monomials.

$$\sqrt[n]{b} = b^{\frac{1}{n}} = \text{ the largest real } n^{th} \text{ root of } b. \tag{10.3}$$

Thus, whereas ± 1 are both 4^{th}-roots of 1, we have defined $\sqrt[4]{1} = 1$.

In order to manipulate $y = b^x$ for rational x, we need to recall some basic facts from algebra.

Important Facts 10.1.1 (Working with rational exponents). *For all positive integers* p *and* q, *and any real number base* $b > 0$, *we have*

$$b^{\frac{p}{q}} = \left(\sqrt[q]{b}\right)^p = \sqrt[q]{b^p}.$$

For any rational numbers r *and* s, *and for all positive bases* a *and* b*:*

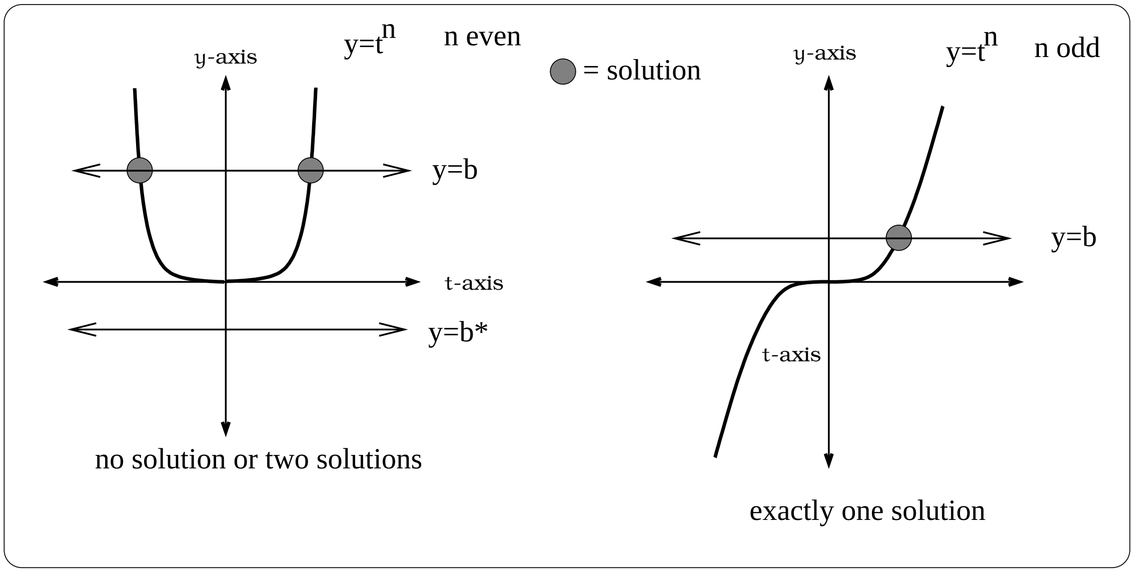


Figure 10.4: How many solutions to $t^n = b$?.

1. *Product of power rule:* $b^r b^s = b^{r+s}$
2. *Power of power rule:* $(b^r)^s = b^{rs}$
3. *Power of product rule:* $(ab)^r = a^r b^r$
4. *Zero exponent rule:* $b^0 = 1$
5. *Negative power rule:* $b^{-r} = \frac{1}{b^r}$

These rules have two important consequences, one theoretical and the other more practical. On the first count, recall that any rational number r can be written in the form $r = \frac{p}{q}$, where p and q are integers. Consequently, using these rules, we see that the expression $y = b^x$ defines a function of x, whenever x is a rational number. On the more practical side of things, using the rules we can calculate and manipulate certain expressions. For example,

$$27^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9;$$
$$8^{-\frac{5}{3}} = \left(\sqrt[3]{8}\right)^{-5} = 2^{-5} = \frac{1}{2^5} = \frac{1}{32}.$$

The sticky point which remains is knowing that $f(x) = b^x$ actually defines a function for *all real values of* x. This is **not** easy to verify and we are simply going to accept it as a fact. The difficulty is that we need the fundamentally new concept of a *limit*, which is the starting point of a *Calculus* course. Once we know the expression does define a function, we can also verify that the rules of Fact 10.1.1 carry through for all real exponent powers. Your calculator should have a "y to the x key", allowing you to calculate expressions such as $\pi^{\sqrt{2}}$ involving non-rational powers.

Here are the key modeling functions we will work with in this Chapter.

Definition 10.1.2. *A function of exponential type has the form*

$$A(x) = A_0 b^x,$$

for some $b > 0$, $b \neq 1$, *and* $A_0 \neq 0$.

We will refer to the formula in Definition 10.1.2 as the *standard exponential form*. Just as with standard forms for quadratic functions, we sometimes need to do a little calculation to put an equation in standard form. The constant A_0 is called the *initial value* of the exponential function; this is because if x represents time, then $A(0) = A_0 b^0 = A_0$ is the value of the function at time $x = 0$; i.e. the initial value of the function.

Example 10.1.3. *Write the equations* $y = 8^{3x}$ *and* $y = 7\left(\frac{1}{2}\right)^{2x-1}$ *in standard exponential form.*

Solution. In both cases, we just use the rules of exponents to maneuver the given equation into standard form:

$$\begin{aligned} y &= 8^{3x} \\ &= (8^3)^x \\ &= 512^x \end{aligned}$$

and

$$\begin{aligned} y &= 7\left(\frac{1}{2}\right)^{2x-1} \\ &= 7\left(\frac{1}{2}\right)^{2x}\left(\frac{1}{2}\right)^{-1} \\ &= 7\left(\left(\frac{1}{2}\right)^2\right)^x 2 \\ &= 14\left(\frac{1}{4}\right)^x \end{aligned}$$

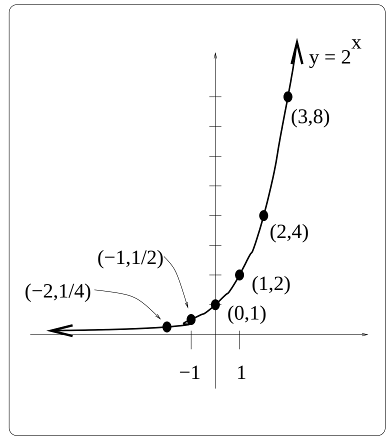
□

10.2 The Functions $y = A_0 b^x$

We know $f(x) = 2^x$ defines a function of x, so we can study basic qualitative features of its graph. The data assembled in the solution of the "Doubling Effect" beginning this Chapter, plus the rules of exponents, produce a number of points on the graph. This graph exhibits four key qualitative features that deserve mention:

x	2^x	**Point on the graph of** $y = 2^x$
⋮	⋮	⋮
-2	1/4	(-2, 1/4)
-1	1/2	(-1, 1/2)
0	1	(0, 1)
1	2	(1, 2)
2	4	(2, 4)
3	8	(3, 8)
⋮	⋮	⋮

(a) Data points from $y = 2^x$.



(b) Graph of $y = 2^x$.

Figure 10.5: Visualizing $y = 2^x$.

- The graph is always above the horizontal axis; i.e. the function values are always positive.
- The graph has y-intercept 1 and is increasing.
- The graph becomes closer and closer to the horizontal axis as we move left; i.e. the x-axis is a **horizontal asymptote** for the left-hand portion of the graph.
- The graph becomes higher and higher above the horizontal axis as we move to the right; i.e., the graph is *unbounded* as we move to the right.

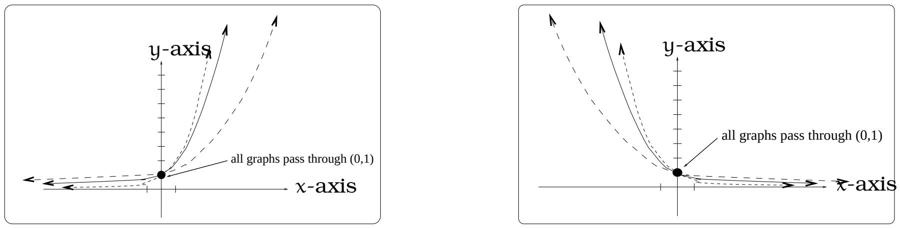
The special case of $y = 2^x$ is representative of the function $y = b^x$, but there are a few subtle points that need to be addressed. First, recall we are always assuming that our base $b > 0$. We will consider three separate cases: $b = 1$, $b > 1$, and $0 < b < 1$.

10.2.1 The case $b = 1$

In the case $b = 1$, we are working with the function $y = 1^x = 1$; this is not too exciting, since the graph is just a horizontal line. We will ignore this case.

10.2.2 The case $b > 1$

If $b > 1$, the graph of the function $y = b^x$ is qualitatively similar to the situation for $b = 2$, which we just considered. The only difference is the exact amount of "concavity" in the graph, but the four features highlighted above are still valid. Figure 10.6(a) indicates how these graphs compare for three different values of b. Functions of this type exhibit what is typically referred to as *exponential growth*; this codifies the fact that the function values grow rapidly as we move to the right along the x-axis.



(a) Graph of $y = b^x$, $b > 1$. (b) Graph of $y = b^x$, $0 < b < 1$.

Figure 10.6: Visualizing cases for b.

10.2.3 The case $0 < b < 1$

We can understand the remaining case $0 < b < 1$, by using the remarks above and our work in Chapter 13. First, with this condition on b, notice that $\frac{1}{b} > 1$, so the graph of $y = \left(\frac{1}{b}\right)^x$ is of the type in Figure 10.6(a). Now, using the rules of exponents:

$$y = \left(\frac{1}{b}\right)^{-x} = \left(\left(\frac{1}{b}\right)^{-1}\right)^x = b^x.$$

By the reflection principle, the graph of $y = \left(\frac{1}{b}\right)^{-x}$ is obtained by reflecting the graph of $y = (\frac{1}{b})^x$ about the y-axis. Putting these remarks together, if $0 < b < 1$, we conclude that the graph of $y = b^x$ will look like Figure 10.6(b). Notice, the graphs in Figure 10.6(b) share qualitative features, mirroring the features outlined previously, with the "asymptote" and "unbounded" portions of the graph interchanged. Graphs of this sort are often said to exhibit *exponential decay*, in the sense that the function values rapidly approach zero as we move to the right along the x-axis.

Important Facts 10.2.1 (Features of Exponential Type Functions). *Let b be a positive real number, not equal to 1. The graph of $y = b^x$ has these four properties:*

1. *The graph is always above the horizontal axis.*
2. *The graph has y-intercept 1.*
3. *If $b > 1$ (resp. $0 < b < 1$), the graph becomes closer and closer to the horizontal axis as we move to the left (resp. move to the right); this says the x-axis is a horizontal asymptote for the left-hand portion of the graph (resp. right-hand portion of the graph).*

4. *If* $b > 1$ *(resp.* $0 < b < 1$*), the graph becomes higher and higher above the horizontal axis as we move to the right (resp. move to the left); this says that the graph is unbounded as we move to the right (resp. move to the left).*

If $A_0 > 0$*, the graph of the function* $y = A_0 b^x$ *is a vertically expanded or compressed version of the graph of* $y = b^x$*. If* $A_0 < 0$*, we additionally reflect about the* x*-axis.*

10.3 Piano Frequency Range

A sound wave will cause your eardrum to move back and forth. In the case of a so-called *pure tone*, this motion is modeled by a function of the form

$$d(t) = A\sin(2\pi f t),$$

where f is called the *frequency*, in units of "periods/unit time", called "Hertz" and abbreviated "Hz". The coefficient A is related to the actual displacement of the eardrum, which is, in turn, related to the loudness of the sound. A person can typically perceive sounds ranging from 20 Hz to 20,000 Hz.

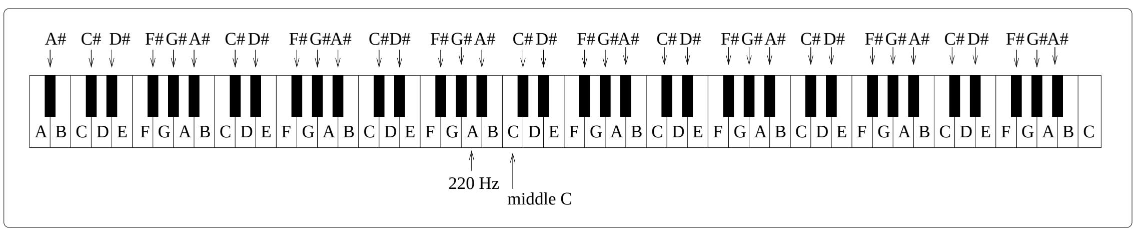


Figure 10.7: A piano keyboard.

A piano keyboard layout is shown in Figure 10.7. The white keys are labelled A, B, C, D, E, F, and G, with the sequence running from left to right and repeating for the length of the keyboard. The black keys fit into this sequence as "sharps", so that the black key between A and B is "A sharp", denoted $A^{\#}$. Thus, starting at any A key, the 12 keys to the right are A, $A^{\#}$, B, C, $C^{\#}$, D, $D^{\#}$, E, F, $F^{\#}$, G, and $G^{\#}$. The sequence then repeats. Notice that between some adjacent pairs of white keys there is no black key.

A piano keyboard is commonly tuned according to a rule requiring that each key (white and black) has a frequency $2^{1/12}$ times the frequency of the key to its immediate left. This makes the ratio of adjacent keys always the same ($2^{1/12}$), and it means that keys 12 keys apart have a ratio of frequencies exactly equal to 2 (since $(2^{1/12})^{12} = 2$). Two such keys are said to be an *o*ctave apart. Assuming that the key A below middle C has a frequency of 220 Hz, we can determine the frequency of every key on the keyboard. For instance, the A# to the right of this key has frequency $220 \times 2^{1/12} = 220 \times 1.059463094... \approx 233.08188\text{Hz}$. The B to the right of this key has frequency $233.08188 \times 2^{1/12} \approx 246.94165\text{Hz}$.

10.4 Exercises

Problem 10.1. Let's brush up on the required calculator skills. Use a calculator to approximate:

(a) 3^{π}

(b) $4^{2+\sqrt{5}}$

(c) π^{π}

(d) $5^{-\sqrt{3}}$

(e) 3^{π^2}

(f) $\sqrt{11^{\pi-7}}$

Problem 10.2. Put each equation in standard exponential form:

(a) $y = 3(2^{-x})$

(b) $y = 4^{-x/2}$

(c) $y = \pi^{\pi x}$

(d) $y = 1\left(\frac{1}{3}\right)^{3+\frac{x}{2}}$

(e) $y = \frac{5}{0.345^{2x-7}}$

(f) $y = 4(0.0003467)^{-0.4x+2}$

Problem 10.3. A colony of yeast cells is estimated to contain 10^6 cells at time $t = 0$. After collecting experimental data in the lab, you decide that the total population of cells at time t hours is given by the function

$$y = 10^6 e^{0.495105t}.$$

(a) How many cells are present after one hour?

(b) (True or False) The population of yeast cells will double every 1.4 hours.

(c) Cherie, another member of your lab, looks at your notebook and says : ...that formula is wrong, my calculations predict the formula for the number of yeast cells is given by the function

$$y = 10^6(2.042727)^{0.693147t}.$$

Should you be worried by Cherie's remark?

(d) Anja, a third member of your lab working with the same yeast cells, took these two measurements: 7.246×10^6 cells after 4 hours; 16.504×10^6 cells after 6 hours. Should you be worried by Anja's results? If Anja's measurements are correct, does your model over estimate or under estimate the number of yeast cells at time t?

Problem 10.4. (a) Find the frequency of middle C.

(b) Find the frequency of A above middle C.

(c) What is the frequency of the lowest note on the keyboard? Is there a way to solve this without simply computing the frequency of every key below $A220$?

(d) The Bosendorfer piano is famous, due in part, to the fact it includes additional keys at the left hand end of the keyboard, extending to the C below the bottom A on a standard keyboard. What is the lowest frequency produced by a Bosendorfer?

Problem 10.5. You have a chess board as pictured, with squares numbered 1 through 64. You also have a huge change jar with an unlimited number of dimes. On the first square you place one dime. On the second square you stack 2 dimes. Then you continue, always *doubling* the number from the previous square.

(a) How many dimes will you have stacked on the 10th square?

(b) How many dimes will you have stacked on the nth square?

(c) How many dimes will you have stacked on the 64th square?

(d) Assuming a dime is 1 mm thick, how high will this last pile be?

(e) The distance from the earth to the sun is approximately 150 million km. Relate the height of the last pile of dimes to this distance.

						63	64
						10	9
1	2	3					8

Problem 10.6. *Myoglobin* and *hemoglobin* are oxygen carrying molecules in the human body. Hemoglobin is found inside red blood cells, which flow from the lungs to the muscles through the bloodstream. Myoglobin is found in muscle cells. The function

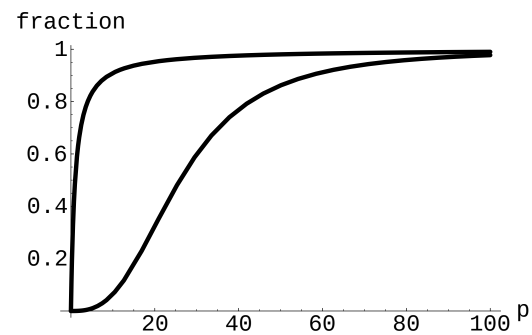
$$Y = M(p) = \frac{p}{1+p}$$

calculates the fraction of myoglobin saturated with oxygen at a given pressure p torrs. For example, at a pressure of 1 torr, $M(1) = 0.5$, which means half of the myoglobin (i.e. 50%) is oxygen saturated. (Note: More precisely, you need to use something called the "partial pressure", but the distinction is not important for this problem.) Likewise, the function

$$Y = H(p) = \frac{p^{2.8}}{26^{2.8} + p^{2.8}}$$

calculates the fraction of hemoglobin saturated with oxygen at a given pressure p.

(a) The graphs of $M(p)$ and $H(p)$ are given below on the domain $0 \leq p \leq 100$; which is which?



(b) If the pressure in the lungs is 100 torrs, what is the level of oxygen saturation of the hemoglobin in the lungs?

(c) The pressure in an active muscle is 20 torrs. What is the level of oxygen saturation of myoglobin in an active muscle? What is the level of hemoglobin in an active muscle?

(d) Define the efficiency of oxygen transport at a given pressure p to be $M(p) - H(p)$. What is the oxygen transport efficiency at 20 torrs? At 40 torrs? At 60 torrs? Sketch the graph of $M(p) - H(p)$; are there conditions under which transport efficiency is maximized (explain)?