Chapter 5

Functions and Graphs

Pictures are certainly important in the work of an architect, but it is perhaps less evident that visual aids can be powerful tools for solving mathematical problems. If we start with an equation and attach a picture, then the mathematics can come to life. This adds a new dimension to both interpreting and solving problems. One of the real triumphs of modern mathematics is a theory connecting pictures and equations via the concept of a graph. This transition from “equation” to “picture” (called graphing) and its usefulness (called graphical analysis) are the theme of the next two sections. The importance of these ideas is HUGE and cannot be overstated. Every moment spent studying these ideas will pay back dividends in this course and in any future mathematics, science or engineering courses.

5.1 Relating Data, Plots and Equations

Imagine you are standing high atop an oceanside cliff and spot a seagull hovering in the air-current. Assuming the gull moves up and down along a vertical line of motion, how can we best describe its location at time $t$ seconds?

There are three different (but closely linked) ways to describe the location of the gull:

- a table of data of the gull’s height above cliff level at various times $t$;
- a plot of the data in a “time” (seconds) vs. “height” (feet) coordinate system;
- an equation relating time $t$ (seconds) and height $s$ (feet).

To make sure we really understand how to pass back and forth between these three descriptive modes, imagine we have tabulated (Figure 5.2) the height of the gull above cliff level at one-second time intervals.
for a 10 second time period. Here, a “negative height” means the gull is below cliff level. We can try to visualize the meaning of this data by plotting these 11 data points \((t, s)\) in a time (sec.) vs. height (ft.) coordinate system.

<table>
<thead>
<tr>
<th>(t) (sec)</th>
<th>(s) (ft)</th>
<th>(t) (sec)</th>
<th>(s) (ft)</th>
<th>(t) (sec)</th>
<th>(s) (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20</td>
<td>4</td>
<td>-10</td>
<td>8</td>
<td>20</td>
</tr>
<tr>
<td>1</td>
<td>6.88</td>
<td>5</td>
<td>-8.12</td>
<td>9</td>
<td>36.88</td>
</tr>
<tr>
<td>2</td>
<td>-2.5</td>
<td>6</td>
<td>-2.5</td>
<td>10</td>
<td>57.5</td>
</tr>
<tr>
<td>3</td>
<td>-8.12</td>
<td>7</td>
<td>6.88</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(a) Symbolic data.

(b) Visual data.

Figure 5.2: Symbolic versus visual view of data.

We can improve the quality of this description by increasing the number of data points. For example, if we tabulate the height of the gull above cliff level at 1/2 second or 1/4 second time intervals (over the same 10 second time period), we might get these two plots:

(a) \(\frac{1}{2}\) second intervals.

(b) \(\frac{1}{4}\) second intervals.

Figure 5.3: Shorter time intervals mean more data points.

We have focused on how to go from data to a plot, but the reverse process is just as easy: A point \((t, s)\) in any of these three plots is interpreted to mean that the gull is \(s\) feet above cliff level at time \(t\) seconds.

Furthermore, increasing the amount of data, we see how the plotted points are “filling in” a portion of a parabola. Of course, it is way too tedious to create longer and longer tables of data. What we really want is a “formula” (think of it as a prescription) that tells us how to produce a data point for the gull’s height at any given time \(t\). If we had such a formula, then we could completely dispense with the tables of data and just use the formula to crank out data points. For example, look at this equation involving the variables \(t\) and \(s\):

\[
s = \frac{15}{8} (t - 4)^2 - 10.
\]
If we plug in \( t = 0, 1, 2, 9, 10 \), then we get \( s = 20, 6.88, -2.5, 36.88, 57.5 \), respectively; this was some of our initial tabulated data. This same equation produces ALL of the data points for the other two plots, using 1/2 second and 1/4 second time intervals. (Granted, we have swept under the rug the issue of "...where the heck the equation comes from..."; that is a consequence of mathematically modeling the motion of this gull. Right now, we are focusing on how the equation relates to the data and the plot, assuming the equation is in front of us to start with.) In addition, it is very important to notice that having this equation produces an infinite number of data points for our gull’s location, since we can plug in any \( t \) value between 0 and 10 and get out a corresponding height \( s \). In other words, the equation is A LOT more powerful than a finite (usually called discrete) collection of tabulated data.

### 5.2 What is a Function?

Our lives are chock full of examples where two changing quantities are related to one another:

- The cost of postage is related to the weight of the item.
- The value of an investment will depend upon the time elapsed.
- The population of cells in a growth medium will be related to the amount of time elapsed.
- The speed of a chemical reaction will be related to the temperature of the reaction vessel.

In all such cases, it would be beneficial to have a “procedure” whereby we can assign a unique output value to any acceptable input value. For example, given the time elapsed (an input value), we would like to predict a unique future value of an investment (the output value). Informally, this leads to the broadest (and hence most applicable) definition of what we will call a function:

**Definition 5.2.1.** A function is a procedure for assigning a unique output to any allowable input.

The key word here is “procedure.” Our discussion of the hovering seagull in 5.1 highlights three ways to produce such a “procedure” using data, plots of curves and equations.

- A table of data, by its very nature, will relate two columns of data: The output and input values are listed as column entries of the table and reading across each row is the “procedure” which relates an input with a unique output.
• Given a curve in Figure 5.4, consider the “procedure” which associates to each $x$ on the horizontal axis the $y$ coordinate of the pictured point $P$ on the curve.

• Given an equation relating two quantities $x$ and $y$, plugging in a particular $x$ value and going through the “procedure” of algebra often produces a unique output value $y$.

### 5.2.1 The definition of a function (equation viewpoint)

Now we focus on giving a precise definition of a function, in the situation when the “procedure” relating two quantities is actually given by an equation. Keep in mind, this is only one of three possible ways to describe a function; we could alternatively use tables of data or the plot of a curve. We focus on the equation viewpoint first, since it is no doubt the most familiar.

If we think of $x$ and $y$ as related physical quantities (e.g. time and distance), then it is sometimes possible (and often desirable) to express one of the variables in terms of the other. For example, by simple arithmetic, the equations

\[
3x + 2y = 4 \quad x^2 - x = \frac{1}{2}y - 4 \quad y\sqrt{x^2 + 1} = 1,
\]

can be rewritten as equivalent equations

\[
y = \frac{1}{2}(4 - 3x) \quad 2x^2 - 2x + 8 = y \quad y = \frac{1}{\sqrt{x^2 + 1}}.
\]

This leads to THE MOST IMPORTANT MATH DEFINITION IN THE WORLD:

**Definition 5.2.2.** A function is a package, consisting of three parts:

- An equation of the form

  \[y = "a \text{ mathematical expression only involving the variable } x,"\]

  which we usually indicate via the shorthand notation $y = f(x)$. This equation has the very special property that each time we plug in an $x$ value, it produces exactly one (a unique) $y$ value. We call the mathematical expression $f(x)$ "the rule".

- A set $D$ of $x$-values we are allowed to plug into $f(x)$, called the "domain" of the function.
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- The set $\mathbb{R}$ of output values $f(x)$, where $x$ varies over the domain, called the "range" of the function.

Any time we have a function $y = f(x)$, we refer to $x$ as the independent variable (the "input data") and $y$ as the dependent variable (the "output data"). This terminology emphasizes the fact that we have freedom in the values of $x$ we plug in, but once we specify an $x$ value, the $y$ value is uniquely determined by the rule $f(x)$.

**Examples 5.2.3.**

(i) The equation $y = -2x + 3$ is in the form $y = f(x)$, where the rule is $f(x) = -2x + 3$. Once we specify a domain of $x$ values, we have a function. For example, we could let the domain be all real numbers.

(ii) Take the same rule $f(x) = -2x + 3$ from (i) and let the domain be all non-negative real numbers. This describes a function. However, the functions $f(x) = -2x + 3$ on the domain of all non-negative real numbers and $f(x) = -2x + 3$ on the domain of all real numbers (from (i)) are different, even though they share the same rule: this is because their domains differ! This example illustrates the idea of what is called a restricted domain. In other words, we started with the function in (i) on the domain of all real numbers, then we "restricted" to the subset of non-negative real numbers.

(iii) The equation $y = b$, where $b$ is a constant, defines a function on the domain of all real numbers, where the rule is $f(x) = b$; we call these the constant functions. Recall, in Chapter 3, we observed that the solutions of the equation $y = b$, plotted in the $xy$ coordinate system, will give a horizontal line. For example, if $b = 0$, you get the horizontal axis.

(iv) Consider the equation $y = \frac{1}{x}$, then the rule $f(x) = \frac{1}{x}$ defines a function, as long as we do not plug in $x = 0$. For example, take the domain to be the non-zero real numbers.

(v) Consider the equation $y = \sqrt{1 - x^2}$. Before we start plugging in $x$ values, we want to know the expression under the radical symbol (square root symbol) is non-negative; this insures the square root is a real number. This amounts to solving an inequality equation: $0 \leq 1 - x^2$; i.e., $-1 \leq x \leq 1$. These remarks show that the rule $f(x) = \sqrt{1 - x^2}$ defines a function, where the domain of $x$ values is $-1 \leq x \leq 1$. 

Graph of $y = b$

Figure 5.5: Constant function.
Typically, the domain of a function $y = f(x)$ will either be the entire number line, an interval on the number line, or a finite union of such intervals. We summarize the notation used to represent intervals in Table 5.1.

<table>
<thead>
<tr>
<th>Description</th>
<th>Symbolic Notation</th>
<th>Picture</th>
</tr>
</thead>
<tbody>
<tr>
<td>All numbers $x$ between $a$ and $b$, $x$ possibly equal to either $a$ or $b$</td>
<td>$a \leq x \leq b$</td>
<td>$a\ b$</td>
</tr>
<tr>
<td>All numbers $x$ between $a$ and $b$, $x \neq a$ and $x \neq b$</td>
<td>$a &lt; x &lt; b$</td>
<td>$a\ b$</td>
</tr>
<tr>
<td>All numbers $x$ between $a$ and $b$, $x \neq b$ and $x$ possibly equal to $a$</td>
<td>$a \leq x &lt; b$</td>
<td>$a\ b$</td>
</tr>
<tr>
<td>All numbers $x$ between $a$ and $b$, $x \neq a$ and $x$ possibly equal to $b$</td>
<td>$a &lt; x \leq b$</td>
<td>$a\ b$</td>
</tr>
</tbody>
</table>

Table 5.1: Interval Notations

We can interpret a function as a “prescription” that takes a given $x$ value (in the domain) and produces a single unique $y$ value (in the range). We need to be really careful and not fall into the trap of thinking that every equation in the world is a function. For example, if we look at this equation

$$x + y^2 = 1$$

and plug in $x = 0$, the equation becomes

$$y^2 = 1.$$ 

This equation has two solutions, $y = \pm 1$, so the conclusion is that plugging in $x = 0$ does NOT produce a single output value. This violates one of the conditions of our function definition, so the equation $x + y^2 = 1$ is NOT a function in the independent variable $x$. Notice, if you were to try and solve this equation for $y$ in terms of $x$, you’d first write $y^2 = 1 - x$ and then take a square root (to isolate $y$); but the square root introduces TWO roots, which is just another way of reflecting the fact there can be two $y$ values attached to a single $x$ value. Alternatively, you can solve the equation for $x$ in terms of $y$, getting $x = 1 - y^2$; this shows the equation does define a function $x = g(y)$ in the independent variable $y$. 
5.2. WHAT IS A FUNCTION?

5.2.2 The definition of a function
(conceptual viewpoint)

Conceptually, you can think of a function as a “process”: An allowable input goes into a “black box” and out pops a unique new value denoted by the symbol $f(x)$. Compare this with the machine making “hula-hoops” in Figure 5.6. While you are problem solving, you will find this to be a useful viewpoint when a function is described in words.

Examples 5.2.4. Here are four examples of relationships that are functions:

(i) **The total amount of water used by a household since midnight on a particular day.** Let $y$ be the total number of gallons of water used by a household between 12:00am and a particular time $t$; we will use time units of “hours.” Given a time $t$, the household will have used a specific (unique) amount of water, call it $S(t)$. Then $y = S(t)$ defines a function in the independent variable $t$ with dependent variable $y$. The domain would be $0 \leq t \leq 24$ and the largest possible value of $S(t)$ on this domain is $S(24)$. This tells us that the range would be the set of values $0 \leq y \leq S(24)$.

(ii) **The height of the center of a basketball as you dribble, depending on time.** Let $s$ be the height of the basketball center at time $t$ seconds after you start dribbling. Given a time $t$, if we freeze the action, the center of the ball has a single unique height above the floor, call it $h(t)$. So, the height of the basketball center is given by a function $s = h(t)$. The domain would be a given interval of time you are dribbling the ball; for example, maybe $0 \leq t \leq 2$ (the first 2 seconds). In this case, the range would be all of the possible heights attained by the center of the basketball during this 2 seconds.

(iii) **The state sales tax due on a taxable item.** Let $T$ be the state tax (in dollars) due on a taxable item that sells for $z$ dollars. Given a taxable item that costs $z$ dollars, the state tax due is a single unique
amount, call it \( W(z) \). So, \( T = W(z) \) is a function, where the independent variable is \( z \). The domain could be taken to be \( 0 \leq z \leq 1,000,000 \), which would cover all items costing up to one-million dollars. The range of the function would be the set of all values \( W(z) \), as \( z \) ranges over the domain.

(iv) **The speed of a chemical reaction depending on the temperature.** Let \( v \) be the speed of a particular chemical reaction and \( T \) the temperature in Celsius \( \circ \)C. Given a particular temperature \( T \), one could experimentally measure the speed of the reaction; there will be a unique speed, call it \( r(T) \). So, \( v = r(T) \) is a function, where the independent variable is \( T \). The domain could be taken to be \( 0 \leq T \leq 100 \), which would cover the range of temperatures between the freezing and boiling points of water. The range of the function would be the set of all speeds \( r(T) \), as \( T \) ranges over the domain.

### 5.3 The Graph of a Function

Let’s start with a concrete example: the function \( f(x) = -2x + 3 \) on the domain of all real numbers. We discussed this Example 5.2.3. Plug in the specific \( x \) values, where \( x = -1, 0, 1, 2 \) and tabulate the resulting \( y \) values of the function:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>point ((x,y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-5</td>
<td>((-1, -5))</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>((0, 3))</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>((1, 1))</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>((2, -1))</td>
</tr>
</tbody>
</table>

(a) Tabulated data.

![Graph of \( y = -2x + 3 \).](image.png)  
(b) Visual data.

**Figure 5.7:** Symbolic versus visual view of data.

This tells us that the points \((0, 3), (1, 1), (2, -1), (-1, 5)\) are solutions of the equation \( y = -2x + 3 \). For example, if \( y = -2x + 3, x = 0, y = 3 \), then \( 3 = -2 \cdot 0 + 3 \) (which is true), or if \( y = -2x + 3, x = 2, y = -1 \), then \( -1 = -2 \cdot 2 + 3 \) (which is true), etc. In general, if we plug in \( x \) we get out \(-2x + 3\), so the point \((x, -2x + 3)\) is a solution to the function equation \( y = f(x) \). We can plot all of these solutions in the \( xy \)-coordinate system. The set of points we obtain, as we vary over all \( x \) in the domain, is called the set of **solutions** of the equation \( y = -2x + 3 \):

\[
\text{Solutions} = \{(x, -2x + 3) \mid x \text{ any real number}\}.
\]

Notice that plotting these points produces a line of slope \( m = -2 \) with \( y \)-intercept 3. In other words, the graph of the function \( f(x) = -2x + 3 \) is
5.4. **THE VERTICAL LINE TEST**

the same as the graph of the equation \( y = -2x + 3 \), as we discussed in Chapter 4.

In general, by definition, we say that a point \((x,y)\) is a solution to the function equation \( y = f(x) \) if plugging \( x \) and \( y \) into the equation gives a true statement.

How can we find ALL the solutions of the equation \( y = f(x) \)? In general, the definition of a function is “rigged” so it is easy to describe all solutions of the equation \( y = f(x) \): Each time we specify an \( x \) value (in the domain), there is only one \( y \) value, namely \( f(x) \). This means the point \( P = (x,f(x)) \) is the ONLY solution to the equation \( y = f(x) \) with first coordinate \( x \). We define the graph of the function \( y = f(x) \) to be the plot of all solutions of this equation (in the \( xy \) coordinate system). It is common to refer to this as either the “graph of \( f(x) \)” or the “graph of \( f \).”

\[
\text{Graph} = \{(x,f(x)) | x \text{ in the domain}\} \quad (5.1)
\]

**Important Procedure 5.3.1. Points on a graph.** The description of the graph of a function gives us a procedure to produce points on the graph AND to test whether a given point is on the graph. On the one hand, if you are given \( u \) in the domain of a function \( y = f(x) \), then you immediately can plot the point \((u,f(u))\) on the graph. On the other hand, if someone gives you a point \((u,v)\), it will be on the graph only if \( v = f(u) \) is true. We illustrate this in Example 5.3.2.

**Example 5.3.2.** The function \( s = h(t) = \frac{15}{8}(t - 4)^2 - 10 \) defines a function in the independent variable \( t \). If we restrict to the domain \( 0 \leq t \leq 10 \), then the discussion in Chapter 7 tells us that the graph is a portion of a parabola: See Figure 5.8. Using the above procedure, you can verify that the data points discussed in the seagull example (in §5.1) all lie on this parabola. On the other hand, the point \((0,0)\) is NOT on the graph, since \( h(0) = 20 \neq 0 \).

![Figure 5.8: s = h(t).](image)

**5.4 The Vertical Line Test**

There is a pictorial aspect of the graph of a function that is very revealing: Since \((x,f(x))\) is the only point on the graph with first coordinate equal to \( x \), a vertical line passing through \( x \) on the \( x \)-axis (with \( x \) in the domain) crosses the graph of \( y = f(x) \) once and only once. This gives us a decisive way to test if a curve is the graph of a function.

**Important Procedure 5.4.1. The vertical line test.** Draw a curve in the \( xy \)-plane and specify a set \( D \) of \( x \)-values. Suppose every vertical line through a value in \( D \) intersects the curve exactly once. Then the curve is
the graph of some function on the domain \( D \). If we can find a single vertical line through some value in \( D \) that intersects the curve more than once, then the curve is not the graph of a function on the domain \( D \).

For example, draw any straight line \( m \) in the plane. By the vertical line test, if the line \( m \) is not vertical, \( m \) is the graph of a function. On the other hand, if the line \( m \) is vertical, then \( m \) is not the graph of a function. These two situations are illustrated in Figure 5.9. As another example, consider the equation \( x^2 + y^2 = 1 \), whose graph is the unit circle and specify the domain \( D \) to be \(-1 \leq x \leq 1\); recall Example 3.2.2. The vertical line passing through the point \((\frac{1}{2}, 0)\) will intersect the unit circle twice; by the vertical line test, the unit circle is not the graph of a function on the domain \(-1 \leq x \leq 1\).

![Figure 5.9: Applying the vertical line test.](image)

5.4.1 Imposed Constraints

In physical problems, it might be natural to constrain (meaning to “limit” or “restrict”) the domain. As an example, suppose the height \( s \) (in feet) of a ball above the ground after \( t \) seconds is given by the function

\[
s = h(t) = -16t^2 + 4.
\]

We could look at the graph of the function in the ts-plane and we will review in Chapter 7 that the graph looks like a parabola. The physical context of this problem makes it natural to only consider the portion of the graph in the first quadrant; why? One way of specifying this quadrant would be to restrict the domain of possible \( t \) values to lie between 0 and \( \frac{1}{2} \); notationally, we would write this constraint as \( 0 \leq t \leq \frac{1}{2} \).

![Figure 5.10: Restricting the domain.](image)

5.5 Linear Functions

A major goal of this course is to discuss several different kinds of functions. The work we did in Chapter 4 actually sets us up to describe one
very useful type of function called a *linear function*. Back in Chapter 4, we discussed how lines in the plane can be described using equations in the variables $x$ and $y$. One of the key conclusions was:

**Important Fact 5.5.1.** A non-vertical line in the plane will be the graph of an equation $y = mx + b$, where $m$ is the slope of the line and $b$ is the $y$-intercept.

Notice that any non-vertical line will satisfy the conditions of the vertical line test, which means it must be the graph of a function. What is the function? The answer is to use the equation in $x$ and $y$ we already obtained in Chapter 4: The rule $f(x) = mx + b$ on some specified domain will have a line of slope $m$ and $y$-intercept $b$ as its graph. We call a function of this form a *linear function*.

**Example 5.5.2.** You are driving 65 mph from the Kansas state line (mile marker 0) to Salina (mile marker 130) along I-35. Describe a linear function that calculates mile marker after $t$ hours. Describe another linear function that will calculate your distance from Salina after $t$ hours.

**Solution.** Define a function $d(t)$ to be the mile marker after $t$ hours. Using “distance=rate×time,” we conclude that $65t$ will be the distance traveled after $t$ hours. Since we started at mile marker 0, $d(t) = 65t$ is the rule for the first function. A reasonable domain would be to take $0 \leq t \leq 2$, since it takes 2 hours to reach Salina.

For the second situation, we need to describe a different function, call it $s(t)$, that calculates your distance from Salina after $t$ hours. To describe the rule of $s(t)$ we can use the previous work:

$$
  s(t) = (\text{mile marker Salina}) - \\
  \hspace{1cm} (\text{your mile marker at } t \text{ hrs.}) \\
  = 130 - d(t) \\
  = 130 - 65t.
$$

For the rule $s(t)$, the best domain would again be $0 \leq t \leq 2$. We have graphed these two functions in the same coordinate system: See Figure 5.11 (Which function goes with which graph?).

### 5.6 Profit Analysis

Let's give a first example of how to interpret the graph of a function in the context of an application.
Example 5.6.1. A software company plans to bring a new product to market. The sales price per unit is $15 and the expense to produce and market \( x \) units is $100(1 + \sqrt{x})$. What is the profit potential?

Two functions control the profit potential of the new software. The first tells us the **gross income**, in dollars, on the sale of \( x \) units. All of the costs involved in developing, supporting, distributing and marketing \( x \) units are controlled by the expense equation (again in dollars):

\[
\begin{align*}
\text{g}(x) &= 15x \\
\text{e}(x) &= 100(1 + \sqrt{x})
\end{align*}
\]

A **profit** will be realized on the sale of \( x \) units whenever the gross income exceeds expenses; i.e., this occurs when \( g(x) > e(x) \). A **loss** occurs on the sale of \( x \) units when expenses exceed gross income; i.e., when \( e(x) > g(x) \). Whenever the sale of \( x \) units yields zero profit (and zero loss), we call \( x \) a **break-even point**; i.e., when \( e(x) = g(x) \).

The above approach is “symbolic.” Let’s see how to study profit and loss visually, by studying the graphs of the two functions \( g(x) \) and \( e(x) \). To begin with, plot the graphs of the two individual functions in the \( xy \)-coordinate system. We will focus on the situation when the sales figures are between 0 to 100 units; so the domain of \( x \) values is the interval \( 0 \leq x \leq 100 \). Given any sales figure \( x \), we can graphically relate three things:

- \( x \) on the horizontal axis;
- a point on the graph of the gross income or expense function;
- \( y \) on the vertical axis.

If \( x = 20 \) units sold, there is a unique point \( P = (20, g(20)) = (20, 300) \) on the gross income graph and a unique point \( Q = (20, e(20)) = (20, 547) \) on the expenses graph. Since the \( y \)-coordinates of \( P \) and \( Q \) are the function values at \( x = 20 \), the height of the point above the horizontal axis is controlled by the function.
If we plot both graphs in the same coordinate system, we can visually study the distance between points on each graph above $x$ on the horizontal axis. In the first part of this plot, the expense graph is above the income graph, showing a loss is realized; the exact amount of the loss will be $e(x) - g(x)$, which is the length of the pictured line segment. Further to the right, the two graphs cross at the point labeled “B”; this is the break-even point; i.e., expense and income agree, so there is zero profit (and zero loss). Finally, to the right of B the income graph is above the expense graph, so there is a profit; the exact amount of the profit will be $g(x) - e(x)$, which is the length of the right-most line segment. Our analysis will be complete once we pin down the break-even point $B$. This amounts to solving the equation $g(x) = e(x)$.

\[
15x = 100(1 + \sqrt{x})
\]
\[
15x - 100 = 100\sqrt{x}
\]
\[
225x^2 - 3000x + 10000 = 10000x
\]
\[
225x^2 - 13000x + 10000 = 0.
\]

Applying the quadratic formula, we get two answers: $x = 0.78$ or 57.

Now, we face a problem: Which of these two solutions is the answer to the original problem? We are going to argue that only the second solution $x = 57$ gives us the break even point. What about the other "solution" at $x = 0.78$? Try plugging $x = 0.78$ into the original equation: $15(0.78) \neq 100(1 + \sqrt{0.78})$. What has happened? Well, when going from the second to the third line, both sides of the equation were squared. Whenever we do this, we run the risk of adding extraneous solutions. What should you do? After solving any equation, look back at your steps and ask yourself whether or not you may have added (or lost) solutions. In particular, be wary when squaring or taking the square root of both sides of an equation. Always check your final answer in the original equation.

We can now compute the coordinates of the break-even point using either function:

$$B = (57, g(57)) = (57, 855) = (57, e(57)).$$
5.7 Exercises

Problem 5.1. For each of the following functions, find the expression for
\[
\frac{f(x + h) - f(x)}{h}
\]
Simplify each of your expressions far enough so that plugging in \( h = 0 \) would be allowed.

(a) \( f(x) = x^2 - 2x \).
(b) \( f(x) = 2x + 3 \).
(c) \( f(x) = x^2 - 3 \).
(d) \( f(x) = 4 - x^2 \).
(e) \( f(x) = -\pi x^2 - \pi^2 \).
(f) \( f(x) = \sqrt{x - 1} \). (Hint: Rationalize the numerator)

Problem 5.2. Here are the graphs of two linear functions on the domain \( 0 \leq x \leq 20 \). Find the formula for each of the rules \( y = f(x) \) and \( y = g(x) \). Find the formula for a NEW function \( v(x) \) that calculates the vertical distance between the two lines at \( x \). Explain in terms of the picture what \( v(x) \) is calculating. What is \( v(5) \)? What is \( v(20) \)? What are the smallest and largest values of \( v(x) \) on the domain \( 0 \leq x \leq 20 \)?

Problem 5.3. Dave leaves his office in Padelford Hall on his way to teach in Gould Hall. Below are several different scenarios. In each case, sketch a plausible (reasonable) graph of the function \( s = d(t) \) which keeps track of Dave’s distance \( s \) from Padelford Hall at time \( t \). Take distance units to be “feet” and time units to be “minutes.” Assume Dave’s path to Gould Hall is along a straight line which is 2400 feet long.

(a) Dave leaves Padelford Hall and walks at a constant speed until he reaches Gould Hall 10 minutes later.
(b) Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute. He then continues on to Gould Hall at the same constant speed he had when he originally left Padelford Hall.
(c) Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute to figure out where he is. Dave then continues on to Gould Hall at twice the constant speed he had when he originally left Padelford Hall.
(d) Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Dave gets confused and stops for 1 minute to figure out where he is. Dave is totally lost, so he simply heads back to his office, walking the same constant speed he had when he originally left Padelford Hall.
(e) Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Dave gets confused and stops for 1 minute to figure out where he is. Dave then continues on to Gould Hall at twice the constant speed he had when he originally left Padelford Hall.
(f) Suppose you want to sketch the graph of a new function \( s = g(t) \) that keeps track of Dave’s distance \( s \) from Gould Hall at time \( t \). How would your graphs change in (a)-(e)?

Problem 5.4. At 5 AM one day, a monk began a trek from his monastery by the sea to the monastery at the top of a mountain. He reached the mountain-top monastery at 11 AM, spent the rest of the day in meditation,
and then slept the night there. In the morning, at 5 AM, he began walking back to the seaside monastery. Though walking downhill should have been faster, he dawdled in the beautiful sunshine, and ending up getting to the seaside monastery at exactly 11 AM.

(a) Was there necessarily a time during each trip when the monk was in exactly the same place on both days? Why or why not?

(b) Suppose the monk walked faster on the second day, and got back at 9 AM. What is your answer to part (a) in this case?

(c) Suppose the monk started later, at 10 AM, and reached the seaside monastery at 3 PM. What is your answer to part (a) in this case?

Problem 5.5. Sketch a reasonable graph for each of the following functions. Specify a reasonable domain and range and state any assumptions you are making. Finally, describe the largest and smallest values of your function.

(a) Height of a person depending on age.

(b) Height of the top of your head as you jump on a pogo stick for 5 seconds.

(c) The amount of postage you must put on a first class letter, depending on the weight of the letter.

(d) Distance of your big toe from the ground as you ride your bike for 10 seconds.

(e) Your height above the water level in a swimming pool after you dive off the high board.

Problem 5.6. Here is a picture of the graph of the function $f(x) = 3x^2 - 3x - 2$.

(a) Explain why we can assume the cable follows the path indicated in the picture;

Recall the procedure 5.3.1 on page 63.

(a) Find the $x$ and $y$ intercepts of the graph.

(b) Find the exact coordinates of all points $(x,y)$ on the graph which have $y$-coordinate equal to 5.

(c) Find the coordinates of all points $(x,y)$ on the graph which have $y$-coordinate equal to -3.

(d) Which of these points is on the graph: $(1, -2), (-1,3), (2.4,8), (\sqrt{3},7 - 3\sqrt{3})$.

(e) Find the exact coordinates of the point $(x,y)$ on the graph with $x = \sqrt{1 + \sqrt{2}}$.

Problem 5.7. After winning the lottery, you decide to buy your own island. The island is located 1 km offshore from a straight portion of the mainland. There is currently no source of electricity on the island, so you want to run a cable from the mainland to the island. An electrical power sub-station is located 4 km from your island’s nearest location to the shore. It costs $50,000 per km to lay a cable in the water and $30,000 per km to lay a cable over the land.

(a) Explain why we can assume the cable follows the path indicated in the picture;
i.e., explain why the path consists of two line segments, rather than a weird curved path AND why it is OK to assume the cable reaches shore to the right of the power station and the left of the island.

(b) Let $x$ be the distance downshore from the power sub-station to where the cable reaches the land. Find a function $f(x)$ in the variable $x$ that computes the cost to lay a cable out to your island.

(c) Make a table of values of $f(x)$, where $x = 0, 1, 2, 3, 4$. Use these calculations to estimate the installation of minimal cost.

**Problem 5.8.** This problem deals with the “mechanical aspects” of working with the rule of a function. For each of the functions listed in (a)-(c), calculate: $f(0)$, $f(-2)$, $f(x + 3)$, $f(\Diamond)$, $f(\Diamond + \triangle)$.

(a) The function $f(x) = \frac{1}{2}(x - 3)$ on the domain of all real numbers.

(b) The function $f(x) = 2x^2 - 6x$ on the domain of all real numbers.

(c) The function $f(x) = 4\pi^2$.

**Problem 5.9.** Which of the curves in Figure 5.14 represent the graph of a function? If the curve is not the graph of a function, describe what goes wrong and how you might “fix it.” When you describe how to “fix” the graph, you are allowed to cut the curve into pieces and such that each piece is the graph of a function. Many of these problems have more than one correct answer.

**Problem 5.10.** Find an EXACT answer for each problem.

(a) Solve for $x$

$$\frac{x}{x + 3} + \frac{5}{x - 7} = \frac{30}{x^2 - 4x - 21}$$

(b) Solve for $x$

$$\sqrt{5x - 4} = \frac{x}{2} + 2$$

(c) Solve for $x$

$$\sqrt{x} + \sqrt{x - 20} = 10$$

(d) Solve for $t$

$$\sqrt{2t - 1} + \sqrt{3t + 3} = 5$$
Figure 5.14: Curves to consider for Problem 5.9