Chapter 7 TECHNIQUES OF INTEGRATION

7.1 Integration by Part

Integration by parts formula:

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)dx.$$

When you use Maple for integration, you need not use the rule for evaluating integrals.

7.1.1 Evaluate integrals

Example 1. Evaluate $\int x\cos(x)\mathrm{d}x$.

```
> int(x*cos(x),x);
```

$$\cos(x) + x\sin(x) \tag{1.1.1}$$

Example 2. Evaluate $\int x\mathrm{e}^{x}\mathrm{d}x$.

```
> int(x*exp(x),x);
```

$$(-1 + x)\,\mathrm{e}^{x} \tag{1.1.2}$$

Example 3. Calculate $\int \mathrm{e}^{x}\sin(x)\mathrm{d}x$.

```
> int(exp(x)*sin(x),x);
```

$$-\frac{1}{2}\,\mathrm{e}^{x}\cos(x) + \frac{1}{2}\,\mathrm{e}^{x}\sin(x) \tag{1.1.3}$$

Example 4. Calculate $\int \sqrt{x}\,\ln(x)\mathrm{d}x$.

```
> int(sqrt(x)*ln(x),x);
```

$$\frac{2}{3}\,x^{(3/2)}\ln(x) - \frac{4}{9}\,x^{(3/2)} \tag{1.1.4}$$

Example 5. Find $\int_0^1 \arctan(x)\mathrm{d}x$.

```
> int(arctan(x),x=0..1);
```

$$\frac{1}{4}\,\pi - \frac{1}{2}\ln(2) \tag{1.1.5}$$

Example 6. Find $\int_0^{\frac{\pi}{4}} x\sin(2x)\mathrm{d}x$.

```
> int(x*sin(2*x),x=0..Pi/4);
```

$$\frac{1}{4} \tag{1.1.6}$$

Exercises

1. Find $\int x^2 \, e^{-x} \mathrm{d}x$.
2. Find $\int \ln(x) \mathrm{d}x$.
3. Find $\int e^x \cos(2x) \mathrm{d}x$.
4. Find $\int_0^1 \arcsin(x) \mathrm{d}x$.
5. Find $\int_0^{\pi} \sin(x) \cos(3x) \mathrm{d}x$.

7.2 Trigonometric Integrals

In this section we consider integrals such as $\int \sin(x)^m \cos(x)^n \, \mathrm{d}x$.

7.2.1 Calculate trigonometric integrals

Example 1. Evaluate $\int \sin(x)^4 \cos(x)^5 \mathrm{d}x$.

```
> int(sin(x)^4*cos(x)^5,x);
```

$$-\frac{1}{9} \sin(x)^3 \cos(x)^6 - \frac{1}{21} \sin(x) \cos(x)^6 + \frac{1}{105} \cos(x)^4 \sin(x) + \frac{4}{315} \cos(x)^2 \sin(x) + \frac{8}{315} \sin(x) \qquad \textbf{(2.1.1)}$$

Example 2. Evaluate $\int \sin(x)^4 \mathrm{d}x$.

```
> int(sin(x)^4,x);
```

$$-\frac{1}{4} \sin(x)^3 \cos(x) - \frac{3}{8} \cos(x) \sin(x) + \frac{3}{8} x \qquad \textbf{(2.1.2)}$$

Example 3. Evaluate $\int \sin(x)^4 \cos(x)^2 \mathrm{d}x$.

```
> int(sin(x)^4*cos(x)^2,x);
```

$$-\frac{1}{6} \sin(x)^3 \cos(x)^3 - \frac{1}{8} \sin(x) \cos(x)^3 + \frac{1}{16} \cos(x) \sin(x) + \frac{1}{16} x \qquad \textbf{(2.1.3)}$$

Example 4. Evaluate $\int \tan(x) \mathrm{d}x$.

```
> int(tan(x),x);
```

$$-\ln(\cos(x)) \qquad \textbf{(2.1.4)}$$

Example 5. Find $\int_0^{\frac{\pi}{4}} \tan(x)^3 \mathrm{d}x$.

```
> int(tan(x)^3,x=0..Pi/4);
```

$$\frac{1}{2} - \frac{1}{2}\ln(2) \tag{2.1.5}$$

Example 6. Find $\int_0^{\pi} \sin(4x)\cos(3x)\mathrm{d}x$.

```
> int(sin(4*x)*cos(3*x),x=0..Pi);
```

$$\frac{8}{7} \tag{2.1.6}$$

Exercises

1. Find $\int \sin(x)^2 \cos(x)^3 \mathrm{d}x$.
2. Find $\int \sin(x)^4 \cos(x)^2 \mathrm{d}x$.
3. Find $\int \tan(x)^2 \mathrm{d}x$.
4. Find $\int_0^{\pi} \sin(x)^6 \mathrm{d}x$.
5. Find $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\cos(x)} \mathrm{d}x$.

7.3 Trigonometric Substitution

To integrate functions involving square root expressions, a useful approach to the integration is trigonometric substitution. Once again, Maple hides all of these substitutions, and you can evaluate these integrals directly.

7.3.1 Evaluate integrals involving square root expressions

Example 1. Evaluate $\int \sqrt{1-x^2}\,\mathrm{d}x$.

```
> int(sqrt(1-x^2),x);
```

$$\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin(x) \tag{3.1.1}$$

Example 2. Evaluate $\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\mathrm{d}x$.

```
> int( x^2/(4-x^2)^(3/2),x);
```

$$\frac{x}{\sqrt{4-x^2}} - \arcsin\left(\frac{1}{2}\,x\right) \tag{3.1.2}$$

Example 3. Evaluate $\int \sqrt{4x^2+20}\,dx$.

```
> int(sqrt(4*x^2+20),x);
```

$$x\sqrt{x^2+5} + 5\,\operatorname{arcsinh}\left(\frac{1}{5}\sqrt{5}\,x\right) \tag{3.1.3}$$

Example 4. Evaluate $\int \frac{1}{(x^2-6x+1)^2}\,dx$.

```
> int(1/(x^2-6*x+1)^2, x);
```

$$-\frac{1}{32}\,\frac{2\,x-6}{x^2-6\,x+1} + \frac{1}{64}\sqrt{2}\,\operatorname{arctanh}\left(\frac{1}{8}\,(2\,x-6)\sqrt{2}\right) \tag{3.1.4}$$

Exercises

1. $\int \frac{x^2}{x^2-2x-3}\,dx$.
2. $\int \frac{2x-3}{(x+1)^2(x-2)}\,dx$.
3. $\int \frac{x^2+1}{(x^2+2x+2)^2}\,dx$.
4. $\int \frac{x+1}{x^2(x^2+4)^2}\,dx$.

7.4 Integrals of Hyperbolic and Inverse Hyperbolic Functions

7.4.1 Integrals of hyperbolic functions

Example 1. Calculate $\int x\cosh(x^2)\,dx$.

```
> int(x*cosh(x^2),x);
```

$$\frac{1}{2}\sinh(x^2) \tag{4.1.1}$$

Example 2. Calculate $\int \sinh^4(x)\cosh^5(x)\,dx$.

```
> Ans:=int(sinh(x)^4*cosh(x)^5,x);
```

$$Ans := \frac{1}{9}\sinh(x)^3\cosh(x)^6 - \frac{1}{21}\sinh(x)\cosh(x)^6 + \frac{1}{105}\sinh(x)\cosh(x)^4 + \frac{4}{315}\sinh(x)\cosh(x)^2 + \frac{8}{315}\sinh(x) \tag{4.1.2}$$

```
> simplify(Ans);
```

$$\frac{1}{315}\sinh(x)^5\left(35\cosh(x)^4+20\cosh(x)^2+8\right) \tag{4.1.3}$$

7.4.2 Integrals of inverse hyperbolic functions

Example 3. Calculate $\int \sqrt{x^2+16}\,\mathrm{d}x$.

```
> int(sqrt(x^2+16),x);
```

$$\frac{1}{2}x\sqrt{x^2+16}+8\,\mathrm{arcsinh}\left(\frac{1}{4}x\right) \tag{4.2.1}$$

Example 4. Evaluate $\int_2^4 \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x$.

```
> ans:=int(1/sqrt(x^2-1),x=2..4);
```

$$ans := -\ln\left(2+\sqrt{3}\right)+\ln\left(4+\sqrt{3}\sqrt{5}\right) \tag{4.2.2}$$

```
> evalf(ans);
```

$$0.746479172 \tag{4.2.3}$$

Example 5. Evaluate $\int_1^9 \frac{1}{x\sqrt{x^4+1}}\,\mathrm{d}x$.

```
> ans:=int(1/(x*sqrt(x^4+1)),x=1..9);
```

$$ans := -\frac{1}{4}\ln\left(\sqrt{2}\sqrt{3281}+1\right)+\frac{1}{4}\ln\left(\sqrt{2}\sqrt{3281}-1\right)+\frac{1}{4}\ln\left(1+\sqrt{2}\right)-\frac{1}{4}\ln\left(\sqrt{2}-1\right) \tag{4.2.4}$$

```
> evalf(ans);
```

$$0.4345141107 \tag{4.2.5}$$

Exercises

1. Calculate $\int x\sinh\left(x^2+1\right)\,\mathrm{d}x$.
2. Calculate $\int \sinh^3(x)\cosh^6(x)\,\mathrm{d}x$.
3. Calculate $\int \frac{\cosh(x)}{\sinh(x)}\,\mathrm{d}x$.
4. Calculate $\int \frac{1}{\sqrt{x^2+16}}\,\mathrm{d}x$.
5. Evaluate $\int \sqrt{x^2-1}\,\mathrm{d}x$.
6. Evaluate $\int_{0.2}^{0.8} \frac{1}{x\sqrt{1-x^2}}\,\mathrm{d}x$.

7.5 The Method of Partial Fractions

When the integrand is a rational function, then it can be represented as a partial fraction decomposition and be evaluated. Once again, Maple will help you do the partial fraction decomposition. You need not do it by yourself.

7.5.1 Evaluate the integrals of rational functions

Example 1. Evaluate $\int \frac{1}{x^2-7x+10}\,dx$.

```
> int(1/(x^2-7*x+10), x);
```

$$-\frac{1}{3}\ln(x-2)+\frac{1}{3}\ln(x-5) \tag{5.1.1}$$

Example 2. Evaluate $\int \frac{(x^2+2)(2x-8)(x+2)}{x-1}\,dx$.

```
> int((x^2+2)/(x-1)*(2*x-8)*(x+2),x);
```

$$\frac{1}{2}x^4-\frac{2}{3}x^3-7x^2-22x-54\ln(x-1) \tag{5.1.2}$$

Example 3. Evaluate $\int \frac{x^3+1}{x^4+1}\,dx$.

```
> int((x^3+1)/(x^4+1),x);
```

$$\frac{1}{8}\sqrt{2}\ln\left(\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}\right)+\frac{1}{4}\sqrt{2}\arctan\left(x\sqrt{2}+1\right)+\frac{1}{4}\sqrt{2}\arctan\left(x\sqrt{2}-1\right)+\frac{1}{4}\ln\left(x^4+1\right) \tag{5.1.3}$$

Example 4. Evaluate $\int \frac{3x-9}{(x-1)(x+1)^2}\,dx$.

```
> int((3*x-9)/((x-1)*(x+1)^2),x);
```

$$-\frac{6}{x+1}-\frac{3}{2}\ln(x-1)+\frac{3}{2}\ln(x+1) \tag{5.1.4}$$

Example 5. Evaluate $\int \frac{4-x}{x(x^2+2)^2}\,dx$.

```
> int((4-x)/(x*(x^2+2)^2),x);
```

$$\frac{1}{8}\,\frac{-2x+8}{x^2+2}-\frac{1}{8}\sqrt{2}\arctan\left(\frac{1}{2}x\sqrt{2}\right)+\ln(x)-\frac{1}{2}\ln\left(x^2+2\right) \tag{5.1.5}$$

7.6 Improper Integrals

The improper integral of $f(x)$ over $[a, \textit{infinity})$ is defined as the limit

$$\int_a^\infty f(x)\,dx=\lim_{R\to\infty}\int_a^R f(x)\,dx.$$

When the limit exists, we say that the improper integral is convergent. Otherwise, it is divergent.

7.6.1 Evaluate improper integrals

Example 1. Show that $\int_2^{\infty} \frac{1}{x^2}\,\mathrm{d}x$ converges and compute its value.

```
> InR:=limit(int(1/x^2,x=2..R),R=infinity);
```

$$InR := \frac{1}{2} \tag{6.1.1}$$

It can be evaluated as follows.

```
> InR:=int(1/x^2,x=2..infinity);
```

$$InR := \frac{1}{2} \tag{6.1.2}$$

Example 2. Determine if $\int_{-\infty}^{-1} \frac{1}{x}\,\mathrm{d}x$ converges.

```
> InR:=int(1/x,x=-infinity..-1);
```

$$InR := -\infty \tag{6.1.3}$$

Hence, it diverges.

Example 3. Determine if $\int_0^{\infty} x\mathrm{e}^{-x}\mathrm{d}x$ converges. If so, find its value.

```
> int(x*exp(-x),x=0..infinity);
```

$$1 \tag{6.1.4}$$

It converges and the value is 1.

Example 4. Determine whether $\int_1^{\infty} \frac{1}{\sqrt{x}+\mathrm{e}^{3x}}\,\mathrm{d}x$ converges or diverges.

```
> int(1/(sqrt(x)+exp(3*x)),x=1..infinity);
```

$$\int_1^{\infty} \frac{1}{\sqrt{x}+\mathrm{e}^{(3x)}}\,\mathrm{d}x \tag{6.1.5}$$

We have $0 \le \frac{1}{\sqrt{x}+\mathrm{e}^{3x}} \le \frac{1}{\mathrm{e}^{3x}}$ and

```
> int(1/exp(3*x),x=1..infinity);
```

$$\frac{1}{3}\,\mathrm{e}^{(-3)} \tag{6.1.6}$$

Hence, $\int_1^{\infty} \frac{1}{\sqrt{x}+\mathrm{e}^{3x}}\,\mathrm{d}x$ converges.

Example 5. Determine whether $\int_0^\infty \frac{1}{\sqrt{1+x^2}}\,dx$ converges or diverges.

```
> int(1/sqrt(1+x^2),x=0..infinity);
```

$$\infty \quad (6.1.7)$$

Hence, $\int_0^\infty \frac{1}{\sqrt{1+x^2}}\,dx$ diverges.

Exercises

1. Show that $\int_2^\infty \frac{1}{x^3}\,dx$ converges and compute its value.

2. Determine if $\int_1^\infty \frac{1}{x+1}\,dx$ converges or diverges.

3. Evaluate $\int_0^\infty e^{-3x}\,dx$.

4. Determine whether $\int_1^\infty \frac{1}{\sqrt{x}+x^2}\,dx$ converges or diverges.

5. Determine whether $\int_3^\infty \frac{1}{\ln(x)}\,dx$ converges or diverges.

7.7 Probability and Integration

7.7.1 Probability

Example 1. Find the constant C for which $p(x) = \frac{C}{1+x^2}$ is a probability density function. Then compute $P(1 \le x \le 4)$.

```
> intp:=int(c/(1+x^2),x=-infinity..infinity);
```

$$intp := c\,\pi \quad (7.1.1)$$

```
> C=solve(intp=1,c);
```

$$C = \frac{1}{\pi} \quad (7.1.2)$$

```
> P1to4:=int(1/(Pi*(1+x^2)),x=1..4);
```

$$P1to4 := -\frac{1}{4}\,\frac{\pi - 4\arctan(4)}{\pi} \quad (7.1.3)$$

```
> evalf(P1to4);
```

$$0.1720208697 \quad (7.1.4)$$

Example 2. Let $r > 0$. Calculate the mean of the exponential probability density $p(t) = \frac{1}{r} e^{-\frac{t}{r}}$ on $[0, \infty]$.

```
> assume(r>0):
> p:=1/r*exp(-t/r);
```

$$p := \frac{e^{-\frac{t}{r\sim}}}{r\sim} \tag{7.1.5}$$

```
> Mu=int(t*p,t=0..infinity);
```

$$\mathrm{M} = r\sim \tag{7.1.6}$$

```
> r:='r':
```

Example 3. Evaluate numerically $\frac{1}{3\sqrt{2\pi}} \int_{14.5}^{\infty} e^{-\frac{(t-10)^2}{18}} \mathrm{d}t$.

```
> evalf(1/(3*sqrt(2*Pi))*int(exp(-(t-10)^2/18),t=14.5..
  infinity));
```

$$0.06680720122 \tag{7.1.7}$$

Note that we have

```
> wholeInt:=simplify(1/(3*sqrt(2*Pi))*(int(exp(-t^2/18),t=0..
  infinity)));
```

$$wholeInt := \frac{1}{2} \tag{7.1.8}$$

Hence, we can also evaluate $\frac{1}{3\sqrt{2\pi}} \int_{14.5}^{\infty} e^{-\frac{(t-10)^2}{18}} \mathrm{d}t$ by the following.

```
> evalf(wholeInt-1/(3*sqrt(2*Pi))*int(exp(-t^2/18),t=0..4.5));
```

$$0.0668072015 \tag{7.1.9}$$

Here, we move the center $t = 10$ to $t = 0$, then the integral limit $t = 14.5$ is replaced by $t = 4.5$.

Exercises

1. Find the constant C for which $p(x) = \frac{C}{(1+x)^3}$ is a probability density function on $[0, \infty]$. Then compute $P(0 \le x \le 1)$.

2. Find the constant C for which $p(x) = C\sqrt{1-x^2}$ is a probability density function on $[-1, 1]$. Then compute $P\left(-\frac{1}{2} \le x \le \frac{1}{2}\right)$.

3. Verify that $p(t) = \frac{1}{50} e^{-\frac{t}{50}}$ satisfies the condition $\int_0^{\infty} p(t)\, \mathrm{d}t = 1$.

4. Calculate the mean of the exponential probability density $p(t) = \frac{1}{2}e^{-\frac{t}{2}}$ on $[0, \infty]$.

5. Calculate the standard deviation σ of the exponential probability density $p(t) = \frac{1}{3}e^{-\frac{t}{3}}$ on $[0, \infty]$. The standard deviation σ is defined by

$$\sigma^2 = \int_0^\infty (t-\mu)^2 p(t)\,\mathrm{d}t$$

and μ is the mean $\mu = \int_0^\infty t\,p(t)\,\mathrm{d}t$.

7.8 Numerical Integration

(1) Trapezoidal Rule $\boldsymbol{T_N}$:

$$\boldsymbol{T_N} = \frac{1}{2}\,\triangle x(y_0 + 2y_1 + \ldots + 2y_{N-1} + y_N),$$

which is the average of the left sum of the right sum.

(2) Midpoint Rule $\boldsymbol{M_N}$:

$$\boldsymbol{M_N} = \Delta x(f(c_1) + f(c_2) + \cdots + f(c_N)), \quad c_i = a + (i - \frac{1}{2})\Delta x.$$

(3) Simpson's Rule $\boldsymbol{S_N}$:

$$\boldsymbol{S_N} = \frac{1}{3}\,\triangle x(y_0 + 4y_1 + 2y_2 + \cdots + 4y_{N-3} + 2y_{N-2} + 4y_{N-1} + y_N).$$

Let $N = 2m$. Then

$$\boldsymbol{S_N} = \frac{1}{3}\boldsymbol{T_m} + \frac{2}{3}\boldsymbol{M_m}$$

(4) Error bound:

$$\text{Error}(\boldsymbol{T_N}) \le \max|f''(x)|\,\frac{(b-a)^3}{12N^2}$$

$$\text{Error}(\boldsymbol{M_N}) \le \max|f''(x)|\,\frac{(b-a)^3}{24N^2}$$

$$\text{Error}(\boldsymbol{S_N}) \le \max|f^{(4)}(x)|\,\frac{(b-a)^5}{180N^4}$$

8.1.1 Numerical integration

Example 1. Calculate $\boldsymbol{T_6}$ and $\boldsymbol{M_6}$ for $\int_1^4 \sqrt{x}\,\mathrm{d}x$.

```
> a:=1: b:=4: f:=x->sqrt(x): N:=6: dx:=(b-a)/N:
> Rsum:=dx*sum(f(a+j*dx),j=1..6): Lsum:= dx*sum(f(a+j*dx),j=0.
  .5):
```

```
> TN:=evalf(0.5*(Rsum+Lsum));
```
$$TN := 4.661488383 \quad (8.1.1)$$

```
> MN:=evalf(dx*sum(f(a+(j-1/2)*dx),j=1..6));
```
$$MN := 4.669244676 \quad (8.1.2)$$

Example 2. Use the error bound to estimate the error for $\boldsymbol{T_6}$ and $\boldsymbol{M_6}$ in **Example 1,** and compare them to the exact errors.

```
> d2f:=D(D(f))(x);
```
$$d2f := -\frac{1}{4}\frac{1}{x^{(3/2)}} \quad (8.1.3)$$

```
> K2:=maximize(abs(d2f),x=a..b);
```
$$K2 := \frac{1}{4} \quad (8.1.4)$$

```
> TNError:=evalf(K2*(b-a)^3/(12*N^2));
```
$$TNError := 0.01562500000 \quad (8.1.5)$$

```
> Exactv:=evalf(int(sqrt(x),x=1..4));
```
$$Exactv := 4.666666667 \quad (8.1.6)$$

```
> TNExactError:=evalf(abs(TN-Exactv));
```
$$TNExactError := 0.005178284 \quad (8.1.7)$$

```
> MNError:=evalf(K2*(b-a)^3/(24*N^2));
```
$$MNError := 0.007812500000 \quad (8.1.8)$$

```
> MNExactError:=evalf(abs(MN-Exactv));
```
$$MNExactError := 0.002578009 \quad (8.1.9)$$

Example 3. Find N such that $\boldsymbol{M_N}$ approximates $\int_0^3 e^{-x^2}\,dx$ with an error of at most 0.0001.

```
> a:=0: b:=3: f:=x->exp(-x^2): er:=0.0001: N:='N':
> K2:=maximize(abs(D(D(f))(x)),x=0..3);
```
$$K2 := 2 \quad (8.1.10)$$

```
> fsolve(K2*(b-a)^3/(24*N^2)=er,N=5);
```
$$150.0000000 \quad (8.1.11)$$

Hence, N must be ≥ 150.

Example 4. Use Simpson's Rule with $N = 8$ to approximate $\int_2^4 \sqrt{1+x^3}\,dx$.

```
> a:=2: b:=4: f:=x->sqrt(1+x^3): N:=8:
> m:=N/2: dx:=(b-a)/m:
> Rsum:=dx*sum(f(a+j*dx),j=1..m): Lsum:=dx*sum(f(a+j*dx),j=0..
  (m-1)):
> Tm:=1/2*(Rsum+Lsum): Mm:=dx*sum(f(a+(j-1/2)*dx),j=1..m):
```

```
> SN:=(1/3*Tm+2/3*Mm):
> SN:=evalf(SN);
```

$$SN := 10.74159295 \tag{8.1.12}$$

Example 5. Find S_8 for $\int_1^3 \frac{1}{x}\,dx$. Then (a) find an error bound; (b) find N such that the error is at most 10^{-6}.

```
> a:=1: b:=3: f:=x->1/x: m:=4:  dx:=(b-a)/m:
> Rsum:=dx*sum(f(a+j*dx),j=1..m): Lsum:=dx*sum(f(a+j*dx),j=0..
  (m-1)):
> Tm:=1/2*(Rsum+Lsum): Mm:=dx*sum(f(a+(j-1/2)*dx),j=1..m):
> SN:=evalf((1/3*Tm+2/3*Mm));
```

$$SN := 1.098725349 \tag{8.1.13}$$

(a) Find an error bound.

```
> K2:=maximize((D@@4)(f)(x),x=a..b);
```

$$K2 := 24 \tag{8.1.14}$$

```
> S8er:=evalf(K2*(b-a)^5/(180*N^4));
```

$$S8er := 0.001041666667 \tag{8.1.15}$$

(b) Find N such that the error is at most 10^{-6}.

```
> N:='N':
> fsolve(K2*(b-a)^5/(180*N^4)=10^(-6), N=5);
```

$$45.44877466 \tag{8.1.16}$$

Hence, we need $N \geq 46$.

Exercises

1. Calculate T_6 and M_6 for $\int_0^2 \sqrt[3]{x}\,dx$.

2. Use the error bound to estimate the error for T_6 and M_6 in **Exercise 1,** and compare them to the exact errors.

3. Find N such that M_N approximates $\int_0^2 e^{-3x^2}\,dx$ with an error of at most 0.0001.

4. Use Simpson's Rule with $N = 8$ to approximate $\int_0^3 \frac{1}{\sqrt{1+x^3}}\,dx$.

5. Find S_8 for $\int_0^1 \frac{1}{1+x^2}\,dx$. Then (a) find an error bound; (b) find N such that the error is at most 10^{-6}.