3 DIFFERENTIATION

3.1 Definition of the Derivative

Preliminary Questions

1. Which of the lines in Figure 10 are tangent to the curve?

FIGURE 10

SOLUTION Lines B and D are tangent to the curve.

2. What are the two ways of writing the difference quotient?

SOLUTION The difference quotient may be written either as

$$\frac{f(x) - f(a)}{x - a}$$

or as

$$\frac{f(a+h) - f(a)}{h}.$$

3. Find a and h such that $\frac{f(a+h) - f(a)}{h}$ is equal to the slope of the secant line between $(3, f(3))$ and $(5, f(5))$.

SOLUTION With $a = 3$ and $h = 2$, $\frac{f(a+h) - f(a)}{h}$ is equal to the slope of the secant line between the points $(3, f(3))$ and $(5, f(5))$ on the graph of $f(x)$.

4. Which derivative is approximated by $\frac{\tan\left(\frac{\pi}{4} + 0.0001\right) - 1}{0.0001}$?

SOLUTION $\frac{\tan(\frac{\pi}{4} + 0.0001) - 1}{0.0001}$ is a good approximation to the derivative of the function $f(x) = \tan x$ at $x = \frac{\pi}{4}$.

5. What do the following quantities represent in terms of the graph of $f(x) = \sin x$?

(a) $\sin 1.3 - \sin 0.9$ **(b)** $\frac{\sin 1.3 - \sin 0.9}{0.4}$ **(c)** $f'(0.9)$

SOLUTION Consider the graph of $y = \sin x$.

(a) The quantity $\sin 1.3 - \sin 0.9$ represents the difference in height between the points $(0.9, \sin 0.9)$ and $(1.3, \sin 1.3)$.

(b) The quantity $\frac{\sin 1.3 - \sin 0.9}{0.4}$ represents the slope of the secant line between the points $(0.9, \sin 0.9)$ and $(1.3, \sin 1.3)$ on the graph.

(c) The quantity $f'(0.9)$ represents the slope of the tangent line to the graph at $x = 0.9$.

Exercises

1. Let $f(x) = 5x^2$. Show that $f(3+h) = 5h^2 + 30h + 45$. Then show that

$$\frac{f(3+h) - f(3)}{h} = 5h + 30$$

and compute $f'(3)$ by taking the limit as $h \to 0$.

SOLUTION With $f(x) = 5x^2$, it follows that

$$f(3+h) = 5(3+h)^2 = 5(9 + 6h + h^2) = 45 + 30h + 5h^2.$$

Using this result, we find

$$\frac{f(3+h) - f(3)}{h} = \frac{45 + 30h + 5h^2 - 5 \cdot 9}{h} = \frac{45 + 30h + 5h^2 - 45}{h} = \frac{30h + 5h^2}{h} = 30 + 5h.$$

As $h \to 0$, $30 + 5h \to 30$, so $f'(3) = 30$.

In Exercises 3–6, compute $f'(a)$ in two ways, using Eq. (1) and Eq. (2).

3. $f(x) = x^2 + 9x, \quad a = 0$

SOLUTION Let $f(x) = x^2 + 9x$. Then

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{(0+h)^2 + 9(0+h) - 0}{h} = \lim_{h \to 0} \frac{9h + h^2}{h} = \lim_{h \to 0} (9 + h) = 9.$$

Alternately,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 + 9x - 0}{x} = \lim_{x \to 0} (x + 9) = 9.$$

5. $f(x) = 3x^2 + 4x + 2, \quad a = -1$

SOLUTION Let $f(x) = 3x^2 + 4x + 2$. Then

$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0} \frac{3(-1+h)^2 + 4(-1+h) + 2 - 1}{h}$$

$$= \lim_{h \to 0} \frac{3h^2 - 2h}{h} = \lim_{h \to 0} (3h - 2) = -2.$$

Alternately,

$$f'(-1) = \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1} \frac{3x^2 + 4x + 2 - 1}{x + 1}$$

$$= \lim_{x \to -1} \frac{(3x+1)(x+1)}{x+1} = \lim_{x \to -1} (3x + 1) = -2.$$

In Exercises 7–10, refer to Figure 11.

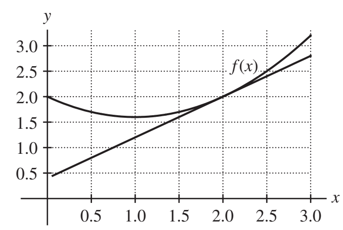


FIGURE 11

7. Find the slope of the secant line through $(2, f(2))$ and $(2.5, f(2.5))$. Is it larger or smaller than $f'(2)$? Explain.

SOLUTION From the graph, it appears that $f(2.5) = 2.5$ and $f(2) = 2$. Thus, the slope of the secant line through $(2, f(2))$ and $(2.5, f(2.5))$ is

$$\frac{f(2.5) - f(2)}{2.5 - 2} = \frac{2.5 - 2}{2.5 - 2} = 1.$$

From the graph, it is also clear that the secant line through $(2, f(2))$ and $(2.5, f(2.5))$ has a larger slope than the tangent line at $x = 2$. In other words, the slope of the secant line through $(2, f(2))$ and $(2.5, f(2.5))$ is larger than $f'(2)$.

9. Estimate $f'(1)$ and $f'(2)$.

SOLUTION From the graph, it appears that the tangent line at $x = 1$ would be horizontal. Thus, $f'(1) \approx 0$. The tangent line at $x = 2$ appears to pass through the points $(0.5, 0.8)$ and $(2, 2)$. Thus

$$f'(2) \approx \frac{2 - 0.8}{2 - 0.5} = 0.8.$$

In Exercises 11–14, refer to Figure 12.

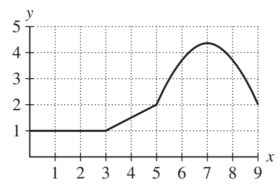


FIGURE 12 Graph of $f(x)$.

11. Determine $f'(a)$ for $a = 1, 2, 4, 7$.

SOLUTION Remember that the value of the derivative of f at $x = a$ can be interpreted as the slope of the line tangent to the graph of $y = f(x)$ at $x = a$. From Figure 12, we see that the graph of $y = f(x)$ is a horizontal line (that is, a line with zero slope) on the interval $0 \le x \le 3$. Accordingly, $f'(1) = f'(2) = 0$. On the interval $3 \le x \le 5$, the graph of $y = f(x)$ is a line of slope $\frac{1}{2}$; thus, $f'(4) = \frac{1}{2}$. Finally, the line tangent to the graph of $y = f(x)$ at $x = 7$ is horizontal, so $f'(7) = 0$.

13. Which is larger, $f'(5.5)$ or $f'(6.5)$?

SOLUTION The line tangent to the graph of $y = f(x)$ at $x = 5.5$ has a larger slope than the line tangent to the graph of $y = f(x)$ at $x = 6.5$. Therefore, $f'(5.5)$ is larger than $f'(6.5)$.

In Exercises 15–18, use the limit definition to calculate the derivative of the linear function.

15. $f(x) = 7x - 9$

SOLUTION

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{7(a+h) - 9 - (7a-9)}{h} = \lim_{h\to 0} 7 = 7.$$

17. $g(t) = 8 - 3t$

SOLUTION

$$\lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = \lim_{h\to 0} \frac{8 - 3(a+h) - (8-3a)}{h} = \lim_{h\to 0} \frac{-3h}{h} = \lim_{h\to 0} (-3) = -3.$$

19. Find an equation of the tangent line at $x = 3$, assuming that $f(3) = 5$ and $f'(3) = 2$?

SOLUTION By definition, the equation of the tangent line to the graph of $f(x)$ at $x = 3$ is $y = f(3) + f'(3)(x-3) = 5 + 2(x-3) = 2x - 1$.

21. Describe the tangent line at an arbitrary point on the "curve" $y = 2x + 8$.

SOLUTION Since $y = 2x + 8$ represents a straight line, the tangent line at any point is the line itself, $y = 2x + 8$.

23. Let $f(x) = \dfrac{1}{x}$. Does $f(-2+h)$ equal $\dfrac{1}{-2+h}$ or $\dfrac{1}{-2} + \dfrac{1}{h}$? Compute the difference quotient at $a = -2$ with $h = 0.5$.

SOLUTION Let $f(x) = \frac{1}{x}$. Then

$$f(-2+h) = \frac{1}{-2+h}.$$

With $a = -2$ and $h = 0.5$, the difference quotient is

$$\frac{f(a+h) - f(a)}{h} = \frac{f(-1.5) - f(-2)}{0.5} = \frac{\frac{1}{-1.5} - \frac{1}{-2}}{0.5} = -\frac{1}{3}.$$

25. Let $f(x) = 1/\sqrt{x}$. Compute $f'(5)$ by showing that

$$\frac{f(5+h) - f(5)}{h} = -\frac{1}{\sqrt{5}\sqrt{5+h}(\sqrt{5+h} + \sqrt{5})}$$

SOLUTION Let $f(x) = 1/\sqrt{x}$. Then

$$\frac{f(5+h)-f(5)}{h} = \frac{\frac{1}{\sqrt{5+h}} - \frac{1}{\sqrt{5}}}{h} = \frac{\sqrt{5}-\sqrt{5+h}}{h\sqrt{5}\sqrt{5+h}}$$

$$= \frac{\sqrt{5}-\sqrt{5+h}}{h\sqrt{5}\sqrt{5+h}}\left(\frac{\sqrt{5}+\sqrt{5+h}}{\sqrt{5}+\sqrt{5+h}}\right)$$

$$= \frac{5-(5+h)}{h\sqrt{5}\sqrt{5+h}(\sqrt{5+h}+\sqrt{5})} = -\frac{1}{\sqrt{5}\sqrt{5+h}(\sqrt{5+h}+\sqrt{5})}.$$

Thus,

$$f'(5) = \lim_{h\to 0}\frac{f(5+h)-f(5)}{h} = \lim_{h\to 0} -\frac{1}{\sqrt{5}\sqrt{5+h}(\sqrt{5+h}+\sqrt{5})}$$

$$= -\frac{1}{\sqrt{5}\,\sqrt{5}(\sqrt{5}+\sqrt{5})} = -\frac{1}{10\sqrt{5}}.$$

In Exercises 27–44, use the limit definition to compute $f'(a)$ and find an equation of the tangent line.

27. $f(x) = 2x^2 + 10x, \quad a = 3$

SOLUTION Let $f(x) = 2x^2 + 10x$. Then

$$f'(3) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{2(3+h)^2 + 10(3+h) - 48}{h}$$

$$= \lim_{h\to 0}\frac{18 + 12h + 2h^2 + 30 + 10h - 48}{h} = \lim_{h\to 0}(22 + 2h) = 22.$$

At $a = 3$, the tangent line is

$$y = f'(3)(x-3) + f(3) = 22(x-3) + 48 = 22x - 18.$$

29. $f(t) = t - 2t^2, \quad a = 3$

SOLUTION Let $f(t) = t - 2t^2$. Then

$$f'(3) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{(3+h) - 2(3+h)^2 - (-15)}{h}$$

$$= \lim_{h\to 0}\frac{3 + h - 18 - 12h - 2h^2 + 15}{h}$$

$$= \lim_{h\to 0}(-11 - 2h) = -11.$$

At $a = 3$, the tangent line is

$$y = f'(3)(t-3) + f(3) = -11(t-3) - 15 = -11t + 18.$$

31. $f(x) = x^3 + x, \quad a = 0$

SOLUTION Let $f(x) = x^3 + x$. Then

$$f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h^3 + h - 0}{h}$$

$$= \lim_{h\to 0}(h^2 + 1) = 1.$$

At $a = 0$, the tangent line is

$$y = f'(0)(x-0) + f(0) = x.$$

33. $f(x) = x^{-1}, \quad a = 8$

SOLUTION Let $f(x) = x^{-1}$. Then

$$f'(8) = \lim_{h\to 0}\frac{f(8+h)-f(8)}{h} = \lim_{h\to 0}\frac{\frac{1}{8+h} - \left(\frac{1}{8}\right)}{h} = \lim_{h\to 0}\frac{\frac{8-8-h}{8(8+h)}}{h} = \lim_{h\to 0}\frac{-h}{(64+8h)h} = -\frac{1}{64}$$

The tangent at $a = 8$ is

$$y = f'(8)(x-8) + f(8) = -\frac{1}{64}(x-8) + \frac{1}{8} = -\frac{1}{64}x + \frac{1}{4}.$$

35. $f(x) = \dfrac{1}{x+3}, \quad a = -2$

SOLUTION Let $f(x) = \frac{1}{x+3}$. Then

$$f'(-2) = \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h\to 0} \frac{\frac{1}{-2+h+3} - 1}{h} = \lim_{h\to 0} \frac{\frac{1}{1+h} - 1}{h} = \lim_{h\to 0} \frac{-h}{h(1+h)} = \lim_{h\to 0} \frac{-1}{1+h} = -1.$$

The tangent line at $a = -2$ is

$$y = f'(-2)(x+2) + f(-2) = -1(x+2) + 1 = -x - 1.$$

37. $f(x) = \sqrt{x+4}, \quad a = 1$

SOLUTION Let $f(x) = \sqrt{x+4}$. Then

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\sqrt{h+5} - \sqrt{5}}{h} = \lim_{h\to 0} \frac{\sqrt{h+5} - \sqrt{5}}{h} \cdot \frac{\sqrt{h+5} + \sqrt{5}}{\sqrt{h+5} + \sqrt{5}}$$
$$= \lim_{h\to 0} \frac{h}{h(\sqrt{h+5} + \sqrt{5})} = \lim_{h\to 0} \frac{1}{\sqrt{h+5} + \sqrt{5}} = \frac{1}{2\sqrt{5}}.$$

The tangent line at $a = 1$ is

$$y = f'(1)(x-1) + f(1) = \frac{1}{2\sqrt{5}}(x-1) + \sqrt{5} = \frac{1}{2\sqrt{5}}x + \frac{9}{2\sqrt{5}}.$$

39. $f(x) = \dfrac{1}{\sqrt{x}}, \quad a = 4$

SOLUTION Let $f(x) = \dfrac{1}{\sqrt{x}}$. Then

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{4+h}} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{\frac{2-\sqrt{4+h}}{2\sqrt{4+h}} \cdot \frac{2+\sqrt{4+h}}{2+\sqrt{4+h}}}{h} = \lim_{h\to 0} \frac{\frac{4-4-h}{4\sqrt{4+h}+2(4+h)}}{h}$$
$$= \lim_{h\to 0} \frac{-1}{4\sqrt{4+h} + 2(4+h)} = -\frac{1}{16}.$$

At $a = 4$ the tangent line is

$$y = f'(4)(x-4) + f(4) = -\frac{1}{16}(x-4) + \frac{1}{2} = -\frac{1}{16}x + \frac{3}{4}.$$

41. $f(t) = \sqrt{t^2+1}, \quad a = 3$

SOLUTION Let $f(t) = \sqrt{t^2+1}$. Then

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\sqrt{10+6h+h^2} - \sqrt{10}}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{10+6h+h^2} - \sqrt{10}}{h} \cdot \frac{\sqrt{10+6h+h^2} + \sqrt{10}}{\sqrt{10+6h+h^2} + \sqrt{10}}$$
$$= \lim_{h\to 0} \frac{6h+h^2}{h(\sqrt{10+6h+h^2} + \sqrt{10})} = \lim_{h\to 0} \frac{6+h}{\sqrt{10+6h+h^2} + \sqrt{10}} = \frac{3}{\sqrt{10}}.$$

The tangent line at $a = 3$ is

$$y = f'(3)(t-3) + f(3) = \frac{3}{\sqrt{10}}(t-3) + \sqrt{10} = \frac{3}{\sqrt{10}}t + \frac{1}{\sqrt{10}}.$$

43. $f(x) = \dfrac{1}{x^2+1}, \quad a = 0$

SOLUTION Let $f(x) = \dfrac{1}{x^2+1}$. Then

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\frac{1}{(0+h)^2+1} - 1}{h} = \lim_{h\to 0} \frac{\frac{-h^2}{h^2+1}}{h} = \lim_{h\to 0} \frac{-h}{h^2+1} = 0.$$

The tangent line at $a = 0$ is

$$y = f(0) + f'(0)(x - 0) = 1 + 0(x - 1) = 1.$$

45. Figure 13 displays data collected by the biologist Julian Huxley (1887–1975) on the average antler weight W of male red deer as a function of age t. Estimate the derivative at $t = 4$. For which values of t is the slope of the tangent line equal to zero? For which values is it negative?

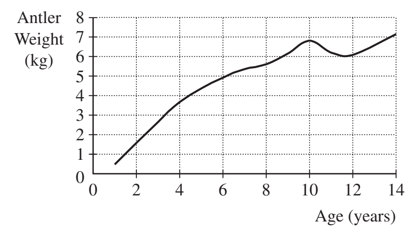
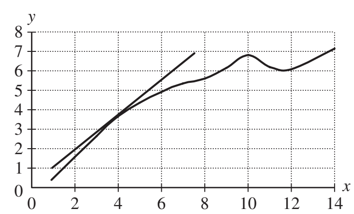


FIGURE 13

SOLUTION Let $W(t)$ denote the antler weight as a function of age. The "tangent line" sketched in the figure below passes through the points $(1, 1)$ and $(6, 5.5)$. Therefore

$$W'(4) \approx \frac{5.5 - 1}{6 - 1} = 0.9 \text{ kg/year}.$$

If the slope of the tangent is zero, the tangent line is horizontal. This appears to happen at roughly $t = 10$ and at $t = 11.6$. The slope of the tangent line is negative when the height of the graph decreases as we move to the right. For the graph in Figure 13, this occurs for $10 < t < 11.6$.



47. GU Let $f(x) = \dfrac{4}{1+2^x}$.

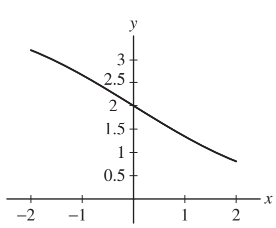
(a) Plot $f(x)$ over $[-2, 2]$. Then zoom in near $x = 0$ until the graph appears straight, and estimate the slope $f'(0)$.

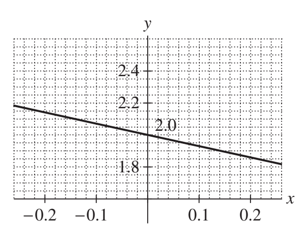
(b) Use (a) to find an approximate equation to the tangent line at $x = 0$. Plot this line and $f(x)$ on the same set of axes.

SOLUTION

(a) The figure below at the left shows the graph of $f(x) = \frac{4}{1+2^x}$ over $[-2, 2]$. The figure below at the right is a close-up near $x = 0$. From the close-up, we see that the graph is nearly straight and passes through the points $(-0.22, 2.15)$ and $(0.22, 1.85)$. We therefore estimate

$$f'(0) \approx \frac{1.85 - 2.15}{0.22 - (-0.22)} = \frac{-0.3}{0.44} = -0.68$$

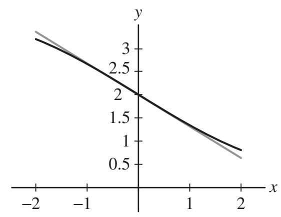




(b) Using the estimate for $f'(0)$ obtained in part (a), the approximate equation of the tangent line is

$$y = f'(0)(x - 0) + f(0) = -0.68x + 2.$$

The figure below shows the graph of $f(x)$ and the approximate tangent line.



49. Determine the intervals along the x-axis on which the derivative in Figure 15 is positive.

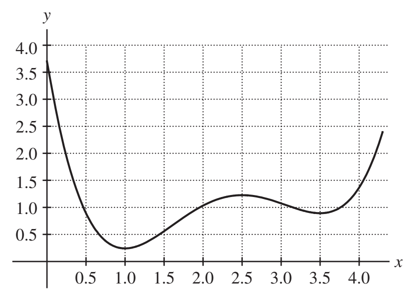


FIGURE 15

SOLUTION The derivative (that is, the slope of the tangent line) is positive when the height of the graph increases as we move to the right. From Figure 15, this appears to be true for $1 < x < 2.5$ and for $x > 3.5$.

In Exercises 51–56, each limit represents a derivative $f'(a)$. Find $f(x)$ and a.

51. $\lim_{h\to 0} \dfrac{(5+h)^3 - 125}{h}$

SOLUTION The difference quotient $\dfrac{(5+h)^3 - 125}{h}$ has the form $\dfrac{f(a+h) - f(a)}{h}$ where $f(x) = x^3$ and $a = 5$.

53. $\lim_{h\to 0} \dfrac{\sin\left(\frac{\pi}{6} + h\right) - 0.5}{h}$

SOLUTION The difference quotient $\dfrac{\sin(\frac{\pi}{6} + h) - .5}{h}$ has the form $\dfrac{f(a+h) - f(a)}{h}$ where $f(x) = \sin x$ and $a = \frac{\pi}{6}$.

55. $\lim_{h\to 0} \dfrac{5^{2+h} - 25}{h}$

SOLUTION The difference quotient $\dfrac{5^{(2+h)} - 25}{h}$ has the form $\dfrac{f(a+h) - f(a)}{h}$ where $f(x) = 5^x$ and $a = 2$.

57. Apply the method of Example 6 to $f(x) = \sin x$ to determine $f'\left(\frac{\pi}{4}\right)$ accurately to four decimal places.

SOLUTION We know that

$$f'(\pi/4) = \lim_{h\to 0} \frac{f(\pi/4 + h) - f(\pi/4)}{h} = \lim_{h\to 0} \frac{\sin(\pi/4 + h) - \sqrt{2}/2}{h}.$$

Creating a table of values of h close to zero:

h	-0.001	-0.0001	-0.00001	0.00001	0.0001	0.001
$\dfrac{\sin(\frac{\pi}{4} + h) - (\sqrt{2}/2)}{h}$	0.7074602	0.7071421	0.7071103	0.7071033	0.7070714	0.7067531

Accurate up to four decimal places, $f'(\frac{\pi}{4}) \approx 0.7071$.

59. For each graph in Figure 16, determine whether $f'(1)$ is larger or smaller than the slope of the secant line between $x = 1$ and $x = 1 + h$ for $h > 0$. Explain.

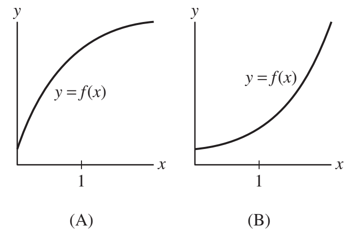


FIGURE 16

SOLUTION

- On curve (A), $f'(1)$ is larger than

$$\frac{f(1+h) - f(1)}{h};$$

the curve is bending downwards, so that the secant line to the right is at a lower angle than the tangent line. We say such a curve is **concave down**, and that its derivative is *decreasing*.

- On curve (B), $f'(1)$ is smaller than

$$\frac{f(1+h) - f(1)}{h};$$

the curve is bending upwards, so that the secant line to the right is at a steeper angle than the tangent line. We say such a curve is **concave up**, and that its derivative is *increasing*.

61. GU Sketch the graph of $f(x) = x^{5/2}$ on [0, 6].

(a) Use the sketch to justify the inequalities for $h > 0$:

$$\frac{f(4) - f(4-h)}{h} \leq f'(4) \leq \frac{f(4+h) - f(4)}{h}$$

(b) Use (a) to compute $f'(4)$ to four decimal places.

(c) Use a graphing utility to plot $f(x)$ and the tangent line at $x = 4$, using your estimate for $f'(4)$.

SOLUTION

(a) The slope of the secant line between points $(4, f(4))$ and $(4 + h, f(4 + h))$ is

$$\frac{f(4+h) - f(4)}{h}.$$

$x^{5/2}$ is a smooth curve increasing at a faster rate as $x \to \infty$. Therefore, if $h > 0$, then the slope of the secant line is greater than the slope of the tangent line at $f(4)$, which happens to be $f'(4)$. Likewise, if $h < 0$, the slope of the secant line is less than the slope of the tangent line at $f(4)$, which happens to be $f'(4)$.

(b) We know that

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{(4+h)^{5/2} - 32}{h}.$$

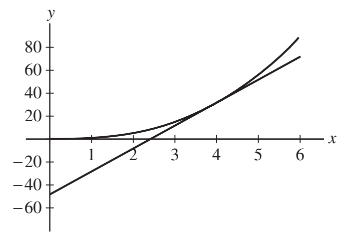
Creating a table with values of h close to zero:

h	-0.0001	-0.00001	0.00001	0.0001
$\frac{(4+h)^{5/2} - 32}{h}$	19.999625	19.99999	20.0000	20.0000375

Thus, $f'(4) \approx 20.0000$.

(c) Using the estimate for $f'(4)$ obtained in part (b), the equation of the line tangent to $f(x) = x^{5/2}$ at $x = 4$ is

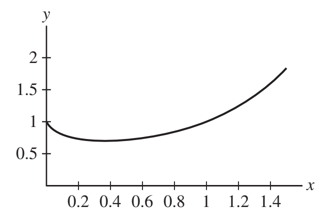
$$y = f'(4)(x - 4) + f(4) = 20(x - 4) + 32 = 20x - 48.$$



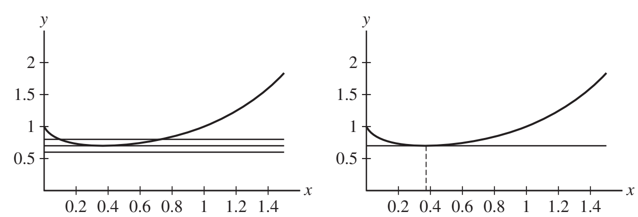
63. GU Use a plot of $f(x) = x^x$ to estimate the value c such that $f'(c) = 0$. Find c to sufficient accuracy so that

$$\left|\frac{f(c+h) - f(c)}{h}\right| \le 0.006 \qquad \text{for} \quad h = \pm 0.001$$

SOLUTION Here is a graph of $f(x) = x^x$ over the interval [0, 1.5].



The graph shows one location with a horizontal tangent line. The figure below at the left shows the graph of $f(x)$ together with the horizontal lines $y = 0.6$, $y = 0.7$ and $y = 0.8$. The line $y = 0.7$ is very close to being tangent to the graph of $f(x)$. The figure below at the right refines this estimate by graphing $f(x)$ and $y = 0.69$ on the same set of axes. The point of tangency has an x-coordinate of roughly 0.37, so $c \approx 0.37$.



We note that

$$\left|\frac{f(0.37 + 0.001) - f(0.37)}{0.001}\right| \approx 0.00491 < 0.006$$

and

$$\left|\frac{f(0.37 - 0.001) - f(0.37)}{0.001}\right| \approx 0.00304 < 0.006,$$

so we have determined c to the desired accuracy.

In Exercises 65–71, estimate derivatives using the ***symmetric difference quotient*** *(SDQ), defined as the average of the difference quotients at h and $-h$:*

$$\frac{1}{2}\left(\frac{f(a+h) - f(a)}{h} + \frac{f(a-h) - f(a)}{-h}\right) = \frac{f(a+h) - f(a-h)}{2h} \qquad \boxed{4}$$

The SDQ usually gives a better approximation to the derivative than the difference quotient.

65. The vapor pressure of water at temperature T (in kelvins) is the atmospheric pressure P at which no net evaporation takes place. Use the following table to estimate $P'(T)$ for $T = 303, 313, 323, 333, 343$ by computing the SDQ given by Eq. (4) with $h = 10$.

T (K)	293	303	313	323	333	343	353
P (atm)	0.0278	0.0482	0.0808	0.1311	0.2067	0.3173	0.4754

SOLUTION Using equation (4),

$$P'(303) \approx \frac{P(313) - P(293)}{20} = \frac{0.0808 - 0.0278}{20} = 0.00265 \text{ atm/K};$$

$$P'(313) \approx \frac{P(323) - P(303)}{20} = \frac{0.1311 - 0.0482}{20} = 0.004145 \text{ atm/K};$$
$$P'(323) \approx \frac{P(333) - P(313)}{20} = \frac{0.2067 - 0.0808}{20} = 0.006295 \text{ atm/K};$$
$$P'(333) \approx \frac{P(343) - P(323)}{20} = \frac{0.3173 - 0.1311}{20} = 0.00931 \text{ atm/K};$$
$$P'(343) \approx \frac{P(353) - P(333)}{20} = \frac{0.4754 - 0.2067}{20} = 0.013435 \text{ atm/K}$$

In Exercises 67 and 68, traffic speed S along a certain road (in km/h) varies as a function of traffic density q (number of cars per km of road). Use the following data to answer the questions:

q (density)	60	70	80	90	100
S (speed)	72.5	67.5	63.5	60	56

67. Estimate $S'(80)$.

SOLUTION Let $S(q)$ be the function determining S given q. Using equation (4) with $h = 10$,

$$S'(80) \approx \frac{S(90) - S(70)}{20} = \frac{60 - 67.5}{20} = -0.375;$$

with $h = 20$,

$$S'(80) \approx \frac{S(100) - S(60)}{40} = \frac{56 - 72.5}{40} = -0.4125;$$

The mean of these two symmetric difference quotients is -0.39375 kph·km/car.

Exercises 69–71: The current (in amperes) at time t (in seconds) flowing in the circuit in Figure 19 is given by Kirchhoff's Law:

$$i(t) = Cv'(t) + R^{-1}v(t)$$

where $v(t)$ is the voltage (in volts), C the capacitance (in farads), and R the resistance (in ohms, Ω).

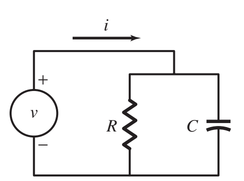


FIGURE 19

69. Calculate the current at $t = 3$ if

$$v(t) = 0.5t + 4 \text{ V}$$

where $C = 0.01$ F and $R = 100\ \Omega$.

SOLUTION Since $v(t)$ is a line with slope 0.5, $v'(t) = 0.5$ volts/s for all t. From the formula, $i(3) = Cv'(3) + (1/R)v(3) = 0.01(0.5) + (1/100)(5.5) = 0.005 + 0.055 = 0.06$ amperes.

71. Assume that $R = 200\ \Omega$ but C is unknown. Use the following data to estimate $v'(4)$ (by an SDQ) and deduce an approximate value for the capacitance C.

t	3.8	3.9	4	4.1	4.2
$v(t)$	388.8	404.2	420	436.2	452.8
$i(t)$	32.34	33.22	34.1	34.98	35.86

SOLUTION Solving $i(4) = Cv'(4) + (1/R)v(4)$ for C yields

$$C = \frac{i(4) - (1/R)v(4)}{v'(4)} = \frac{34.1 - \frac{420}{200}}{v'(4)}.$$

To compute C, we first approximate $v'(4)$. Taking $h = 0.1$, we find

$$v'(4) \approx \frac{v(4.1) - v(3.9)}{0.2} = \frac{436.2 - 404.2}{0.2} = 160.$$

Plugging this in to the equation above yields

$$C \approx \frac{34.1 - 2.1}{160} = 0.2 \text{ farads.}$$

Further Insights and Challenges

73. Explain how the symmetric difference quotient defined by Eq. (4) can be interpreted as the slope of a secant line.

SOLUTION The symmetric difference quotient

$$\frac{f(a+h) - f(a-h)}{2h}$$

is the slope of the secant line connecting the points $(a - h, f(a - h))$ and $(a + h, f(a + h))$ on the graph of f; the difference in the function values is divided by the difference in the x-values.

75. Show that if $f(x)$ is a quadratic polynomial, then the SDQ at $x = a$ (for any $h \neq 0$) is *equal* to $f'(a)$. Explain the graphical meaning of this result.

SOLUTION Let $f(x) = px^2 + qx + r$ be a quadratic polynomial. We compute the SDQ at $x = a$.

$$\begin{aligned} \frac{f(a+h) - f(a-h)}{2h} &= \frac{p(a+h)^2 + q(a+h) + r - (p(a-h)^2 + q(a-h) + r)}{2h} \\ &= \frac{pa^2 + 2pah + ph^2 + qa + qh + r - pa^2 + 2pah - ph^2 - qa + qh - r}{2h} \\ &= \frac{4pah + 2qh}{2h} = \frac{2h(2pa + q)}{2h} = 2pa + q \end{aligned}$$

Since this doesn't depend on h, the limit, which is equal to $f'(a)$, is also $2pa + q$. Graphically, this result tells us that the secant line to a parabola passing through points chosen symmetrically about $x = a$ is always parallel to the tangent line at $x = a$.

3.2 The Derivative as a Function

Preliminary Questions

1. What is the slope of the tangent line through the point $(2, f(2))$ if $f'(x) = x^3$?

SOLUTION The slope of the tangent line through the point $(2, f(2))$ is given by $f'(2)$. Since $f'(x) = x^3$, it follows that $f'(2) = 2^3 = 8$.

2. Evaluate $(f - g)'(1)$ and $(3f + 2g)'(1)$ assuming that $f'(1) = 3$ and $g'(1) = 5$.

SOLUTION $(f - g)'(1) = f'(1) - g'(1) = 3 - 5 = -2$ and $(3f + 2g)'(1) = 3f'(1) + 2g'(1) = 3(3) + 2(5) = 19$.

3. To which of the following does the Power Rule apply?

(a) $f(x) = x^2$
(b) $f(x) = 2^e$
(c) $f(x) = x^e$
(d) $f(x) = e^x$
(e) $f(x) = x^x$
(f) $f(x) = x^{-4/5}$

SOLUTION

(a) Yes. x^2 is a power function, so the Power Rule can be applied.
(b) Yes. 2^e is a constant function, so the Power Rule can be applied.
(c) Yes. x^e is a power function, so the Power Rule can be applied.
(d) No. e^x is an exponential function (the base is constant while the exponent is a variable), so the Power Rule does not apply.
(e) No. x^x is not a power function because both the base and the exponent are variable, so the Power Rule does not apply.
(f) Yes. $x^{-4/5}$ is a power function, so the Power Rule can be applied.

4. Choose (a) or (b). The derivative does not exist if the tangent line is: (a) horizontal (b) vertical.

SOLUTION The derivative does not exist when: (b) the tangent line is vertical. At a horizontal tangent, the derivative is zero.

5. Which property distinguishes $f(x) = e^x$ from all other exponential functions $g(x) = b^x$?

SOLUTION The line tangent to $f(x) = e^x$ at $x = 0$ has slope equal to 1.

Exercises

In Exercises 1–6, compute $f'(x)$ using the limit definition.

1. $f(x) = 3x - 7$

SOLUTION Let $f(x) = 3x - 7$. Then,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{3(x+h) - 7 - (3x-7)}{h} = \lim_{h\to 0} \frac{3h}{h} = 3.$$

3. $f(x) = x^3$

SOLUTION Let $f(x) = x^3$. Then,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2.$$

5. $f(x) = x - \sqrt{x}$

SOLUTION Let $f(x) = x - \sqrt{x}$. Then,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x + h - \sqrt{x+h} - (x - \sqrt{x})}{h} = 1 - \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= 1 - \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = 1 - \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = 1 - \frac{1}{2\sqrt{x}}.$$

In Exercises 7–14, use the Power Rule to compute the derivative.

7. $\left.\dfrac{d}{dx} x^4\right|_{x=-2}$

SOLUTION $\dfrac{d}{dx}\left(x^4\right) = 4x^3$ so $\left.\dfrac{d}{dx} x^4\right|_{x=-2} = 4(-2)^3 = -32.$

9. $\left.\dfrac{d}{dt} t^{2/3}\right|_{t=8}$

SOLUTION $\dfrac{d}{dt}\left(t^{2/3}\right) = \dfrac{2}{3}t^{-1/3}$ so $\left.\dfrac{d}{dt} t^{2/3}\right|_{t=8} = \dfrac{2}{3}(8)^{-1/3} = \dfrac{1}{3}.$

11. $\dfrac{d}{dx} x^{0.35}$

SOLUTION $\dfrac{d}{dx}\left(x^{0.35}\right) = 0.35(x^{0.35-1}) = 0.35x^{-0.65}.$

13. $\dfrac{d}{dt} t^{\sqrt{17}}$

SOLUTION $\dfrac{d}{dt}\left(t^{\sqrt{17}}\right) = \sqrt{17}t^{\sqrt{17}-1}$

In Exercises 15–18, compute $f'(x)$ and find an equation of the tangent line to the graph at $x = a$.

15. $f(x) = x^4, \quad a = 2$

SOLUTION Let $f(x) = x^4$. Then, by the Power Rule, $f'(x) = 4x^3$. The equation of the tangent line to the graph of $f(x)$ at $x = 2$ is

$$y = f'(2)(x-2) + f(2) = 32(x-2) + 16 = 32x - 48.$$

17. $f(x) = 5x - 32\sqrt{x}, \quad a = 4$

SOLUTION Let $f(x) = 5x - 32x^{1/2}$. Then $f'(x) = 5 - 16x^{-1/2}$. In particular, $f'(4) = -3$. The tangent line at $x = 4$ is

$$y = f'(4)(x-4) + f(4) = -3(x-4) - 44 = -3x - 32.$$

19. Calculate:
(a) $\frac{d}{dx}12e^x$ **(b)** $\frac{d}{dt}(25t - 8e^t)$ **(c)** $\frac{d}{dt}e^{t-3}$

Hint for (c): Write e^{t-3} as $e^{-3}e^t$.

SOLUTION

(a) $\frac{d}{dx}12e^x = 12\frac{d}{dx}e^x = 12e^x$.

(b) $\frac{d}{dt}(25t - 8e^t) = 25\frac{d}{dt}t - 8\frac{d}{dt}e^t = 25 - 8e^t$.

(c) $\frac{d}{dt}e^{t-3} = e^{-3}\frac{d}{dt}e^t = e^{-3} \cdot e^t = e^{t-3}$.

In Exercises 21–32, calculate the derivative.

21. $f(x) = 2x^3 - 3x^2 + 5$

SOLUTION $\frac{d}{dx}\left(2x^3 - 3x^2 + 5\right) = 6x^2 - 6x$.

23. $f(x) = 4x^{5/3} - 3x^{-2} - 12$

SOLUTION $\frac{d}{dx}\left(4x^{5/3} - 3x^{-2} - 12\right) = \frac{20}{3}x^{2/3} + 6x^{-3}$.

25. $g(z) = 7z^{-5/14} + z^{-5} + 9$

SOLUTION $\frac{d}{dz}\left(7z^{-5/14} + z^{-5} + 9\right) = -\frac{5}{2}z^{-19/14} - 5z^{-6}$.

27. $f(s) = \sqrt[4]{s} + \sqrt[3]{s}$

SOLUTION $f(s) = \sqrt[4]{s} + \sqrt[3]{s} = s^{1/4} + s^{1/3}$. In this form, we can apply the Sum and Power Rules.

$$\frac{d}{ds}\left(s^{1/4} + s^{1/3}\right) = \frac{1}{4}(s^{(1/4)-1}) + \frac{1}{3}(s^{(1/3)-1}) = \frac{1}{4}s^{-3/4} + \frac{1}{3}s^{-2/3}.$$

29. $g(x) = e^2$

SOLUTION Because e^2 is a constant, $\frac{d}{dx}e^2 = 0$.

31. $h(t) = 5e^{t-3}$

SOLUTION $\frac{d}{dt}5e^{t-3} = 5e^{-3}\frac{d}{dt}e^t = 5e^{-3}e^t = 5e^{t-3}$.

In Exercises 33–36, calculate the derivative by expanding or simplifying the function.

33. $P(s) = (4s - 3)^2$

SOLUTION $P(s) = (4s - 3)^2 = 16s^2 - 24s + 9$. Thus,

$$\frac{dP}{ds} = 32s - 24.$$

35. $g(x) = \frac{x^2 + 4x^{1/2}}{x^2}$

SOLUTION $g(x) = \frac{x^2 + 4x^{1/2}}{x^2} = 1 + 4x^{-3/2}$. Thus,

$$\frac{dg}{dx} = -6x^{-5/2}.$$

In Exercises 37–42, calculate the derivative indicated.

37. $\left.\frac{dT}{dC}\right|_{C=8}$, $T = 3C^{2/3}$

SOLUTION With $T(C) = 3C^{2/3}$, we have $\frac{dT}{dC} = 2C^{-1/3}$. Therefore,

$$\left.\frac{dT}{dC}\right|_{C=8} = 2(8)^{-1/3} = 1.$$

39. $\left.\dfrac{ds}{dz}\right|_{z=2}, \quad s = 4z - 16z^2$

SOLUTION With $s = 4z - 16z^2$, we have $\dfrac{ds}{dz} = 4 - 32z$. Therefore,

$$\left.\frac{ds}{dz}\right|_{z=2} = 4 - 32(2) = -60.$$

41. $\left.\dfrac{dr}{dt}\right|_{t=4}, \quad r = t - e^t$

SOLUTION With $r = t - e^t$, we have $\dfrac{dr}{dt} = 1 - e^t$. Therefore,

$$\left.\frac{dr}{dt}\right|_{t=4} = 1 - e^4.$$

43. Match the functions in graphs (A)–(D) with their derivatives (I)–(III) in Figure 13. Note that two of the functions have the same derivative. Explain why.

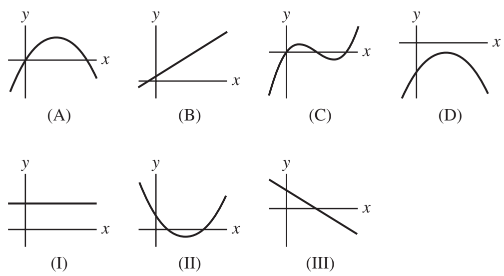


FIGURE 13

SOLUTION

- Consider the graph in (A). On the left side of the graph, the slope of the tangent line is positive but on the right side the slope of the tangent line is negative. Thus the derivative should transition from positive to negative with increasing x. This matches the graph in (III).
- Consider the graph in (B). This is a linear function, so its slope is constant. Thus the derivative is constant, which matches the graph in (I).
- Consider the graph in (C). Moving from left to right, the slope of the tangent line transitions from positive to negative then back to positive. The derivative should therefore be negative in the middle and positive to either side. This matches the graph in (II).
- Consider the graph in (D). On the left side of the graph, the slope of the tangent line is positive but on the right side the slope of the tangent line is negative. Thus the derivative should transition from positive to negative with increasing x. This matches the graph in (III).

Note that the functions whose graphs are shown in (A) and (D) have the same derivative. This happens because the graph in (D) is just a vertical translation of the graph in (A), which means the two functions differ by a constant. The derivative of a constant is zero, so the two functions end up with the same derivative.

45. Assign the labels $f(x)$, $g(x)$, and $h(x)$ to the graphs in Figure 15 in such a way that $f'(x) = g(x)$ and $g'(x) = h(x)$.

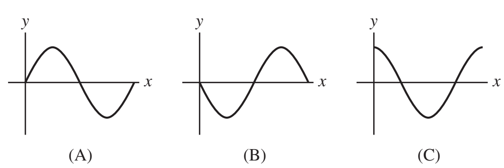


FIGURE 15

SOLUTION Consider the graph in (A). Moving from left to right, the slope of the tangent line is positive over the first quarter of the graph, negative in the middle half and positive again over the final quarter. The derivative of this function must therefore be negative in the middle and positive on either side. This matches the graph in (C).

Now focus on the graph in (C). The slope of the tangent line is negative over the left half and positive on the right half. The derivative of this function therefore needs to be negative on the left and positive on the right. This description matches the graph in (B).

We should therefore label the graph in (A) as $f(x)$, the graph in (B) as $h(x)$, and the graph in (C) as $g(x)$. Then $f'(x) = g(x)$ and $g'(x) = h(x)$.

47. Use the table of values of $f(x)$ to determine which of (A) or (B) in Figure 17 is the graph of $f'(x)$. Explain.

x	0	0.5	1	1.5	2	2.5	3	3.5	4
$f(x)$	10	55	98	139	177	210	237	257	268

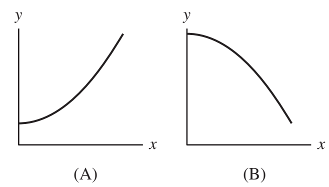


FIGURE 17 Which is the graph of $f'(x)$?

SOLUTION The increment between successive x values in the table is a constant 0.5 but the increment between successive $f(x)$ values decreases from 45 to 43 to 41 to 38 and so on. Thus the difference quotients decrease with increasing x, suggesting that $f'(x)$ decreases as a function of x. Because the graph in (B) depicts a decreasing function, (B) might be the graph of the derivative of $f(x)$.

49. Compute the derivatives, where c is a constant.

(a) $\frac{d}{dt}\,ct^3$ **(b)** $\frac{d}{dy}\,(9c^2y^3 - 24c)$ **(c)** $\frac{d}{dz}\,(5z + 4cz^2)$

SOLUTION

(a) $\frac{d}{dt}ct^3 = 3ct^2$.

(b) $\frac{d}{dz}(5z + 4cz^2) = 5 + 8cz$.

(c) $\frac{d}{dy}(9c^2y^3 - 24c) = 27c^2y^2$.

51. Find the points on the graph of $y = x^2 + 3x - 7$ at which the slope of the tangent line is equal to 4.

SOLUTION Let $y = x^2 + 3x - 7$. Solving $dy/dx = 2x + 3 = 4$ yields $x = \frac{1}{2}$.

53. Determine a and b such that $p(x) = x^2 + ax + b$ satisfies $p(1) = 0$ and $p'(1) = 4$.

SOLUTION Let $p(x) = x^2 + ax + b$ satisfy $p(1) = 0$ and $p'(1) = 4$. Now, $p'(x) = 2x + a$. Therefore $0 = p(1) = 1 + a + b$ and $4 = p'(1) = 2 + a$; i.e., $a = 2$ and $b = -3$.

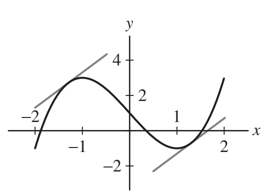
55. Let $f(x) = x^3 - 3x + 1$. Show that $f'(x) \geq -3$ for all x and that, for every $m > -3$, there are precisely two points where $f'(x) = m$. Indicate the position of these points and the corresponding tangent lines for one value of m in a sketch of the graph of $f(x)$.

SOLUTION Let $P = (a, b)$ be a point on the graph of $f(x) = x^3 - 3x + 1$.

- The derivative satisfies $f'(x) = 3x^2 - 3 \geq -3$ since $3x^2$ is nonnegative.
- Suppose the slope m of the tangent line is greater than -3. Then $f'(a) = 3a^2 - 3 = m$, whence

$$a^2 = \frac{m+3}{3} > 0 \quad \text{and thus} \quad a = \pm\sqrt{\frac{m+3}{3}}.$$

- The two parallel tangent lines with slope 2 are shown with the graph of $f(x)$ here.



57. Compute the derivative of $f(x) = x^{3/2}$ using the limit definition. *Hint:* Show that

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h}\left(\frac{1}{\sqrt{(x+h)^3} + \sqrt{x^3}}\right)$$

SOLUTION Once we have the difference of square roots, we multiply by the conjugate to solve the problem.

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^{3/2} - x^{3/2}}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)^3} - \sqrt{x^3}}{h} \left(\frac{\sqrt{(x+h)^3} + \sqrt{x^3}}{\sqrt{(x+h)^3} + \sqrt{x^3}}\right)$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \left(\frac{1}{\sqrt{(x+h)^3} + \sqrt{x^3}}\right).$$

The first factor of the expression in the last line is clearly the limit definition of the derivative of x^3, which is $3x^2$. The second factor can be evaluated, so

$$\frac{d}{dx} x^{3/2} = 3x^2 \frac{1}{2\sqrt{x^3}} = \frac{3}{2} x^{1/2}.$$

59. Let $f(x) = xe^x$. Use the limit definition to compute $f'(0)$, and find the equation of the tangent line at $x = 0$.

SOLUTION Let $f(x) = xe^x$. Then $f(0) = 0$, and

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{he^h - 0}{h} = \lim_{h\to 0} e^h = 1.$$

The equation of the tangent line is

$$y = f'(0)(x-0) + f(0) = 1(x-0) + 0 = x.$$

61. Biologists have observed that the pulse rate P (in beats per minute) in animals is related to body mass (in kilograms) by the approximate formula $P = 200m^{-1/4}$. This is one of many *allometric scaling laws* prevalent in biology. Is $|dP/dm|$ an increasing or decreasing function of m? Find an equation of the tangent line at the points on the graph in Figure 18 that represent goat ($m = 33$) and man ($m = 68$).

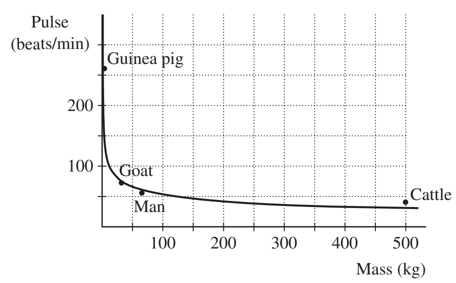


FIGURE 18

SOLUTION $dP/dm = -50m^{-5/4}$. For $m > 0$, $|dP/dm| = |50m^{-5/4}|$. $|dP/dm| \to 0$ as m gets larger; $|dP/dm|$ gets smaller as m gets bigger.

For each $m = c$, the equation of the tangent line to the graph of P at m is

$$y = P'(c)(m - c) + P(c).$$

For a goat ($m = 33$ kg), $P(33) = 83.445$ beats per minute (bpm) and

$$\frac{dP}{dm} = -50(33)^{-5/4} \approx -0.63216 \text{ bpm/kg}.$$

Hence, $y = -0.63216(m - 33) + 83.445$.

For a man ($m = 68$ kg), we have $P(68) = 69.647$ bpm and

$$\frac{dP}{dm} = -50(68)^{-5/4} \approx -0.25606 \text{ bpm/kg}.$$

Hence, the tangent line has formula $y = -0.25606(m - 68) + 69.647$.

63. The Clausius–Clapeyron Law relates the *vapor pressure* of water P (in atmospheres) to the temperature T (in kelvins):

$$\frac{dP}{dT} = k\frac{P}{T^2}$$

where k is a constant. Estimate dP/dT for $T = 303, 313, 323, 333, 343$ using the data and the approximation

$$\frac{dP}{dT} \approx \frac{P(T+10) - P(T-10)}{20}$$

T (K)	293	303	313	323	333	343	353
P (atm)	0.0278	0.0482	0.0808	0.1311	0.2067	0.3173	0.4754

Do your estimates seem to confirm the Clausius–Clapeyron Law? What is the approximate value of k?

SOLUTION Using the indicated approximation to the first derivative, we calculate

$$P'(303) \approx \frac{P(313) - P(293)}{20} = \frac{0.0808 - 0.0278}{20} = 0.00265 \text{ atm/K};$$
$$P'(313) \approx \frac{P(323) - P(303)}{20} = \frac{0.1311 - 0.0482}{20} = 0.004145 \text{ atm/K};$$
$$P'(323) \approx \frac{P(333) - P(313)}{20} = \frac{0.2067 - 0.0808}{20} = 0.006295 \text{ atm/K};$$
$$P'(333) \approx \frac{P(343) - P(323)}{20} = \frac{0.3173 - 0.1311}{20} = 0.00931 \text{ atm/K};$$
$$P'(343) \approx \frac{P(353) - P(333)}{20} = \frac{0.4754 - 0.2067}{20} = 0.013435 \text{ atm/K}$$

If the Clausius–Clapeyron law is valid, then $\frac{T^2}{P}\frac{dP}{dT}$ should remain constant as T varies. Using the data for vapor pressure and temperature and the approximate derivative values calculated above, we find

T (K)	303	313	323	333	343
$\frac{T^2}{P}\frac{dP}{dT}$	5047.59	5025.76	5009.54	4994.57	4981.45

These values are roughly constant, suggesting that the Clausius–Clapeyron law is valid, and that $k \approx 5000$.

65. In the setting of Exercise 64, show that the point of tangency is the midpoint of the segment of L lying in the first quadrant.

SOLUTION In the previous exercise, we saw that the tangent line to the hyperbola $xy = 1$ or $y = \frac{1}{x}$ at $x = a$ has y-intercept $P = (0, \frac{2}{a})$ and x-intercept $Q = (2a, 0)$. The midpoint of the line segment connecting P and Q is thus

$$\left(\frac{0 + 2a}{2}, \frac{\frac{2}{a} + 0}{2}\right) = \left(a, \frac{1}{a}\right),$$

which is the point of tangency.

67. Make a rough sketch of the graph of the derivative of the function in Figure 20(A).

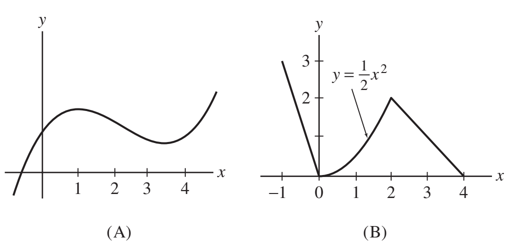
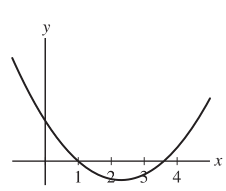


FIGURE 20

SOLUTION The graph has a tangent line with negative slope approximately on the interval (1, 3.6), and has a tangent line with a positive slope elsewhere. This implies that the derivative must be negative on the interval (1, 3.6) and positive elsewhere. The graph may therefore look like this:



69. Sketch the graph of $f(x) = x\,|x|$. Then show that $f'(0)$ exists.

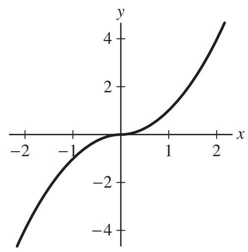
SOLUTION For $x < 0$, $f(x) = -x^2$, and $f'(x) = -2x$. For $x > 0$, $f(x) = x^2$, and $f'(x) = 2x$. At $x = 0$, we find

$$\lim_{h\to 0+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0+} \frac{h^2}{h} = 0$$

and

$$\lim_{h\to 0-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0-} \frac{-h^2}{h} = 0.$$

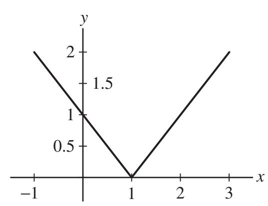
Because the two one-sided limits exist and are equal, it follows that $f'(0)$ exists and is equal to zero. Here is the graph of $f(x) = x|x|$.



In Exercises 71–76, find the points c (if any) such that $f'(c)$ does not exist.

71. $f(x) = |x - 1|$

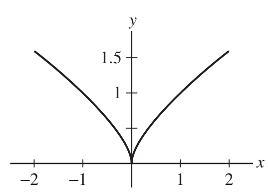
SOLUTION



Here is the graph of $f(x) = |x - 1|$. Its derivative does not exist at $x = 1$. At that value of x there is a sharp corner.

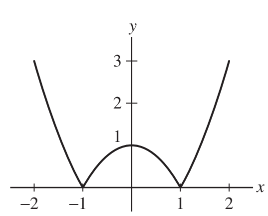
73. $f(x) = x^{2/3}$

SOLUTION Here is the graph of $f(x) = x^{2/3}$. Its derivative does not exist at $x = 0$. At that value of x, there is a sharp corner or "cusp".



75. $f(x) = |x^2 - 1|$

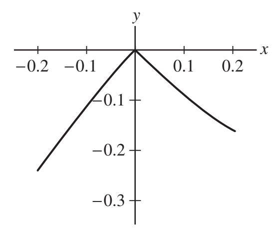
SOLUTION Here is the graph of $f(x) = \left|x^2 - 1\right|$. Its derivative does not exist at $x = -1$ or at $x = 1$. At these values of x, the graph has sharp corners.



GU *In Exercises 77–82, zoom in on a plot of $f(x)$ at the point $(a, f(a))$ and state whether or not $f(x)$ appears to be differentiable at $x = a$. If it is nondifferentiable, state whether the tangent line appears to be vertical or does not exist.*

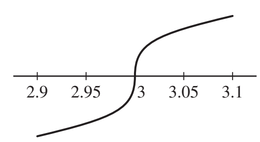
77. $f(x) = (x-1)|x|, \quad a = 0$

SOLUTION The graph of $f(x) = (x-1)|x|$ for x near 0 is shown below. Because the graph has a sharp corner at $x = 0$, it appears that f is not differentiable at $x = 0$. Moreover, the tangent line does not exist at this point.



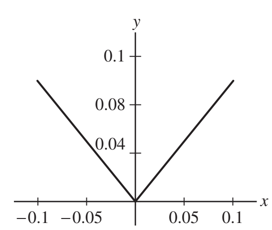
79. $f(x) = (x-3)^{1/3}, \quad a = 3$

SOLUTION The graph of $f(x) = (x-3)^{1/3}$ for x near 3 is shown below. From this graph, it appears that f is not differentiable at $x = 3$. Moreover, the tangent line appears to be vertical.



81. $f(x) = |\sin x|, \quad a = 0$

SOLUTION The graph of $f(x) = |\sin x|$ for x near 0 is shown below. Because the graph has a sharp corner at $x = 0$, it appears that f is not differentiable at $x = 0$. Moreover, the tangent line does not exist at this point.



83. GU Plot the derivative $f'(x)$ of $f(x) = 2x^3 - 10x^{-1}$ for $x > 0$ (set the bounds of the viewing box appropriately) and observe that $f'(x) > 0$. What does the positivity of $f'(x)$ tell us about the graph of $f(x)$ itself? Plot $f(x)$ and confirm this conclusion.

SOLUTION Let $f(x) = 2x^3 - 10x^{-1}$. Then $f'(x) = 6x^2 + 10x^{-2}$. The graph of $f'(x)$ is shown in the figure below at the left and it is clear that $f'(x) > 0$ for all $x > 0$. The positivity of $f'(x)$ tells us that the graph of $f(x)$ is increasing for $x > 0$. This is confirmed in the figure below at the right, which shows the graph of $f(x)$.

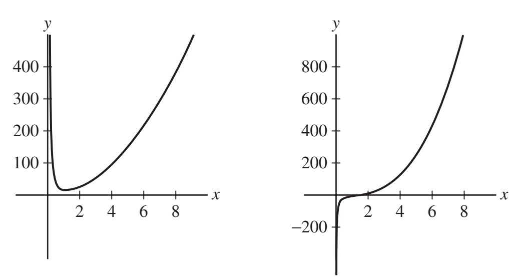


FIGURE 22

Exercises 85–88 refer to Figure 23. Length QR is called the subtangent *at P, and length RT is called the* subnormal.

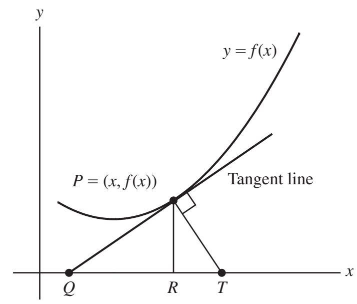


FIGURE 23

85. Calculate the subtangent of

$$f(x) = x^2 + 3x \quad \text{at } x = 2$$

SOLUTION Let $f(x) = x^2 + 3x$. Then $f'(x) = 2x + 3$, and the equation of the tangent line at $x = 2$ is

$$y = f'(2)(x - 2) + f(2) = 7(x - 2) + 10 = 7x - 4.$$

This line intersects the x-axis at $x = \frac{4}{7}$. Thus Q has coordinates $(\frac{4}{7}, 0)$, R has coordinates $(2, 0)$ and the subtangent is

$$2 - \frac{4}{7} = \frac{10}{7}.$$

87. Prove in general that the subnormal at P is $|f'(x)f(x)|$.

SOLUTION The slope of the tangent line at P is $f'(x)$. The slope of the line normal to the graph at P is then $-1/f'(x)$, and the normal line intersects the x-axis at the point T with coordinates $(x + f(x)f'(x), 0)$. The point R has coordinates $(x, 0)$, so the subnormal is

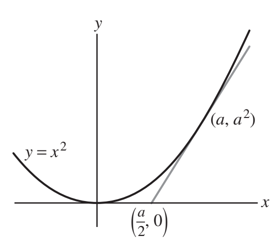
$$|x + f(x)f'(x) - x| = |f(x)f'(x)|.$$

89. Prove the following theorem of Apollonius of Perga (the Greek mathematician born in 262 BCE who gave the parabola, ellipse, and hyperbola their names): The subtangent of the parabola $y = x^2$ at $x = a$ is equal to $a/2$.

SOLUTION Let $f(x) = x^2$. The tangent line to f at $x = a$ is

$$y = f'(a)(x - a) + f(a) = 2a(x - a) + a^2 = 2ax - a^2.$$

The x-intercept of this line (where $y = 0$) is $\frac{a}{2}$ as claimed.



91. Formulate and prove a generalization of Exercise 90 for $y = x^n$.

SOLUTION Let $f(x) = x^n$. Then $f'(x) = nx^{n-1}$, and the equation of the tangent line t $x = a$ is

$$y = f'(a)(x - a) + f(a) = na^{n-1}(x - a) + a^n = na^{n-1}x - (n - 1)a^n.$$

This line intersects the x-axis at $x = (n - 1)a/n$. Thus, Q has coordinates $((n - 1)a/n, 0)$, R has coordinates $(a, 0)$ and the subtangent is

$$a - \frac{n-1}{n}a = \frac{1}{n}a.$$

Further Insights and Challenges

93. A vase is formed by rotating $y = x^2$ around the y-axis. If we drop in a marble, it will either touch the bottom point of the vase or be suspended above the bottom by touching the sides (Figure 25). How small must the marble be to touch the bottom?

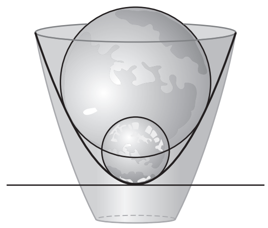


FIGURE 25

SOLUTION Suppose a circle is tangent to the parabola $y = x^2$ at the point (t, t^2). The slope of the parabola at this point is $2t$, so the slope of the radius of the circle at this point is $-\frac{1}{2t}$ (since it is perpendicular to the tangent line of the circle). Thus the center of the circle must be where the line given by $y = -\frac{1}{2t}(x - t) + t^2$ crosses the y-axis. We can find the y-coordinate by setting $x = 0$: we get $y = \frac{1}{2} + t^2$. Thus, the radius extends from $(0, \frac{1}{2} + t^2)$ to (t, t^2) and

$$r = \sqrt{\left(\frac{1}{2} + t^2 - t^2\right)^2 + t^2} = \sqrt{\frac{1}{4} + t^2}.$$

This radius is greater than $\frac{1}{2}$ whenever $t > 0$; so, if a marble has radius $> 1/2$ it sits on the edge of the vase, but if it has radius $\leq 1/2$ it rolls all the way to the bottom.

95. Negative Exponents Let n be a whole number. Use the Power Rule for x^n to calculate the derivative of $f(x) = x^{-n}$ by showing that

$$\frac{f(x+h) - f(x)}{h} = \frac{-1}{x^n(x+h)^n} \frac{(x+h)^n - x^n}{h}$$

SOLUTION Let $f(x) = x^{-n}$ where n is a positive integer.

- The difference quotient for f is

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^{-n} - x^{-n}}{h} = \frac{\frac{1}{(x+h)^n} - \frac{1}{x^n}}{h} = \frac{\frac{x^n - (x+h)^n}{x^n(x+h)^n}}{h}$$
$$= \frac{-1}{x^n(x+h)^n} \frac{(x+h)^n - x^n}{h}.$$

- Therefore,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{-1}{x^n(x+h)^n} \frac{(x+h)^n - x^n}{h}$$
$$= \lim_{h\to 0} \frac{-1}{x^n(x+h)^n} \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = -x^{-2n} \frac{d}{dx}\left(x^n\right).$$

- From above, we continue: $f'(x) = -x^{-2n} \frac{d}{dx}\left(x^n\right) = -x^{-2n} \cdot nx^{n-1} = -nx^{-n-1}$. Since n is a positive integer, $k = -n$ is a negative integer and we have $\frac{d}{dx}\left(x^k\right) = \frac{d}{dx}\left(x^{-n}\right) = -nx^{-n-1} = kx^{k-1}$; i.e. $\frac{d}{dx}\left(x^k\right) = kx^{k-1}$ for negative integers k.

97. Infinitely Rapid Oscillations Define

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Show that $f(x)$ is continuous at $x = 0$ but $f'(0)$ does not exist (see Figure 24).

SOLUTION Let $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. As $x \to 0$,

$$|f(x) - f(0)| = \left|x\sin\left(\frac{1}{x}\right) - 0\right| = |x|\left|\sin\left(\frac{1}{x}\right)\right| \to 0$$

since the values of the sine lie between -1 and 1. Hence, by the Squeeze Theorem, $\lim_{x\to 0} f(x) = f(0)$ and thus f is continuous at $x = 0$.

As $x \to 0$, the difference quotient at $x = 0$,

$$\frac{f(x) - f(0)}{x - 0} = \frac{x\sin\left(\frac{1}{x}\right) - 0}{x - 0} = \sin\left(\frac{1}{x}\right)$$

does *not* converge to a limit since it oscillates infinitely through every value between -1 and 1. Accordingly, $f'(0)$ does not exist.

3.3 Product and Quotient Rules

Preliminary Questions

1. Are the following statements true or false? If false, state the correct version.

(a) fg denotes the function whose value at x is $f(g(x))$.

(b) f/g denotes the function whose value at x is $f(x)/g(x)$.

(c) The derivative of the product is the product of the derivatives.

(d) $\left.\frac{d}{dx}(fg)\right|_{x=4} = f(4)g'(4) - g(4)f'(4)$

(e) $\left.\frac{d}{dx}(fg)\right|_{x=0} = f(0)g'(0) + g(0)f'(0)$

SOLUTION

(a) False. The notation fg denotes the function whose value at x is $f(x)g(x)$.

(b) True.

(c) False. The derivative of a product fg is $f'(x)g(x) + f(x)g'(x)$.

(d) False. $\left.\frac{d}{dx}(fg)\right|_{x=4} = f(4)g'(4) + g(4)f'(4)$.

(e) True.

2. Find $(f/g)'(1)$ if $f(1) = f'(1) = g(1) = 2$ and $g'(1) = 4$.

SOLUTION $\frac{d}{dx}(f/g)\big|_{x=1} = [g(1)f'(1) - f(1)g'(1)]/g(1)^2 = [2(2) - 2(4)]/2^2 = -1$.

3. Find $g(1)$ if $f(1) = 0$, $f'(1) = 2$, and $(fg)'(1) = 10$.

SOLUTION $(fg)'(1) = f(1)g'(1) + f'(1)g(1)$, so $10 = 0 \cdot g'(1) + 2g(1)$ and $g(1) = 5$.

Exercises

In Exercises 1–6, use the Product Rule to calculate the derivative.

1. $f(x) = x^3(2x^2 + 1)$

SOLUTION Let $f(x) = x^3(2x^2 + 1)$. Then

$$f'(x) = x^3\frac{d}{dx}(2x^2 + 1) + (2x^2 + 1)\frac{d}{dx}x^3 = x^3(4x) + (2x^2 + 1)(3x^2) = 10x^4 + 3x^2.$$

3. $f(x) = x^2e^x$

SOLUTION Let $f(x) = x^2e^x$. Then

$$f'(x) = x^2\frac{d}{dx}e^x + e^x\frac{d}{dx}x^2 = x^2e^x + e^x(2x) = e^x(x^2 + 2x).$$

5. $\left.\dfrac{dh}{ds}\right|_{s=4}, \quad h(s) = (s^{-1/2} + 2s)(7 - s^{-1})$

SOLUTION Let $h(s) = (s^{-1/2} + 2s)(7 - s^{-1})$. Then

$$\frac{dh}{ds} = (s^{-1/2} + 2s)\frac{d}{dx}(7 - s^{-1}) + (7 - s^{-1})\frac{d}{ds}\left(s^{-1/2} + 2s\right)$$
$$= (s^{-1/2} + 2s)(s^{-2}) + (7 - s^{-1})\left(-\frac{1}{2}s^{-3/2} + 2\right) = -\frac{7}{2}s^{-3/2} + \frac{3}{2}s^{-5/2} + 14.$$

Therefore,

$$\left.\frac{dh}{ds}\right|_{s=4} = -\frac{7}{2}(4)^{-3/2} + \frac{3}{2}(4)^{-5/2} + 14 = \frac{871}{64}.$$

In Exercises 7–12, use the Quotient Rule to calculate the derivative.

7. $f(x) = \dfrac{x}{x-2}$

SOLUTION Let $f(x) = \frac{x}{x-2}$. Then

$$f'(x) = \frac{(x-2)\frac{d}{dx}x - x\frac{d}{dx}(x-2)}{(x-2)^2} = \frac{(x-2) - x}{(x-2)^2} = \frac{-2}{(x-2)^2}.$$

9. $\left.\dfrac{dg}{dt}\right|_{t=-2}, \quad g(t) = \dfrac{t^2+1}{t^2-1}$

SOLUTION Let $g(t) = \dfrac{t^2+1}{t^2-1}$. Then

$$\frac{dg}{dt} = \frac{(t^2-1)\frac{d}{dt}(t^2+1) - (t^2+1)\frac{d}{dt}(t^2-1)}{(t^2-1)^2} = \frac{(t^2-1)(2t) - (t^2+1)(2t)}{(t^2-1)^2} = -\frac{4t}{(t^2-1)^2}.$$

Therefore,

$$\left.\frac{dg}{dt}\right|_{t=-2} = -\frac{4(-2)}{((-2)^2-1)^2} = \frac{8}{9}.$$

11. $g(x) = \dfrac{1}{1+e^x}$

SOLUTION Let $g(x) = \dfrac{1}{1+e^x}$. Then

$$\frac{dg}{dx} = \frac{(1+e^x)\frac{d}{dx}1 - 1\frac{d}{dx}(1+e^x)}{(1+e^x)^2} = \frac{(1+e^x)(0) - e^x}{(1+e^x)^2} = -\frac{e^x}{(1+e^x)^2}.$$

In Exercises 13–16, calculate the derivative in two ways. First use the Product or Quotient Rule; then rewrite the function algebraically and apply the Power Rule directly.

13. $f(t) = (2t+1)(t^2-2)$

SOLUTION Let $f(t) = (2t+1)(t^2-2)$. Then, using the Product Rule,

$$f'(t) = (2t+1)(2t) + (t^2-2)(2) = 6t^2 + 2t - 4.$$

Multiplying out first, we find $f(t) = 2t^3 + t^2 - 4t - 2$. Therefore, $f'(t) = 6t^2 + 2t - 4$.

15. $h(t) = \dfrac{t^2-1}{t-1}$

SOLUTION Let $h(t) = \frac{t^2-1}{t-1}$. Using the quotient rule,

$$f'(t) = \frac{(t-1)(2t) - (t^2-1)(1)}{(t-1)^2} = \frac{t^2-2t+1}{(t-1)^2} = 1$$

for $t \neq 1$. Simplifying first, we find for $t \neq 1$,

$$h(t) = \frac{(t-1)(t+1)}{(t-1)} = t + 1.$$

Hence $h'(t) = 1$ for $t \neq 1$.

In Exercises 17–38, calculate the derivative.

17. $f(x) = (x^3 + 5)(x^3 + x + 1)$

SOLUTION Let $f(x) = (x^3 + 5)(x^3 + x + 1)$. Then

$$f'(x) = (x^3 + 5)(3x^2 + 1) + (x^3 + x + 1)(3x^2) = 6x^5 + 4x^3 + 18x^2 + 5.$$

19. $\left.\dfrac{dy}{dx}\right|_{x=3}, \quad y = \dfrac{1}{x + 10}$

SOLUTION Let $y = \frac{1}{x+10}$. Using the quotient rule:

$$\frac{dy}{dx} = \frac{(x + 10)(0) - 1(1)}{(x + 10)^2} = -\frac{1}{(x + 10)^2}.$$

Therefore,

$$\left.\frac{dy}{dx}\right|_{x=3} = -\frac{1}{(3 + 10)^2} = -\frac{1}{169}.$$

21. $f(x) = (\sqrt{x} + 1)(\sqrt{x} - 1)$

SOLUTION Let $f(x) = (\sqrt{x} + 1)(\sqrt{x} - 1)$. Multiplying through first yields $f(x) = x - 1$ for $x \geq 0$. Therefore, $f'(x) = 1$ for $x \geq 0$. If we carry out the product rule on $f(x) = (x^{1/2} + 1)(x^{1/2} - 1)$, we get

$$f'(x) = (x^{1/2} + 1)\left(\frac{1}{2}(x^{-1/2})\right) + (x^{1/2} - 1)\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2} + \frac{1}{2}x^{-1/2} + \frac{1}{2} - \frac{1}{2}x^{-1/2} = 1.$$

23. $\left.\dfrac{dy}{dx}\right|_{x=2}, \quad y = \dfrac{x^4 - 4}{x^2 - 5}$

SOLUTION Let $y = \dfrac{x^4 - 4}{x^2 - 5}$. Then

$$\frac{dy}{dx} = \frac{\left(x^2 - 5\right)\left(4x^3\right) - \left(x^4 - 4\right)(2x)}{\left(x^2 - 5\right)^2} = \frac{2x^5 - 20x^3 + 8x}{\left(x^2 - 5\right)^2}.$$

Therefore,

$$\left.\frac{dy}{dx}\right|_{x=2} = \frac{2(2)^5 - 20(2)^3 + 8(2)}{(2^2 - 5)^2} = -80.$$

25. $\left.\dfrac{dz}{dx}\right|_{x=1}, \quad z = \dfrac{1}{x^3 + 1}$

SOLUTION Let $z = \frac{1}{x^3+1}$. Using the quotient rule:

$$\frac{dz}{dx} = \frac{(x^3 + 1)(0) - 1(3x^2)}{(x^3 + 1)^2} = -\frac{3x^2}{(x^3 + 1)^2}.$$

Therefore,

$$\left.\frac{dz}{dx}\right|_{x=1} = -\frac{3(1)^2}{(1^3 + 1)^2} = -\frac{3}{4}.$$

27. $h(t) = \dfrac{t}{(t + 1)(t^2 + 1)}$

SOLUTION Let $h(t) = \dfrac{t}{(t + 1)(t^2 + 1)} = \dfrac{t}{t^3 + t^2 + t + 1}$. Then

$$h'(t) = \frac{\left(t^3 + t^2 + t + 1\right)(1) - t\left(3t^2 + 2t + 1\right)}{\left(t^3 + t^2 + t + 1\right)^2} = \frac{-2t^3 - t^2 + 1}{\left(t^3 + t^2 + t + 1\right)^2}.$$

29. $f(t) = 3^{1/2} \cdot 5^{1/2}$

SOLUTION Let $f(t) = \sqrt{3}\sqrt{5}$. Then $f'(t) = 0$, since $f(t)$ is a *constant* function!

31. $f(x) = (x+3)(x-1)(x-5)$

SOLUTION Let $f(x) = (x+3)(x-1)(x-5)$. Using the Product Rule inside the Product Rule with a first factor of $(x+3)$ and a second factor of $(x-1)(x-5)$, we find

$$f'(x) = (x+3)\left((x-1)(1) + (x-5)(1)\right) + (x-1)(x-5)(1) = 3x^2 - 6x - 13.$$

Alternatively,

$$f(x) = (x+3)\left(x^2 - 6x + 5\right) = x^3 - 3x^2 - 13x + 15.$$

Therefore, $f'(x) = 3x^2 - 6x - 13$.

33. $f(x) = \dfrac{e^x}{x+1}$

SOLUTION Let $f(x) = \dfrac{e^x}{(e^x+1)(x+1)}$. Then

$$f'(x) = \frac{(e^x+1)(x+1)e^x - e^x\left((e^x+1)(1) + (x+1)e^x\right)}{(e^x+1)^2(x+1)^2} = \frac{e^x(x-e^x)}{(e^x+1)^2(x+1)^2}.$$

35. $g(z) = \left(\dfrac{z^2-4}{z-1}\right)\left(\dfrac{z^2-1}{z+2}\right)$ *Hint:* Simplify first.

SOLUTION Let

$$g(z) = \left(\frac{z^2-4}{z-1}\right)\left(\frac{z^2-1}{z+2}\right) = \left(\frac{(z+2)(z-2)}{z-1}\right)\left(\frac{(z+1)(z-1)}{z+2}\right) = (z-2)(z+1)$$

for $z \neq -2$ and $z \neq 1$. Then,

$$g'(z) = (z+1)(1) + (z-2)(1) = 2z - 1.$$

37. $\dfrac{d}{dt}\left(\dfrac{xt-4}{t^2-x}\right)$ (x constant)

SOLUTION Let $f(t) = \frac{xt-4}{t^2-x}$. Using the quotient rule:

$$f'(t) = \frac{(t^2-x)(x) - (xt-4)(2t)}{(t^2-x)^2} = \frac{xt^2 - x^2 - 2xt^2 + 8t}{(t^2-x)^2} = \frac{-xt^2 + 8t - x^2}{(t^2-x)^2}.$$

In Exercises 39–42, calculate the derivative using the values:

$f(4)$	$f'(4)$	$g(4)$	$g'(4)$
10	-2	5	-1

39. $(fg)'(4)$ and $(f/g)'(4)$.

SOLUTION Let $h = fg$ and $H = f/g$. Then $h' = fg' + gf'$ and $H' = \frac{gf' - fg'}{g^2}$. Finally,

$$h'(4) = f(4)g'(4) + g(4)f'(4) = (10)(-1) + (5)(-2) = -20,$$

and

$$H'(4) = \frac{g(4)f'(4) - f(4)g'(4)}{(g(4))^2} = \frac{(5)(-2) - (10)(-1)}{(5)^2} = 0.$$

41. $G'(4)$, where $G(x) = g(x)^2$.

SOLUTION Let $G(x) = g(x)^2 = g(x)g(x)$. Then $G'(x) = g(x)g'(x) + g(x)g'(x) = 2g(x)g'(x)$, and

$$G'(4) = 2g(4)g'(4) = 2(5)(-1) = -10.$$

43. Calculate $F'(0)$, where

$$F(x) = \frac{x^9 + x^8 + 4x^5 - 7x}{x^4 - 3x^2 + 2x + 1}$$

Hint: Do not calculate $F'(x)$. Instead, write $F(x) = f(x)/g(x)$ and express $F'(0)$ directly in terms of $f(0)$, $f'(0)$, $g(0)$, $g'(0)$.

SOLUTION Taking the hint, let

$$f(x) = x^9 + x^8 + 4x^5 - 7x$$

and let

$$g(x) = x^4 - 3x^2 + 2x + 1.$$

Then $F(x) = \frac{f(x)}{g(x)}$. Now,

$$f'(x) = 9x^8 + 8x^7 + 20x^4 - 7 \quad \text{and} \quad g'(x) = 4x^3 - 6x + 2.$$

Moreover, $f(0) = 0$, $f'(0) = -7$, $g(0) = 1$, and $g'(0) = 2$.

Using the quotient rule:

$$F'(0) = \frac{g(0)f'(0) - f(0)g'(0)}{(g(0))^2} = \frac{-7 - 0}{1} = -7.$$

45. Use the Product Rule to calculate $\frac{d}{dx}e^{2x}$.

SOLUTION Note that $e^{2x} = e^x \cdot e^x$. Therefore

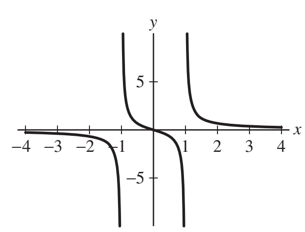
$$\frac{d}{dx}e^{2x} = \frac{d}{dx}(e^x \cdot e^x) = e^x \cdot e^x + e^x \cdot e^x = 2e^{2x}.$$

47. GU Plot $f(x) = x/(x^2 - 1)$ (in a suitably bounded viewing box). Use the plot to determine whether $f'(x)$ is positive or negative on its domain $\{x : x \neq \pm 1\}$. Then compute $f'(x)$ and confirm your conclusion algebraically.

SOLUTION Let $f(x) = \frac{x}{x^2 - 1}$. The graph of $f(x)$ is shown below. From this plot, we see that $f(x)$ is decreasing on its domain $\{x : x \neq \pm 1\}$. Consequently, $f'(x)$ must be negative. Using the quotient rule, we find

$$f'(x) = \frac{(x^2 - 1)(1) - x(2x)}{(x^2 - 1)^2} = -\frac{x^2 + 1}{(x^2 - 1)^2},$$

which is negative for all $x \neq \pm 1$.



49. Find $a > 0$ such that the tangent line to the graph of

$$f(x) = x^2 e^{-x} \quad \text{at } x = a$$

passes through the origin (Figure 4).

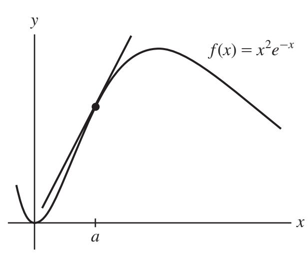


FIGURE 4

SOLUTION Let $f(x) = x^2e^{-x}$. Then $f(a) = a^2e^{-a}$,

$$f'(x) = -x^2e^{-x} + 2xe^{-x} = e^{-x}(2x - x^2),$$

$f'(a) = (2a - a^2)e^{-a}$, and the equation of the tangent line to f at $x = a$ is

$$y = f'(a)(x - a) + f(a) = (2a - a^2)e^{-a}(x - a) + a^2e^{-a}.$$

For this line to pass through the origin, we must have

$$0 = (2a - a^2)e^{-a}(-a) + a^2e^{-a} = e^{-a}\left(a^2 - 2a^2 + a^3\right) = a^2e^{-a}(a - 1).$$

Thus, $a = 0$ or $a = 1$. The only value $a > 0$ such that the tangent line to $f(x) = x^2e^{-x}$ passes through the origin is therefore $a = 1$.

51. The revenue per month earned by the Couture clothing chain at time t is $R(t) = N(t)S(t)$, where $N(t)$ is the number of stores and $S(t)$ is average revenue per store per month. Couture embarks on a two-part campaign: (A) to build new stores at a rate of 5 stores per month, and (B) to use advertising to increase average revenue per store at a rate of \$10,000 per month. Assume that $N(0) = 50$ and $S(0) = \$150{,}000$.

(a) Show that total revenue will increase at the rate

$$\frac{dR}{dt} = 5S(t) + 10{,}000N(t)$$

Note that the two terms in the Product Rule correspond to the separate effects of increasing the number of stores on the one hand, and the average revenue per store on the other.

(b) Calculate $\left.\frac{dR}{dt}\right|_{t=0}$.

(c) If Couture can implement only one leg (A or B) of its expansion at $t = 0$, which choice will grow revenue most rapidly?

SOLUTION

(a) Given $R(t) = N(t)S(t)$, it follows that

$$\frac{dR}{dt} = N(t)S'(t) + S(t)N'(t).$$

We are told that $N'(t) = 5$ stores per month and $S'(t) = 10{,}000$ dollars per month. Therefore,

$$\frac{dR}{dt} = 5S(t) + 10{,}000N(t).$$

(b) Using part (a) and the given values of $N(0)$ and $S(0)$, we find

$$\left.\frac{dR}{dt}\right|_{t=0} = 5(150{,}000) + 10{,}000(50) = 1{,}250{,}000.$$

(c) From part (b), we see that of the two terms contributing to total revenue growth, the term $5S(0)$ is larger than the term $10{,}000N(0)$. Thus, if only one leg of the campaign can be implemented, it should be part A: increase the number of stores by 5 per month.

53. The curve $y = 1/(x^2 + 1)$ is called the *witch of Agnesi* (Figure 6) after the Italian mathematician Maria Agnesi (1718–1799), who wrote one of the first books on calculus. This strange name is the result of a mistranslation of the Italian word *la versiera*, meaning "that which turns." Find equations of the tangent lines at $x = \pm 1$.

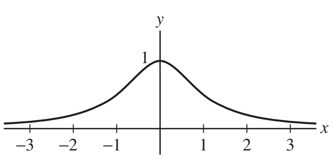


FIGURE 6 The witch of Agnesi.

SOLUTION Let $f(x) = \frac{1}{x^2 + 1}$. Then $f'(x) = \frac{(x^2 + 1)(0) - 1(2x)}{(x^2 + 1)^2} = -\frac{2x}{\left(x^2 + 1\right)^2}$.

- At $x = -1$, the tangent line is

$$y = f'(-1)(x + 1) + f(-1) = \frac{1}{2}(x + 1) + \frac{1}{2} = \frac{1}{2}x + 1.$$

- At $x = 1$, the tangent line is

$$y = f'(1)(x-1) + f(1) = -\frac{1}{2}(x-1) + \frac{1}{2} = -\frac{1}{2}x + 1.$$

55. Use the Product Rule to show that $(f^2)' = 2ff'$.

SOLUTION Let $g = f^2 = ff$. Then $g' = \left(f^2\right)' = (ff)' = ff' + ff' = 2ff'$.

Further Insights and Challenges

57. Let f, g, h be differentiable functions. Show that $(fgh)'(x)$ is equal to

$$f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x)$$

Hint: Write fgh as $f(gh)$.

SOLUTION Let $p = fgh$. Then

$$p' = (fgh)' = f\left(gh' + hg'\right) + ghf' = f'gh + fg'h + fgh'.$$

59. Derivative of the Reciprocal Use the limit definition to prove

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f^2(x)}$$ 7

Hint: Show that the difference quotient for $1/f(x)$ is equal to

$$\frac{f(x) - f(x+h)}{hf(x)f(x+h)}$$

SOLUTION Let $g(x) = \frac{1}{f(x)}$. We then compute the derivative of $g(x)$ using the difference quotient:

$$g'(x) = \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{f(x+h)} - \frac{1}{f(x)}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x) - f(x+h)}{f(x)f(x+h)}\right)$$
$$= -\lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)\left(\frac{1}{f(x)f(x+h)}\right).$$

We can apply the rule of products for limits. The first parenthetical expression is the difference quotient definition of $f'(x)$. The second can be evaluated at $h = 0$ to give $\frac{1}{(f(x))^2}$. Hence

$$g'(x) = \frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f^2(x)}.$$

61. Use the limit definition of the derivative to prove the following special case of the Product Rule:

$$\frac{d}{dx}(xf(x)) = xf'(x) + f(x)$$

SOLUTION First note that because $f(x)$ is differentiable, it is also continuous. It follows that

$$\lim_{h\to 0} f(x+h) = f(x).$$

Now we tackle the derivative:

$$\frac{d}{dx}(xf(x)) = \lim_{h\to 0}\frac{(x+h)f(x+h) - f(x)}{h} = \lim_{h\to 0}\left(x\frac{f(x+h) - f(x)}{h} + f(x+h)\right)$$
$$= x\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} f(x+h)$$
$$= xf'(x) + f(x).$$

63. The Power Rule Revisited If you are familiar with *proof by induction*, use induction to prove the Power Rule for all whole numbers n. Show that the Power Rule holds for $n = 1$; then write x^n as $x \cdot x^{n-1}$ and use the Product Rule.

SOLUTION Let k be a positive integer. If $k = 1$, then $x^k = x$. Note that

$$\frac{d}{dx}\left(x^1\right) = \frac{d}{dx}(x) = 1 = 1x^0.$$

Hence the Power Rule holds for $k = 1$. Assume it holds for $k = n$ where $n \geq 2$. Then for $k = n + 1$, we have

$$\frac{d}{dx}\left(x^k\right) = \frac{d}{dx}\left(x^{n+1}\right) = \frac{d}{dx}\left(x \cdot x^n\right) = x\frac{d}{dx}\left(x^n\right) + x^n\frac{d}{dx}(x)$$
$$= x \cdot nx^{n-1} + x^n \cdot 1 = (n+1)x^n = kx^{k-1}$$

Accordingly, the Power Rule holds for all positive integers by induction.

Exercises 64 and 65: A basic fact of algebra states that c is a root of a polynomial $f(x)$ if and only if $f(x) = (x - c)g(x)$ for some polynomial $g(x)$. We say that c is a ***multiple root*** *if $f(x) = (x - c)^2h(x)$, where $h(x)$ is a polynomial.*

65. Use Exercise 64 to determine whether $c = -1$ is a multiple root:
(a) $x^5 + 2x^4 - 4x^3 - 8x^2 - x + 2$
(b) $x^4 + x^3 - 5x^2 - 3x + 2$

SOLUTION
(a) To show that -1 is a multiple root of

$$f(x) = x^5 + 2x^4 - 4x^3 - 8x^2 - x + 2,$$

it suffices to check that $f(-1) = f'(-1) = 0$. We have $f(-1) = -1 + 2 + 4 - 8 + 1 + 2 = 0$ and

$$f'(x) = 5x^4 + 8x^3 - 12x^2 - 16x - 1$$
$$f'(-1) = 5 - 8 - 12 + 16 - 1 = 0$$

(b) Let $f(x) = x^4 + x^3 - 5x^2 - 3x + 2$. Then $f'(x) = 4x^3 + 3x^2 - 10x - 3$. Because

$$f(-1) = 1 - 1 - 5 + 3 + 2 = 0$$

but

$$f'(-1) = -4 + 3 + 10 - 3 = 6 \neq 0,$$

it follows that $x = -1$ is a root of f, but not a multiple root.

67. According to Eq. (6) in Section 3.2, $\frac{d}{dx}b^x = m(b)\,b^x$. Use the Product Rule to show that $m(ab) = m(a) + m(b)$.

SOLUTION

$$m(ab)(ab)^x = \frac{d}{dx}(ab)^x = \frac{d}{dx}\left(a^x b^x\right) = a^x\frac{d}{dx}b^x + b^x\frac{d}{dx}a^x = m(b)a^x b^x + m(a)a^x b^x = (m(a) + m(b))(ab)^x.$$

Thus, $m(ab) = m(a) + m(b)$.

3.4 Rates of Change

Preliminary Questions

1. Which units might be used for each rate of change?
(a) Pressure (in atmospheres) in a water tank with respect to depth
(b) The rate of a chemical reaction (change in concentration with respect to time with concentration in moles per liter)

SOLUTION
(a) The rate of change of pressure with respect to depth might be measured in atmospheres/meter.
(b) The reaction rate of a chemical reaction might be measured in moles/(liter·hour).

2. Two trains travel from New Orleans to Memphis in 4 hours. The first train travels at a constant velocity of 90 mph, but the velocity of the second train varies. What was the second train's average velocity during the trip?

SOLUTION Since both trains travel the same distance in the same amount of time, they have the same average velocity: 90 mph.

3. Estimate $f(26)$, assuming that $f(25) = 43$, $f'(25) = 0.75$.

SOLUTION $f(x) \approx f(25) + f'(25)(x - 25)$, so $f(26) \approx 43 + 0.75(26 - 25) = 43.75$.

4. The population $P(t)$ of Freedonia in 2009 was $P(2009) = 5$ million.
(a) What is the meaning of $P'(2009)$?
(b) Estimate $P(2010)$ if $P'(2009) = 0.2$.

SOLUTION
(a) Because $P(t)$ measures the population of Freedonia as a function of time, the derivative $P'(2009)$ measures the rate of change of the population of Freedonia in the year 2009.
(b) $P(2010) \approx P(2009) + P'(2010)$. Thus, if $P'(2009) = 0.2$, then $P(2009) \approx 5.2$ million.

Exercises

In Exercises 1–8, find the rate of change.

1. Area of a square with respect to its side s when $s = 5$.

SOLUTION Let the area be $A = f(s) = s^2$. Then the rate of change of A with respect to s is $d/ds(s^2) = 2s$. When $s = 5$, the area changes at a rate of 10 square units per unit increase. (Draw a 5×5 square on graph paper and trace the area added by increasing each side length by 1, excluding the corner, to see what this means.)

3. Cube root $\sqrt[3]{x}$ with respect to x when $x = 1, 8, 27$.

SOLUTION Let $f(x) = \sqrt[3]{x}$. Writing $f(x) = x^{1/3}$, we see the rate of change of $f(x)$ with respect to x is given by $f'(x) = \frac{1}{3}x^{-2/3}$. The requested rates of change are given in the table that follows:

c	ROC of $f(x)$ with respect to x at $x = c$.
1	$f'(1) = \frac{1}{3}(1) = \frac{1}{3}$
8	$f'(8) = \frac{1}{3}(8^{-2/3}) = \frac{1}{3}(\frac{1}{4}) = \frac{1}{12}$
27	$f'(27) = \frac{1}{3}(27^{-2/3}) = \frac{1}{3}(\frac{1}{9}) = \frac{1}{27}$

5. The diameter of a circle with respect to radius.

SOLUTION The relationship between the diameter d of a circle and its radius r is $d = 2r$. The rate of change of the diameter with respect to the radius is then $d' = 2$.

7. Volume V of a cylinder with respect to radius if the height is equal to the radius.

SOLUTION The volume of the cylinder is $V = \pi r^2 h = \pi r^3$. Thus $dV/dr = 3\pi r^2$.

In Exercises 9–11, refer to Figure 10, the graph of distance $s(t)$ from the origin as a function of time for a car trip.

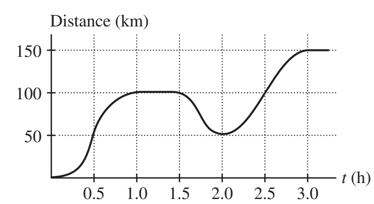


FIGURE 10 Distance from the origin versus time for a car trip.

9. Find the average velocity over each interval.

(a) [0, 0.5] **(b)** [0.5, 1] **(c)** [1, 1.5] **(d)** [1, 2]

SOLUTION

(a) The average velocity over the interval [0, 0.5] is

$$\frac{50 - 0}{0.5 - 0} = 100 \text{ km/hour}.$$

(b) The average velocity over the interval [0.5, 1] is

$$\frac{100 - 50}{1 - 0.5} = 100 \text{ km/hour}.$$

(c) The average velocity over the interval [1, 1.5] is

$$\frac{100 - 100}{1.5 - 1} = 0 \text{ km/hour}.$$

(d) The average velocity over the interval [1, 2] is

$$\frac{50 - 100}{2 - 1} = -50 \text{ km/hour}.$$

11. Match the descriptions (i)–(iii) with the intervals (a)–(c).

(i) Velocity increasing

(ii) Velocity decreasing

(iii) Velocity negative

(a) [0, 0.5]
(b) [2.5, 3]
(c) [1.5, 2]

SOLUTION

(a) **(i)** : The distance curve is increasing, and is also *bending* upward, so that distance is increasing at an increasing rate.
(b) **(ii)** : Over the interval [2.5, 3], the distance curve is flattening, showing that the car is slowing down; that is, the velocity is decreasing.
(c) **(iii)** : The distance curve is decreasing, so the tangent line has negative slope; this means the velocity is negative.

13. Use Figure 3 from Example 1 to estimate the instantaneous rate of change of Martian temperature with respect to time (in degrees Celsius per hour) at $t = 4$ AM.

SOLUTION The segment of the temperature graph around $t = 4$ AM appears to be a straight line passing through roughly (1:36, −70) and (4:48, −75). The instantaneous rate of change of Martian temperature with respect to time at $t = 4$ AM is therefore approximately

$$\frac{dT}{dt} = \frac{-75 - (-70)}{3.2} = -1.5625^\circ\text{C/hour}.$$

15. The velocity (in cm/s) of blood molecules flowing through a capillary of radius 0.008 cm is $v = 6.4 \times 10^{-8} - 0.001r^2$, where r is the distance from the molecule to the center of the capillary. Find the rate of change of velocity with respect to r when $r = 0.004$ cm.

SOLUTION The rate of change of the velocity of the blood molecules is $v'(r) = -0.002r$. When $r = 0.004$ cm, this rate is -8×10^{-6} 1/s.

17. Use Figure 12 to estimate dT/dh at $h = 30$ and 70, where T is atmospheric temperature (in degrees Celsius) and h is altitude (in kilometers). Where is dT/dh equal to zero?

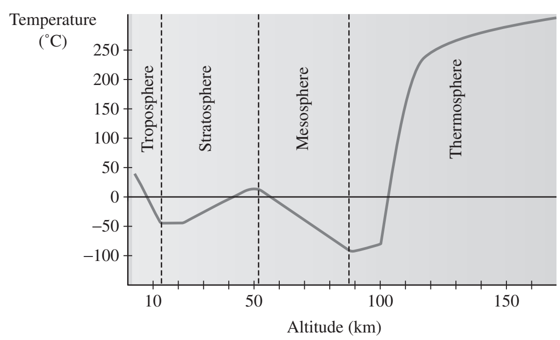


FIGURE 12 Atmospheric temperature versus altitude.

SOLUTION At $h = 30$ km, the graph of atmospheric temperature appears to be linear passing through the points (23, −50) and (40, 0). The slope of this segment of the graph is then

$$\frac{0 - (-50)}{40 - 23} = \frac{50}{17} = 2.94;$$

so

$$\left.\frac{dT}{dh}\right|_{h=30} \approx 2.94^\circ\text{C/km}.$$

At $h = 70$ km, the graph of atmospheric temperature appears to be linear passing through the points (58, 0) and (88, −100). The slope of this segment of the graph is then

$$\frac{-100 - 0}{88 - 58} = \frac{-100}{30} = -3.33;$$

so

$$\left.\frac{dT}{dh}\right|_{h=70} \approx -3.33^\circ\text{C/km}.$$

$\frac{dT}{dh} = 0$ at those points where the tangent line on the graph is horizontal. This appears to happen over the interval [13, 23], and near the points $h = 50$ and $h = 90$.

19. Calculate the rate of change of escape velocity $v_{\text{esc}} = (2.82 \times 10^7)r^{-1/2}$ m/s with respect to distance r from the center of the earth.

SOLUTION The rate that escape velocity changes is $v'_{\text{esc}}(r) = -1.41 \times 10^7 r^{-3/2}$.

21. The position of a particle moving in a straight line during a 5-s trip is $s(t) = t^2 - t + 10$ cm. Find a time t at which the instantaneous velocity is equal to the average velocity for the entire trip.

SOLUTION Let $s(t) = t^2 - t + 10$, $0 \le t \le 5$, with s in centimeters (cm) and t in seconds (s). The average velocity over the t-interval $[0, 5]$ is

$$\frac{s(5) - s(0)}{5 - 0} = \frac{30 - 10}{5} = 4 \text{ cm/s}.$$

The (instantaneous) velocity is $v(t) = s'(t) = 2t - 1$. Solving $2t - 1 = 4$ yields $t = \frac{5}{2}$ s, the time at which the instantaneous velocity equals the calculated average velocity.

23. A particle moving along a line has position $s(t) = t^4 - 18t^2$ m at time t seconds. At which times does the particle pass through the origin? At which times is the particle instantaneously motionless (that is, it has zero velocity)?

SOLUTION The particle passes through the origin when $s(t) = t^4 - 18t^2 = t^2(t^2 - 18) = 0$. This happens when $t = 0$ seconds and when $t = 3\sqrt{2} \approx 4.24$ seconds. With $s(t) = t^4 - 18t^2$, it follows that $v(t) = s'(t) = 4t^3 - 36t = 4t(t^2 - 9)$. The particle is therefore instantaneously motionless when $t = 0$ seconds and when $t = 3$ seconds.

25. A bullet is fired in the air vertically from ground level with an initial velocity 200 m/s. Find the bullet's maximum velocity and maximum height.

SOLUTION We employ Galileo's formula, $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2 = 200t - 4.9t^2$, where the time t is in seconds (s) and the height s is in meters (m). The velocity is $v(t) = 200 - 9.8t$. The maximum velocity of 200 m/s occurs at $t = 0$. This is the initial velocity. The bullet reaches its maximum height when $v(t) = 200 - 9.8t = 0$; i.e., when $t \approx 20.41$ s. At this point, the height is 2040.82 m.

27. A ball tossed in the air vertically from ground level returns to earth 4 s later. Find the initial velocity and maximum height of the ball.

SOLUTION Galileo's formula gives $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2 = v_0 t - 4.9t^2$, where the time t is in seconds (s) and the height s is in meters (m). When the ball hits the ground after 4 seconds its height is 0. Solve $0 = s(4) = 4v_0 - 4.9(4)^2$ to obtain $v_0 = 19.6$ m/s. The ball reaches its maximum height when $s'(t) = 0$, that is, when $19.6 - 9.8t = 0$, or $t = 2$ s. At this time, $t = 2$ s,

$$s(2) = 0 + 19.6(2) - \frac{1}{2}(9.8)(4) = 19.6 \text{ m}.$$

29. Show that for an object falling according to Galileo's formula, the average velocity over any time interval $[t_1, t_2]$ is equal to the average of the instantaneous velocities at t_1 and t_2.

SOLUTION The simplest way to proceed is to compute both values and show that they are equal. The average velocity over $[t_1, t_2]$ is

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{(s_0 + v_0 t_2 - \frac{1}{2}gt_2^2) - (s_0 + v_0 t_1 - \frac{1}{2}gt_1^2)}{t_2 - t_1} = \frac{v_0(t_2 - t_1) + \frac{g}{2}({t_2}^2 - {t_1}^2)}{t_2 - t_1}$$
$$= \frac{v_0(t_2 - t_1)}{t_2 - t_1} - \frac{g}{2}(t_2 + t_1) = v_0 - \frac{g}{2}(t_2 + t_1)$$

Whereas the average of the instantaneous velocities at the beginning and end of $[t_1, t_2]$ is

$$\frac{s'(t_1) + s'(t_2)}{2} = \frac{1}{2}\Big((v_0 - gt_1) + (v_0 - gt_2)\Big) = \frac{1}{2}(2v_0) - \frac{g}{2}(t_2 + t_1) = v_0 - \frac{g}{2}(t_2 + t_1).$$

The two quantities are the same.

31. By Faraday's Law, if a conducting wire of length ℓ meters moves at velocity v m/s perpendicular to a magnetic field of strength B (in teslas), a voltage of size $V = -B\ell v$ is induced in the wire. Assume that $B = 2$ and $\ell = 0.5$.

(a) Calculate dV/dv.

(b) Find the rate of change of V with respect to time t if $v = 4t + 9$.

SOLUTION

(a) Assuming that $B = 2$ and $l = 0.5$, $V = -2(.5)v = -v$. Therefore,

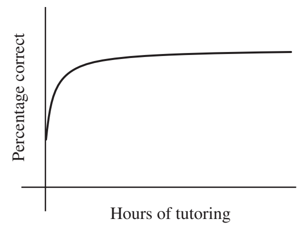
$$\frac{dV}{dv} = -1.$$

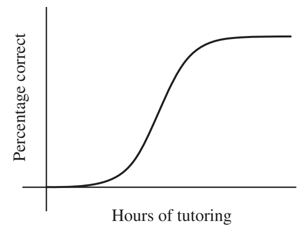
(b) If $v = 4t + 9$, then $V = -2(.5)(4t + 9) = -(4t + 9)$. Therefore, $\frac{dV}{dt} = -4$.

33. Ethan finds that with h hours of tutoring, he is able to answer correctly $S(h)$ percent of the problems on a math exam. Which would you expect to be larger: $S'(3)$ or $S'(30)$? Explain.

SOLUTION One possible graph of $S(h)$ is shown in the figure below on the left. This graph indicates that in the early hours of working with the tutor, Ethan makes rapid progress in learning the material but eventually approaches either the limit of his ability to learn the material or the maximum possible score on the exam. In this scenario, $S'(3)$ would be larger than $S'(30)$.

An alternative graph of $S(h)$ is shown below on the right. Here, in the early hours of working with the tutor little progress is made (perhaps the tutor is assessing how much Ethan already knows, his learning style, his personality, etc.). This is followed by a period of rapid improvement and finally a leveling off as Ethan reaches his maximum score. In this scenario, $S'(3)$ and $S'(30)$ might be roughly equal.





35. To determine drug dosages, doctors estimate a person's body surface area (BSA) (in meters squared) using the formula $\text{BSA} = \sqrt{hm}/60$, where h is the height in centimeters and m the mass in kilograms. Calculate the rate of change of BSA with respect to mass for a person of constant height $h = 180$. What is this rate at $m = 70$ and $m = 80$? Express your result in the correct units. Does BSA increase more rapidly with respect to mass at lower or higher body mass?

SOLUTION Assuming constant height $h = 180$ cm, let $f(m) = \sqrt{hm}/60 = \frac{\sqrt{5}}{10}\sqrt{m}$ be the formula for body surface area in terms of weight. The rate of change of BSA with respect to mass is

$$f'(m) = \frac{\sqrt{5}}{10}\left(\frac{1}{2}m^{-1/2}\right) = \frac{\sqrt{5}}{20\sqrt{m}}.$$

If $m = 70$ kg, this is

$$f'(70) = \frac{\sqrt{5}}{20\sqrt{70}} = \frac{\sqrt{14}}{280} \approx 0.0133631\frac{\text{m}^2}{\text{kg}}.$$

If $m = 80$ kg,

$$f'(80) = \frac{\sqrt{5}}{20\sqrt{80}} = \frac{1}{20\sqrt{16}} = \frac{1}{80}\frac{\text{m}^2}{\text{kg}}.$$

Because the rate of change of BSA depends on $1/\sqrt{m}$, it is clear that BSA increases more rapidly at lower body mass.

37. The tangent lines to the graph of $f(x) = x^2$ grow steeper as x increases. At what rate do the slopes of the tangent lines increase?

SOLUTION Let $f(x) = x^2$. The slopes s of the tangent lines are given by $s = f'(x) = 2x$. The rate at which these slopes are increasing is $ds/dx = 2$.

In Exercises 39–46, use Eq. (3) to estimate the unit change.

39. Estimate $\sqrt{2} - \sqrt{1}$ and $\sqrt{101} - \sqrt{100}$. Compare your estimates with the actual values.

SOLUTION Let $f(x) = \sqrt{x} = x^{1/2}$. Then $f'(x) = \frac{1}{2}(x^{-1/2})$. We are using the derivative to estimate the average rate of change. That is,

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} \approx f'(x),$$

so that

$$\sqrt{x+h} - \sqrt{x} \approx hf'(x).$$

Thus, $\sqrt{2} - \sqrt{1} \approx 1f'(1) = \frac{1}{2}(1) = \frac{1}{2}$. The actual value, to six decimal places, is 0.414214. Also, $\sqrt{101} - \sqrt{100} \approx 1f'(100) = \frac{1}{2}\left(\frac{1}{10}\right) = 0.05$. The actual value, to six decimal places, is 0.0498756.

41. Let $F(s) = 1.1s + 0.05s^2$ be the stopping distance as in Example 3. Calculate $F(65)$ and estimate the increase in stopping distance if speed is increased from 65 to 66 mph. Compare your estimate with the actual increase.

SOLUTION Let $F(s) = 1.1s + .05s^2$ be as in Example 3. $F'(s) = 1.1 + 0.1s$.

- Then $F(65) = 282.75$ ft and $F'(65) = 7.6$ ft/mph.
- $F'(65) \approx F(66) - F(65)$ is approximately equal to the change in stopping distance per 1 mph increase in speed when traveling at 65 mph. Increasing speed from 65 to 66 therefore increases stopping distance by approximately 7.6 ft.
- The actual increase in stopping distance when speed increases from 65 mph to 66 mph is $F(66) - F(65) = 290.4 - 282.75 = 7.65$ feet, which differs by less than one percent from the estimate found using the derivative.

43. The dollar cost of producing x bagels is $C(x) = 300 + 0.25x - 0.5(x/1000)^3$. Determine the cost of producing 2000 bagels and estimate the cost of the 2001st bagel. Compare your estimate with the actual cost of the 2001st bagel.

SOLUTION Expanding the power of 3 yields

$$C(x) = 300 + 0.25x - 5 \times 10^{-10}x^3.$$

This allows us to get the derivative $C'(x) = 0.25 - 1.5 \times 10^{-9}x^2$. The cost of producing 2000 bagels is

$$C(2000) = 300 + 0.25(2000) - 0.5(2000/1000)^3 = 796$$

dollars. The cost of the 2001st bagel is, by definition, $C(2001) - C(2000)$. By the derivative estimate, $C(2001) - C(2000) \approx C'(2000)(1)$, so the cost of the 2001st bagel is approximately

$$C'(2000) = 0.25 - 1.5 \times 10^{-9}(2000^2) = \$0.244.$$

$C(2001) = 796.244$, so the *exact* cost of the 2001st bagel is indistinguishable from the estimated cost. The function is very nearly linear at this point.

45. Demand for a commodity generally decreases as the price is raised. Suppose that the demand for oil (per capita per year) is $D(p) = 900/p$ barrels, where p is the dollar price per barrel. Find the demand when $p = \$40$. Estimate the decrease in demand if p rises to \$41 and the increase if p declines to \$39.

SOLUTION $D(p) = 900p^{-1}$, so $D'(p) = -900p^{-2}$. When the price is \$40 a barrel, the per capita demand is $D(40) = 22.5$ barrels per year. With an increase in price from \$40 to \$41 a barrel, the change in demand $D(41) - D(40)$ is approximately $D'(40) = -900(40^{-2}) = -0.5625$ barrels a year. With a decrease in price from \$40 to \$39 a barrel, the change in demand $D(39) - D(40)$ is approximately $-D'(40) = +0.5625$. An increase in oil prices of a dollar leads to a decrease in demand of 0.5625 barrels a year, and a decrease of a dollar leads to an *increase* in demand of 0.5625 barrels a year.

47. According to Stevens' Law in psychology, the perceived magnitude of a stimulus is proportional (approximately) to a power of the actual intensity I of the stimulus. Experiments show that the *perceived brightness* B of a light satisfies $B = kI^{2/3}$, where I is the light intensity, whereas the *perceived heaviness* H of a weight W satisfies $H = kW^{3/2}$ (k is a constant that is different in the two cases). Compute dB/dI and dH/dW and state whether they are increasing or decreasing functions. Then explain the following statements:

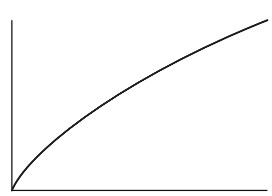
(a) A one-unit increase in light intensity is felt more strongly when I is small than when I is large.

(b) Adding another pound to a load W is felt more strongly when W is large than when W is small.

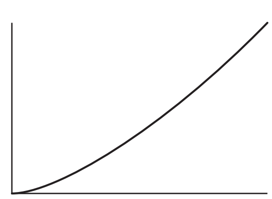
SOLUTION

(a) $dB/dI = \dfrac{2k}{3}I^{-1/3} = \dfrac{2k}{3I^{1/3}}$.

As I increases, dB/dI shrinks, so that the rate of change of perceived intensity decreases as the actual intensity increases. Increased light intensity has a *diminished return* in perceived intensity. A sketch of B against I is shown: See that the height of the graph increases more slowly as you move to the right.

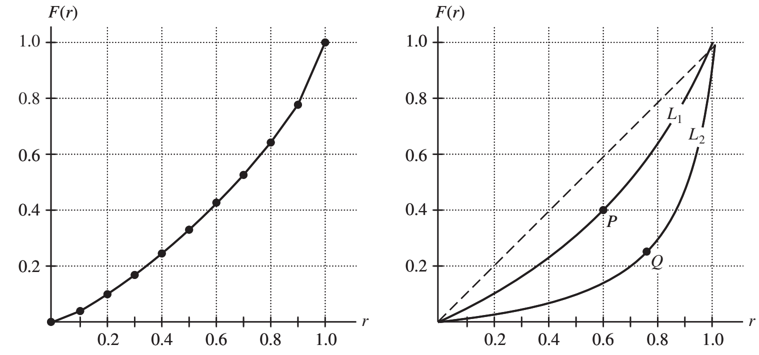


(b) $dH/dW = \frac{3k}{2}W^{1/2}$. As W increases, dH/dW increases as well, so that the rate of change of perceived weight increases as weight increases. A sketch of H against W is shown: See that the graph becomes steeper as you move to the right.



Further Insights and Challenges

*Exercises 49–51: The **Lorenz curve** $y = F(r)$ is used by economists to study income distribution in a given country (see Figure 14). By definition, $F(r)$ is the fraction of the total income that goes to the bottom rth part of the population, where $0 \le r \le 1$. For example, if $F(0.4) = 0.245$, then the bottom 40% of households receive 24.5% of the total income. Note that $F(0) = 0$ and $F(1) = 1$.*



(A) Lorenz curve for Sweden in 2004

(B) Two Lorenz curves: The tangent lines at P and Q have slope 1.

FIGURE 14

49. Our goal is to find an interpretation for $F'(r)$. The average income for a group of households is the total income going to the group divided by the number of households in the group. The national average income is $A = T/N$, where N is the total number of households and T is the total income earned by the entire population.

(a) Show that the average income among households in the bottom rth part is equal to $(F(r)/r)A$.

(b) Show more generally that the average income of households belonging to an interval $[r, r + \Delta r]$ is equal to

$$\left(\frac{F(r + \Delta r) - F(r)}{\Delta r}\right) A$$

(c) Let $0 \le r \le 1$. A household belongs to the $100r$th percentile if its income is greater than or equal to the income of $100r$ % of all households. Pass to the limit as $\Delta r \to 0$ in (b) to derive the following interpretation: A household in the $100r$th percentile has income $F'(r)A$. In particular, a household in the $100r$th percentile receives more than the national average if $F'(r) > 1$ and less if $F'(r) < 1$.

(d) For the Lorenz curves L_1 and L_2 in Figure 14(B), what percentage of households have above-average income?

SOLUTION

(a) The total income among households in the bottom rth part is $F(r)T$ and there are rN households in this part of the population. Thus, the average income among households in the bottom rth part is equal to

$$\frac{F(r)T}{rN} = \frac{F(r)}{r} \cdot \frac{T}{N} = \frac{F(r)}{r} A.$$

(b) Consider the interval $[r, r + \Delta r]$. The total income among households between the bottom rth part and the bottom $r + \Delta r$-th part is $F(r + \Delta r)T - F(r)T$. Moreover, the number of households covered by this interval is $(r + \Delta r)N - rN = \Delta r N$. Thus, the average income of households belonging to an interval $[r, r + \Delta r]$ is equal to

$$\frac{F(r + \Delta r)T - F(r)T}{\Delta r N} = \frac{F(r + \Delta r) - F(r)}{\Delta r} \cdot \frac{T}{N} = \frac{F(r + \Delta r) - F(r)}{\Delta r} A.$$

(c) Take the result from part (b) and let $\Delta r \to 0$. Because

$$\lim_{\Delta r \to 0} \frac{F(r + \Delta r) - F(r)}{\Delta r} = F'(r),$$

we find that a household in the $100r$th percentile has income $F'(r)A$.

(d) The point P in Figure 14(B) has an r-coordinate of 0.6, while the point Q has an r-coordinate of roughly 0.75. Thus, on curve L_1, 40% of households have $F'(r) > 1$ and therefore have above-average income. On curve L_2, roughly 25% of households have above-average income.

51. Use Exercise 49 (c) to prove:

(a) $F'(r)$ is an increasing function of r.

(b) Income is distributed equally (all households have the same income) if and only if $F(r) = r$ for $0 \le r \le 1$.

SOLUTION

(a) Recall from Exercise 49 (c) that $F'(r)A$ is the income of a household in the $100r$-th percentile. Suppose $0 \le r_1 < r_2 \le 1$. Because $r_2 > r_1$, a household in the $100r_2$-th percentile must have income at least as large as a household in the $100r_1$-th percentile. Thus, $F'(r_2)A \ge F'(r_1)A$, or $F'(r_2) \ge F'(r_1)$. This implies $F'(r)$ is an increasing function of r.

(b) If $F(r) = r$ for $0 \le r \le 1$, then $F'(r) = 1$ and households in all percentiles have income equal to the national average; that is, income is distributed equally. Alternately, if income is distributed equally (all households have the same income), then $F'(r) = 1$ for $0 \le r \le 1$. Thus, F must be a linear function in r with slope 1. Moreover, the condition $F(0) = 0$ requires the F intercept of the line to be 0. Hence, $F(r) = 1 \cdot r + 0 = r$.

In Exercises 53 and 54, the average cost per unit at production level x is defined as $C_{avg}(x) = C(x)/x$, *where* $C(x)$ *is the cost function. Average cost is a measure of the efficiency of the production process.*

53. Show that $C_{avg}(x)$ is equal to the slope of the line through the origin and the point $(x, C(x))$ on the graph of $C(x)$. Using this interpretation, determine whether average cost or marginal cost is greater at points A, B, C, D in Figure 15.

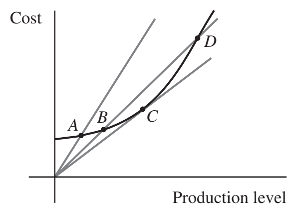


FIGURE 15 Graph of $C(x)$.

SOLUTION By definition, the slope of the line through the origin and $(x, C(x))$, that is, between $(0, 0)$ and $(x, C(x))$ is

$$\frac{C(x) - 0}{x - 0} = \frac{C(x)}{x} = C_{av}.$$

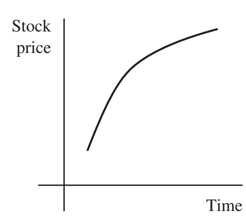
At point A, average cost is greater than marginal cost, as the line from the origin to A is steeper than the curve at this point (we see this because the line, tracing from the origin, crosses the curve from below). At point B, the average cost is still greater than the marginal cost. At the point C, the average cost and the marginal cost are nearly the same, since the tangent line and the line from the origin are nearly the same. The line from the origin to D crosses the cost curve from above, and so is less steep than the tangent line to the curve at D; the average cost at this point is less than the marginal cost.

3.5 Higher Derivatives

Preliminary Questions

1. On September 4, 2003, the *Wall Street Journal* printed the headline "Stocks Go Higher, Though the Pace of Their Gains Slows." Rephrase this headline as a statement about the first and second time derivatives of stock prices and sketch a possible graph.

SOLUTION Because stocks are going higher, stock prices are increasing and the first derivative of stock prices must therefore be positive. On the other hand, because the pace of gains is slowing, the second derivative of stock prices must be negative.



2. True or false? The third derivative of position with respect to time is zero for an object falling to earth under the influence of gravity. Explain.

SOLUTION This statement is true. The acceleration of an object falling to earth under the influence of gravity is constant; hence, the second derivative of position with respect to time is constant. Because the third derivative is just the derivative of the second derivative and the derivative of a constant is zero, it follows that the third derivative is zero.

3. Which type of polynomial satisfies $f'''(x) = 0$ for all x?

SOLUTION The third derivative of all quadratic polynomials (polynomials of the form $ax^2 + bx + c$ for some constants a, b and c) is equal to 0 for all x.

4. What is the millionth derivative of $f(x) = e^x$?

SOLUTION Every derivative of $f(x) = e^x$ is e^x.

Exercises

In Exercises 1–16, calculate y'' and y'''.

1. $y = 14x^2$

SOLUTION Let $y = 14x^2$. Then $y' = 28x$, $y'' = 28$, and $y''' = 0$.

3. $y = x^4 - 25x^2 + 2x$

SOLUTION Let $y = x^4 - 25x^2 + 2x$. Then $y' = 4x^3 - 50x + 2$, $y'' = 12x^2 - 50$, and $y''' = 24x$.

5. $y = \dfrac{4}{3}\pi r^3$

SOLUTION Let $y = \frac{4}{3}\pi r^3$. Then $y' = 4\pi r^2$, $y'' = 8\pi r$, and $y''' = 8\pi$.

7. $y = 20t^{4/5} - 6t^{2/3}$

SOLUTION Let $y = 20t^{4/5} - 6t^{2/3}$. Then $y' = 16t^{-1/5} - 4t^{-1/3}$, $y'' = -\frac{16}{5}t^{-6/5} + \frac{4}{3}t^{-4/3}$, and $y''' = \frac{96}{25}t^{-11/15} - \frac{16}{9}t^{-7/3}$.

9. $y = z - \dfrac{4}{z}$

SOLUTION Let $y = z - 4z^{-1}$. Then $y' = 1 + 4z^{-2}$, $y'' = -8z^{-3}$, and $y''' = 24z^{-4}$.

11. $y = \theta^2(2\theta + 7)$

SOLUTION Let $y = \theta^2(2\theta + 7) = 2\theta^3 + 7\theta^2$. Then $y' = 6\theta^2 + 14\theta$, $y'' = 12\theta + 14$, and $y''' = 12$.

13. $y = \dfrac{x-4}{x}$

SOLUTION Let $y = \frac{x-4}{x} = 1 - 4x^{-1}$. Then $y' = 4x^{-2}$, $y'' = -8x^{-3}$, and $y''' = 24x^{-4}$.

15. $y = x^5 e^x$

SOLUTION Let $y = x^5 e^x$. Then

$$\begin{aligned}
y' &= x^5e^x + 5x^4e^x = (x^5 + 5x^4)e^x \\
y'' &= (x^5 + 5x^4)e^x + (5x^4 + 20x^3)e^x = (x^5 + 10x^4 + 20x^3)e^x \\
y''' &= (x^5 + 10x^4 + 20x^3)e^x + (5x^4 + 40x^3 + 60x^2)e^x = (x^5 + 15x^4 + 60x^3 + 60x^2)e^x.
\end{aligned}$$

In Exercises 17–26, calculate the derivative indicated.

17. $f^{(4)}(1), \quad f(x) = x^4$

SOLUTION Let $f(x) = x^4$. Then $f'(x) = 4x^3$, $f''(x) = 12x^2$, $f'''(x) = 24x$, and $f^{(4)}(x) = 24$. Thus $f^{(4)}(1) = 24$.

19. $\left.\dfrac{d^2y}{dt^2}\right|_{t=1}, \quad y = 4t^{-3} + 3t^2$

SOLUTION Let $y = 4t^{-3} + 3t^2$. Then $\frac{dy}{dt} = -12t^{-4} + 6t$ and $\frac{d^2y}{dt^2} = 48t^{-5} + 6$. Hence

$$\left.\frac{d^2y}{dt^2}\right|_{t=1} = 48(1)^{-5} + 6 = 54.$$

21. $\left.\dfrac{d^4x}{dt^4}\right|_{t=16}, \quad x = t^{-3/4}$

SOLUTION Let $x(t) = t^{-3/4}$. Then $\frac{dx}{dt} = -\frac{3}{4}t^{-7/4}$, $\frac{d^2x}{dt^2} = \frac{21}{16}t^{-11/4}$, $\frac{d^3x}{dt^3} = -\frac{231}{64}t^{-15/4}$, and $\frac{d^4x}{dt^4} = \frac{3465}{256}t^{-19/4}$. Thus

$$\left.\frac{d^4x}{dt^4}\right|_{t=16} = \frac{3465}{256}16^{-19/4} = \frac{3465}{134217728}.$$

23. $f'''(-3), \quad f(x) = 4e^x - x^3$

SOLUTION Let $f(x) = 4e^x - x^3$. Then $f'(x) = 4e^x - 3x^2$, $f''(x) = 4e^x - 6x$, $f'''(x) = 4e^x - 6$, and $f'''(-3) = 4e^{-3} - 6$.

25. $h''(1), \quad h(w) = \sqrt{w}e^w$

SOLUTION Let $h(w) = \sqrt{w}e^w = w^{1/2}e^w$. Then

$$h'(w) = w^{1/2}e^w + e^w\left(\frac{1}{2}w^{-1/2}\right) = \left(w^{1/2} + \frac{1}{2}w^{-1/2}\right)e^w$$

and

$$h''(w) = \left(w^{1/2} + \frac{1}{2}w^{-1/2}\right)e^w + e^w\left(\frac{1}{2}w^{-1/2} - \frac{1}{4}w^{-3/2}\right) = \left(w^{1/2} + w^{-1/2} - \frac{1}{4}w^{-3/2}\right)e^w.$$

Thus, $h''(1) = \frac{7}{4}e$.

27. Calculate $y^{(k)}(0)$ for $0 \le k \le 5$, where $y = x^4 + ax^3 + bx^2 + cx + d$ (with a, b, c, d the constants).

SOLUTION Applying the power, constant multiple, and sum rules at each stage, we get (note $y^{(0)}$ is y by convention):

k	$y^{(k)}$
0	$x^4 + ax^3 + bx^2 + cx + d$
1	$4x^3 + 3ax^2 + 2bx + c$
2	$12x^2 + 6ax + 2b$
3	$24x + 6a$
4	24
5	0

from which we get $y^{(0)}(0) = d$, $y^{(1)}(0) = c$, $y^{(2)}(0) = 2b$, $y^{(3)}(0) = 6a$, $y^{(4)}(0) = 24$, and $y^{(5)}(0) = 0$.

29. Use the result in Example 3 to find $\frac{d^6}{dx^6}x^{-1}$.

SOLUTION The equation in Example 3 indicates that

$$\frac{d^6}{dx^6}x^{-1} = (-1)^6 6! x^{-6-1}.$$

$(-1)^6 = 1$ and $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$, so

$$\frac{d^6}{dx^6}x^{-1} = 720x^{-7}.$$

In Exercises 31–36, find a general formula for $f^{(n)}(x)$.

31. $f(x) = x^{-2}$

SOLUTION $f'(x) = -2x^{-3}$, $f''(x) = 6x^{-4}$, $f'''(x) = -24x^{-5}$, $f^{(4)}(x) = 5 \cdot 24x^{-6}, \ldots$. From this we can conclude that the nth derivative can be written as $f^{(n)}(x) = (-1)^n(n+1)!x^{-(n+2)}$.

33. $f(x) = x^{-1/2}$

SOLUTION $f'(x) = \frac{-1}{2}x^{-3/2}$. We will avoid simplifying numerators and denominators to find the pattern:

$$f''(x) = \frac{-3}{2}\frac{-1}{2}x^{-5/2} = (-1)^2\frac{3 \times 1}{2^2}x^{-5/2}$$

$$f'''(x) = -\frac{5}{2}\frac{3 \times 1}{2^2}x^{-7/2} = (-1)^3\frac{5 \times 3 \times 1}{2^3}x^{-7/2}$$

$$\vdots$$

$$f^{(n)}(x) = (-1)^n\frac{(2n-1) \times (2n-3) \times \ldots \times 1}{2^n}x^{-(2n+1)/2}.$$

35. $f(x) = xe^{-x}$

SOLUTION Let $f(x) = xe^{-x}$. Then

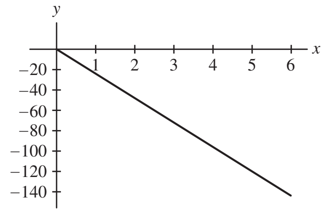
$$f'(x) = x(-e^{-x}) + e^{-x} = (1 - x)e^{-x} = -(x - 1)e^{-x}$$
$$f''(x) = (1 - x)(-e^{-x}) - e^{-x} = (x - 2)e^{-x}$$
$$f'''(x) = (x - 2)(-e^{-x}) + e^{-x} = (3 - x)e^{-x} = -(x - 3)e^{-x}$$

From this we conclude that the nth derivative can be written as $f^{(n)}(x) = (-1)^n(x - n)e^{-x}$.

37. **(a)** Find the acceleration at time $t = 5$ min of a helicopter whose height is $s(t) = 300t - 4t^3$ m.
(b) Plot the acceleration $h''(t)$ for $0 \le t \le 6$. How does this graph show that the helicopter is slowing down during this time interval?

SOLUTION

(a) Let $s(t) = 300t - 4t^3$, with t in minutes and s in meters. The velocity is $v(t) = s'(t) = 300 - 12t^2$ and acceleration is $a(t) = s''(t) = -24t$. Thus $a(5) = -120$ m/min^2.
(b) The acceleration of the helicopter for $0 \le t \le 6$ is shown in the figure below. As the acceleration of the helicopter is negative, the velocity of the helicopter must be decreasing. Because the velocity is positive for $0 \le t \le 6$, the helicopter is slowing down.



39. Figure 5 shows f, f', and f''. Determine which is which.

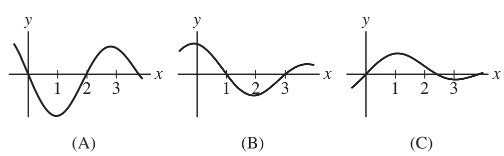


FIGURE 5

SOLUTION (a) f'' (b) f' (c) f.

The tangent line to (c) is horizontal at $x = 1$ and $x = 3$, where (b) has roots. The tangent line to (b) is horizontal at $x = 2$ and $x = 0$, where (a) has roots.

41. Figure 7 shows the graph of the position s of an object as a function of time t. Determine the intervals on which the acceleration is positive.

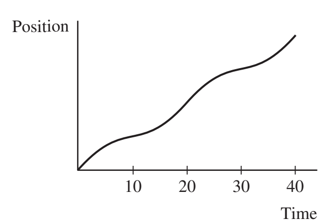


FIGURE 7

SOLUTION Roughly from time 10 to time 20 and from time 30 to time 40. The acceleration is positive over the same intervals over which the graph is bending upward.

43. Find all values of n such that $y = x^n$ satisfies

$$x^2y'' - 2xy' = 4y$$

SOLUTION We have $y' = nx^{n-1}$, $y'' = n(n - 1)x^{n-2}$, so that

$$x^2y'' - 2xy' = x^2(n(n - 1)x^{n-2}) - 2xnx^{n-1} = (n^2 - 3n)x^n = (n^2 - 3n)y$$

Thus the equation is satisfied if and only if $n^2 - 3n = 4$, so that $n^2 - 3n - 4 = 0$. This happens for $n = -1, 4$.

45. According to one model that takes into account air resistance, the acceleration $a(t)$ (in m/s^2) of a skydiver of mass m in free fall satisfies

$$a(t) = -9.8 + \frac{k}{m}v(t)^2$$

where $v(t)$ is velocity (negative since the object is falling) and k is a constant. Suppose that $m = 75$ kg and $k = 14$ kg/m.

(a) What is the object's velocity when $a(t) = -4.9$?

(b) What is the object's velocity when $a(t) = 0$? This velocity is the object's terminal velocity.

SOLUTION Solving $a(t) = -9.8 + \frac{k}{m}v(t)^2$ for the velocity and taking into account that the velocity is negative since the object is falling, we find

$$v(t) = -\sqrt{\frac{m}{k}(a(t) + 9.8)} = -\sqrt{\frac{75}{14}(a(t) + 9.8)}.$$

(a) Substituting $a(t) = -4.9$ into the above formula for the velocity, we find

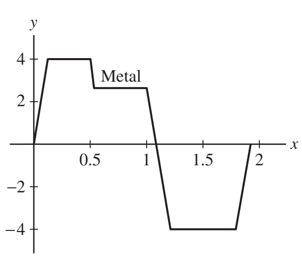
$$v(t) = -\sqrt{\frac{75}{14}(4.9)} = -\sqrt{26.25} = -5.12 \text{ m/s}.$$

(b) When $a(t) = 0$,

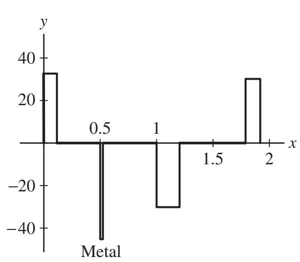
$$v(t) = -\sqrt{\frac{75}{14}(9.8)} = -\sqrt{52.5} = -7.25 \text{ m/s}.$$

47. A servomotor controls the vertical movement of a drill bit that will drill a pattern of holes in sheet metal. The maximum vertical speed of the drill bit is 4 in./s, and while drilling the hole, it must move no more than 2.6 in./s to avoid warping the metal. During a cycle, the bit begins and ends at rest, quickly approaches the sheet metal, and quickly returns to its initial position after the hole is drilled. Sketch possible graphs of the drill bit's vertical velocity and acceleration. Label the point where the bit enters the sheet metal.

SOLUTION There will be multiple cycles, each of which will be more or less identical. Let $v(t)$ be the *downward* vertical velocity of the drill bit, and let $a(t)$ be the vertical acceleration. From the narrative, we see that $v(t)$ can be no greater than 4 and no greater than 2.6 while drilling is taking place. During each cycle, $v(t) = 0$ initially, $v(t)$ goes to 4 quickly. When the bit hits the sheet metal, $v(t)$ goes down to 2.6 quickly, at which it stays until the sheet metal is drilled through. As the drill pulls out, it reaches maximum non-drilling upward speed ($v(t) = -4$) quickly, and maintains this speed until it returns to rest. A possible plot follows:



A graph of the acceleration is extracted from this graph:



In Exercises 48 and 49, refer to the following. In a 1997 study, Boardman and Lave related the traffic speed S on a two-lane road to traffic density Q (number of cars per mile of road) by the formula

$$S = 2882Q^{-1} - 0.052Q + 31.73$$

for $60 \le Q \le 400$ *(Figure 9).*

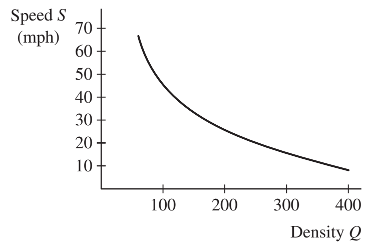


FIGURE 9 Speed as a function of traffic density.

49. (a) Explain intuitively why we should expect that $dS/dQ < 0$.

(b) Show that $d^2S/dQ^2 > 0$. Then use the fact that $dS/dQ < 0$ and $d^2S/dQ^2 > 0$ to justify the following statement: *A one-unit increase in traffic density slows down traffic more when Q is small than when Q is large*.

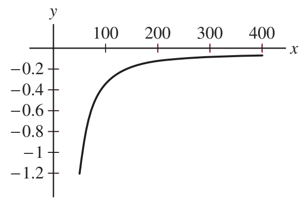
(c) GU Plot dS/dQ. Which property of this graph shows that $d^2S/dQ^2 > 0$?

SOLUTION

(a) Traffic speed must be reduced when the road gets more crowded so we expect dS/dQ to be negative. This is indeed the case since $dS/dQ = -0.052 - 2882/Q^2 < 0$.

(b) The decrease in speed due to a one-unit increase in density is approximately dS/dQ (a negative number). Since $d^2S/dQ^2 = 5764Q^{-3} > 0$ is positive, this tells us that dS/dQ gets larger as Q increases—and a negative number which gets larger is getting closer to zero. So the decrease in speed is smaller when Q is larger, that is, a one-unit increase in traffic density has a smaller effect when Q is large.

(c) dS/dQ is plotted below. The fact that this graph is increasing shows that $d^2S/dQ^2 > 0$.



51. CAS Let $f(x) = \dfrac{x+2}{x-1}$. Use a computer algebra system to compute the $f^{(k)}(x)$ for $1 \le k \le 4$. Can you find a general formula for $f^{(k)}(x)$?

SOLUTION Let $f(x) = \dfrac{x+2}{x-1}$. Using a computer algebra system,

$$f'(x) = -\frac{3}{(x-1)^2} = (-1)^1 \frac{3 \cdot 1}{(x-1)^{1+1}};$$

$$f''(x) = \frac{6}{(x-1)^3} = (-1)^2 \frac{3 \cdot 2 \cdot 1}{(x-1)^{2+1}};$$

$$f'''(x) = -\frac{18}{(x-1)^4} = (-1)^3 \frac{3 \cdot 3!}{(x-1)^{3+1}}; \text{ and}$$

$$f^{(4)}(x) = \frac{72}{(x-1)^5} = (-1)^4 \frac{3 \cdot 4!}{(x-1)^{4+1}}.$$

From the pattern observed above, we conjecture

$$f^{(k)}(x) = (-1)^k \frac{3 \cdot k!}{(x-1)^{k+1}}.$$

s

Further Insights and Challenges

53. What is $p^{(99)}(x)$ for $p(x)$ as in Exercise 52?

SOLUTION First note that for any integer $n \le 98$,

$$\frac{d^{99}}{dx^{99}} x^n = 0.$$

Now, if we expand $p(x)$, we find

$$p(x) = x^{99} + \text{ terms of degree at most 98;}$$

therefore,

$$\frac{d^{99}}{dx^{99}} p(x) = \frac{d^{99}}{dx^{99}}(x^{99} + \text{ terms of degree at most 98}) = \frac{d^{99}}{dx^{99}} x^{99}$$

Using logic similar to that used to compute the derivative in Example (3), we compute:

$$\frac{d^{99}}{dx^{99}}(x^{99}) = 99 \times 98 \times \ldots 1,$$

so that $\frac{d^{99}}{dx^{99}} p(x) = 99!$.

55. Use the Product Rule to find a formula for $(fg)'''$ and compare your result with the expansion of $(a+b)^3$. Then try to guess the general formula for $(fg)^{(n)}$.

SOLUTION Continuing from Exercise 54, we have

$$h''' = f''g' + gf''' + 2(f'g'' + g'f'') + fg''' + g''f' = f'''g + 3f''g' + 3f'g'' + fg'''$$

The binomial theorem gives

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3$$

and more generally

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where the binomial coefficients are given by

$$\binom{n}{k} = \frac{k(k-1)\cdots(k-n+1)}{n!}.$$

Accordingly, the general formula for $(fg)^{(n)}$ is given by

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)},$$

where $p^{(k)}$ is the kth derivative of p (or p itself when $k = 0$).

3.6 Trigonometric Functions

Preliminary Questions

1. Determine the sign (+ or −) that yields the correct formula for the following:

(a) $\frac{d}{dx}(\sin x + \cos x) = \pm \sin x \pm \cos x$

(b) $\frac{d}{dx} \sec x = \pm \sec x \tan x$

(c) $\frac{d}{dx} \cot x = \pm \csc^2 x$

SOLUTION The correct formulas are

(a) $\frac{d}{dx}(\sin x + \cos x) = -\sin x + \cos x$

(b) $\frac{d}{dx} \sec x = \sec x \tan x$

(c) $\frac{d}{dx} \cot x = -\csc^2 x$

2. Which of the following functions can be differentiated using the rules we have covered so far?

(a) $y = 3\cos x \cot x$ **(b)** $y = \cos(x^2)$ **(c)** $y = e^x \sin x$

SOLUTION

(a) $3\cos x \cot x$ is a product of functions whose derivatives are known. This function can therefore be differentiated using the Product Rule.

(b) $\cos(x^2)$ is a composition of the functions $\cos x$ and x^2. We have not yet discussed how to differentiate composite functions.

(c) $x^2 \cos x$ is a product of functions whose derivatives are known. This function can therefore be differentiated using the Product Rule.

3. Compute $\frac{d}{dx}(\sin^2 x + \cos^2 x)$ without using the derivative formulas for $\sin x$ and $\cos x$.

SOLUTION Recall that $\sin^2 x + \cos^2 x = 1$ for all x. Thus,

$$\frac{d}{dx}(\sin^2 x + \cos^2 x) = \frac{d}{dx}1 = 0.$$

4. How is the addition formula used in deriving the formula $(\sin x)' = \cos x$?

SOLUTION The difference quotient for the function $\sin x$ involves the expression $\sin(x + h)$. The addition formula for the sine function is used to expand this expression as $\sin(x + h) = \sin x \cos h + \sin h \cos x$.

Exercises

In Exercises 1–4, find an equation of the tangent line at the point indicated.

1. $y = \sin x, \quad x = \frac{\pi}{4}$

SOLUTION Let $f(x) = \sin x$. Then $f'(x) = \cos x$ and the equation of the tangent line is

$$y = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right).$$

3. $y = \tan x, \quad x = \frac{\pi}{4}$

SOLUTION Let $f(x) = \tan x$. Then $f'(x) = \sec^2 x$ and the equation of the tangent line is

$$y = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = 2\left(x - \frac{\pi}{4}\right) + 1 = 2x + 1 - \frac{\pi}{2}.$$

In Exercises 5–24, compute the derivative.

5. $f(x) = \sin x \cos x$

SOLUTION Let $f(x) = \sin x \cos x$. Then

$$f'(x) = \sin x(-\sin x) + \cos x(\cos x) = -\sin^2 x + \cos^2 x.$$

7. $f(x) = \sin^2 x$

SOLUTION Let $f(x) = \sin^2 x = \sin x \sin x$. Then

$$f'(x) = \sin x(\cos x) + \sin x(\cos x) = 2\sin x \cos x.$$

9. $H(t) = \sin t \sec^2 t$

SOLUTION Let $H(t) = \sin t \sec^2 t$. Then

$$\begin{aligned} H'(t) &= \sin t \frac{d}{dt}(\sec t \cdot \sec t) + \sec^2 t(\cos t) \\ &= \sin t(\sec t \sec t \tan t + \sec t \sec t \tan t) + \sec t \\ &= 2\sin t \sec^2 t \tan t + \sec t. \end{aligned}$$

11. $f(\theta) = \tan\theta \sec\theta$

SOLUTION Let $f(\theta) = \tan\theta\sec\theta$. Then

$$f'(\theta) = \tan\theta\sec\theta\tan\theta + \sec\theta\sec^2\theta = \sec\theta\tan^2\theta + \sec^3\theta = \left(\tan^2\theta + \sec^2\theta\right)\sec\theta.$$

13. $f(x) = (2x^4 - 4x^{-1})\sec x$

SOLUTION Let $f(x) = (2x^4 - 4x^{-1})\sec x$. Then

$$f'(x) = (2x^4 - 4x^{-1})\sec x\tan x + \sec x(8x^3 + 4x^{-2}).$$

15. $y = \dfrac{\sec\theta}{\theta}$

SOLUTION Let $y = \dfrac{\sec\theta}{\theta}$. Then

$$y' = \frac{\theta\sec\theta\tan\theta - \sec\theta}{\theta^2}.$$

17. $R(y) = \dfrac{3\cos y - 4}{\sin y}$

SOLUTION Let $R(y) = \dfrac{3\cos y - 4}{\sin y}$. Then

$$R'(y) = \frac{\sin y(-3\sin y) - (3\cos y - 4)(\cos y)}{\sin^2 y} = \frac{4\cos y - 3(\sin^2 y + \cos^2 y)}{\sin^2 y} = \frac{4\cos y - 3}{\sin^2 y}.$$

19. $f(x) = \dfrac{1 + \tan x}{1 - \tan x}$

SOLUTION Let $f(x) = \dfrac{1 + \tan x}{1 - \tan x}$. Then

$$f'(x) = \frac{(1 - \tan x)\sec^2 x - (1 + \tan x)\left(-\sec^2 x\right)}{(1 - \tan x)^2} = \frac{2\sec^2 x}{(1 - \tan x)^2}.$$

21. $f(x) = e^x \sin x$

SOLUTION Let $f(x) = e^x\sin x$. Then $f'(x) = e^x\cos x + \sin x e^x = e^x(\cos x + \sin x)$.

23. $f(\theta) = e^\theta(5\sin\theta - 4\tan\theta)$

SOLUTION Let $f(\theta) = e^\theta(5\sin\theta - 4\tan\theta)$. Then

$$\begin{aligned} f'(\theta) &= e^\theta(5\cos\theta - 4\sec^2\theta) + e^\theta(5\sin\theta - 4\tan\theta) \\ &= e^\theta(5\sin\theta + 5\cos\theta - 4\tan\theta - 4\sec^2\theta). \end{aligned}$$

In Exercises 25–34, find an equation of the tangent line at the point specified.

25. $y = x^3 + \cos x, \quad x = 0$

SOLUTION Let $f(x) = x^3 + \cos x$. Then $f'(x) = 3x^2 - \sin x$ and $f'(0) = 0$. The tangent line at $x = 0$ is

$$y = f'(0)(x - 0) + f(0) = 0(x) + 1 = 1.$$

27. $y = \sin x + 3\cos x, \quad x = 0$

SOLUTION Let $f(x) = \sin x + 3\cos x$. Then $f'(x) = \cos x - 3\sin x$ and $f'(0) = 1$. The tangent line at $x = 0$ is

$$y = f'(0)(x - 0) + f(0) = x + 3.$$

29. $y = 2(\sin\theta + \cos\theta), \quad \theta = \frac{\pi}{3}$

SOLUTION Let $f(\theta) = 2(\sin\theta + \cos\theta)$. Then $f'(\theta) = 2(\cos\theta - \sin\theta)$ and $f'(\frac{\pi}{3}) = 1 - \sqrt{3}$. The tangent line at $x = \frac{\pi}{3}$ is

$$y = f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right) = (1 - \sqrt{3})\left(x - \frac{\pi}{3}\right) + 1 + \sqrt{3}.$$

31. $y = e^x \cos x, \quad x = 0$

SOLUTION Let $f(x) = e^x \cos x$. Then

$$f'(x) = e^x(-\sin x) + e^x \cos x = e^x(\cos x - \sin x),$$

and $f'(0) = e^0(\cos 0 - \sin 0) = 1$. Thus, the equation of the tangent line is

$$y = f'(0)(x - 0) + f(0) = x + 1.$$

33. $y = e^t(1 - \cos t), \quad t = \frac{\pi}{2}$

SOLUTION Let $f(t) = e^t(1 - \cos t)$. Then

$$f'(t) = e^t \sin t + e^t(1 - \cos t) = e^t(1 + \sin t - \cos t),$$

and $f'(\frac{\pi}{2}) = 2e^{\pi/2}$. The tangent line at $x = \frac{\pi}{2}$ is

$$y = f'\left(\frac{\pi}{2}\right)\left(t - \frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 2e^{\pi/2}\left(t - \frac{\pi}{2}\right) + e^{\pi/2}.$$

In Exercises 35–37, use Theorem 1 to verify the formula.

35. $\frac{d}{dx} \cot x = -\csc^2 x$

SOLUTION $\cot x = \frac{\cos x}{\sin x}$. Using the quotient rule and the derivative formulas, we compute:

$$\frac{d}{dx} \cot x = \frac{d}{dx} \frac{\cos x}{\sin x} = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x.$$

37. $\frac{d}{dx} \csc x = -\csc x \cot x$

SOLUTION Since $\csc x = \frac{1}{\sin x}$, we can apply the quotient rule and the two known derivatives to get:

$$\frac{d}{dx} \csc x = \frac{d}{dx} \frac{1}{\sin x} = \frac{\sin x(0) - 1(\cos x)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x} \frac{1}{\sin x} = -\cot x \csc x.$$

In Exercises 39–42, calculate the higher derivative.

39. $f''(\theta), \quad f(\theta) = \theta \sin \theta$

SOLUTION Let $f(\theta) = \theta \sin \theta$. Then

$$f'(\theta) = \theta \cos \theta + \sin \theta$$
$$f''(\theta) = \theta(-\sin \theta) + \cos \theta + \cos \theta = -\theta \sin \theta + 2 \cos \theta.$$

41. $y'', \quad y''', \quad y = \tan x$

SOLUTION Let $y = \tan x$. Then $y' = \sec^2 x$ and by the Chain Rule,

$$y'' = = \frac{d}{dx} \sec^2 x = 2(\sec x)(\sec x \tan x) = 2 \sec^2 x \tan x$$
$$y''' = 2 \sec^2 x(\sec^2 x) + (2 \sec^2 x \tan x) \tan x = 2 \sec^4 + 4 \sec^4 x \tan^2 x$$

43. Calculate the first five derivatives of $f(x) = \cos x$. Then determine $f^{(8)}$ and $f^{(37)}$.

SOLUTION Let $f(x) = \cos x$.

- Then $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$, and $f^{(5)}(x) = -\sin x$.
- Accordingly, the successive derivatives of f cycle among

$$\{-\sin x, -\cos x, \sin x, \cos x\}$$

  in that order. Since 8 is a multiple of 4, we have $f^{(8)}(x) = \cos x$.
- Since 36 is a multiple of 4, we have $f^{(36)}(x) = \cos x$. Therefore, $f^{(37)}(x) = -\sin x$.

45. Find the values of x between 0 and 2π where the tangent line to the graph of $y = \sin x \cos x$ is horizontal.

SOLUTION Let $y = \sin x \cos x$. Then

$$y' = (\sin x)(-\sin x) + (\cos x)(\cos x) = \cos^2 x - \sin^2 x.$$

When $y' = 0$, we have $\sin x = \pm \cos x$. In the interval $[0, 2\pi]$, this occurs when $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

47. GU Let $g(t) = t - \sin t$.

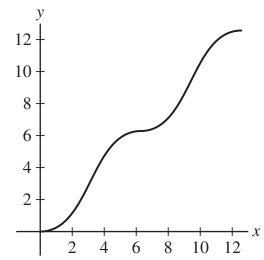
(a) Plot the graph of g with a graphing utility for $0 \le t \le 4\pi$.

(b) Show that the slope of the tangent line is nonnegative. Verify this on your graph.

(c) For which values of t in the given range is the tangent line horizontal?

SOLUTION Let $g(t) = t - \sin t$.

(a) Here is a graph of g over the interval $[0, 4\pi]$.

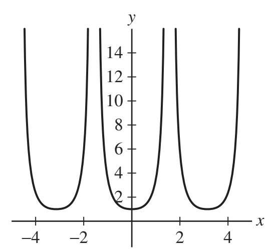


(b) Since $g'(t) = 1 - \cos t \ge 0$ for all t, the slope of the tangent line to g is always nonnegative.

(c) In the interval $[0, 4\pi]$, the tangent line is horizontal when $t = 0, 2\pi, 4\pi$.

49. Show that no tangent line to the graph of $f(x) = \tan x$ has zero slope. What is the least slope of a tangent line? Justify by sketching the graph of $(\tan x)'$.

SOLUTION Let $f(x) = \tan x$. Then $f'(x) = \sec^2 x = \frac{1}{\cos^2 x}$. Note that $f'(x) = \frac{1}{\cos^2 x}$ has numerator 1; the equation $f'(x) = 0$ therefore has no solution. Because the maximum value of $\cos^2 x$ is 1, the minimum value of $f'(x) = \frac{1}{\cos^2 x}$ is 1. Hence, the least slope for a tangent line to $\tan x$ is 1. Here is a graph of f'.



51. The horizontal range R of a projectile launched from ground level at an angle θ and initial velocity v_0 m/s is $R = (v_0^2/9.8)\sin\theta\cos\theta$. Calculate $dR/d\theta$. If $\theta = 7\pi/24$, will the range increase or decrease if the angle is increased slightly? Base your answer on the sign of the derivative.

SOLUTION Let $R(\theta) = (v_0^2/9.8)\sin\theta\cos\theta$.

$$\frac{dR}{d\theta} = R'(\theta) = (v_0^2/9.8)(-\sin^2\theta + \cos^2\theta).$$

If $\theta = 7\pi/24$, $\frac{\pi}{4} < \theta < \frac{\pi}{2}$, so $|\sin\theta| > |\cos\theta|$, and $dR/d\theta < 0$ (numerically, $dR/d\theta = -0.0264101 v_0^2$). At this point, increasing the angle will *decrease* the range.

Further Insights and Challenges

53. Use the limit definition of the derivative and the addition law for the cosine function to prove that $(\cos x)' = -\sin x$.

SOLUTION Let $f(x) = \cos x$. Then

$$f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h\to 0} \left((-\sin x)\frac{\sin h}{h} + (\cos x)\frac{\cos h - 1}{h} \right) = (-\sin x)\cdot 1 + (\cos x)\cdot 0 = -\sin x.$$

55. Verify the following identity and use it to give another proof of the formula $(\sin x)' = \cos x$.

$$\sin(x+h) - \sin x = 2\cos\left(x + \tfrac{1}{2}h\right)\sin\left(\tfrac{1}{2}h\right)$$

Hint: Use the addition formula to prove that $\sin(a+b) - \sin(a-b) = 2\cos a \sin b$.

SOLUTION Recall that

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

and

$$\sin(a-b) = \sin a \cos b - \cos a \sin b.$$

Subtracting the second identity from the first yields

$$\sin(a+b) - \sin(a-b) = 2\cos a \sin b.$$

If we now set $a = x + \frac{h}{2}$ and $b = \frac{h}{2}$, then the previous equation becomes

$$\sin(x+h) - \sin x = 2\cos\left(x + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right).$$

Finally, we use the limit definition of the derivative of $\sin x$ to obtain

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\cdot \lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} = \cos x \cdot 1 = \cos x.$$

In other words, $\dfrac{d}{dx}(\sin x) = \cos x$.

57. Let $f(x) = x\sin x$ and $g(x) = x\cos x$.

(a) Show that $f'(x) = g(x) + \sin x$ and $g'(x) = -f(x) + \cos x$.

(b) Verify that $f''(x) = -f(x) + 2\cos x$ and $g''(x) = -g(x) - 2\sin x$.

(c) By further experimentation, try to find formulas for all higher derivatives of f and g. *Hint:* The kth derivative depends on whether $k = 4n$, $4n+1$, $4n+2$, or $4n+3$.

SOLUTION Let $f(x) = x\sin x$ and $g(x) = x\cos x$.

(a) We examine first derivatives: $f'(x) = x\cos x + (\sin x)\cdot 1 = g(x) + \sin x$ and $g'(x) = (x)(-\sin x) + (\cos x)\cdot 1 = -f(x) + \cos x$; i.e., $f'(x) = g(x) + \sin x$ and $g'(x) = -f(x) + \cos x$.

(b) Now look at second derivatives: $f''(x) = g'(x) + \cos x = -f(x) + 2\cos x$ and $g''(x) = -f'(x) - \sin x = -g(x) - 2\sin x$; i.e., $f''(x) = -f(x) + 2\cos x$ and $g''(x) = -g(x) - 2\sin x$.

(c)
- The third derivatives are $f'''(x) = -f'(x) - 2\sin x = -g(x) - 3\sin x$ and $g'''(x) = -g'(x) - 2\cos x = f(x) - 3\cos x$; i.e., $f'''(x) = -g(x) - 3\sin x$ and $g'''(x) = f(x) - 3\cos x$.
- The fourth derivatives are $f^{(4)}(x) = -g'(x) - 3\cos x = f(x) - 4\cos x$ and $g^{(4)}(x) = f'(x) + 3\sin x = g(x) + 4\sin x$; i.e., $f^{(4)} = f(x) - 4\cos x$ and $g^{(4)}(x) = g(x) + 4\sin x$.
- We can now see the pattern for the derivatives, which are summarized in the following table. Here $n = 0, 1, 2, \ldots$

k	$4n$	$4n+1$	$4n+2$	$4n+3$
$f^{(k)}(x)$	$f(x) - k\cos x$	$g(x) + k\sin x$	$-f(x) + k\cos x$	$-g(x) - k\sin x$
$g^{(k)}(x)$	$g(x) + k\sin x$	$-f(x) + k\cos x$	$-g(x) - k\sin x$	$f(x) - k\cos x$

3.7 The Chain Rule

Preliminary Questions

1. Identify the outside and inside functions for each of these composite functions.

(a) $y = \sqrt{4x + 9x^2}$

(b) $y = \tan(x^2 + 1)$

(c) $y = \sec^5 x$

(d) $y = (1 + e^x)^4$

SOLUTION

(a) The outer function is $\sqrt{x}$, and the inner function is $4x + 9x^2$.
(b) The outer function is $\tan x$, and the inner function is $x^2 + 1$.
(c) The outer function is x^5, and the inner function is $\sec x$.
(d) The outer function is x^4, and the inner function is $1 + e^x$.

2. Which of the following can be differentiated easily *without* using the Chain Rule?

(a) $y = \tan(7x^2 + 2)$ **(b)** $y = \dfrac{x}{x+1}$
(c) $y = \sqrt{x} \cdot \sec x$ **(d)** $y = \sqrt{x \cos x}$
(e) $y = xe^x$ **(f)** $y = e^{\sin x}$

SOLUTION The function $\frac{x}{x+1}$ can be differentiated using the Quotient Rule, and the functions $\sqrt{x} \cdot \sec x$ and xe^x can be differentiated using the Product Rule. The functions $\tan(7x^2 + 2)$, $\sqrt{x \cos x}$ and $e^{\sin x}$ require the Chain Rule.

3. Which is the derivative of $f(5x)$?

(a) $5f'(x)$ **(b)** $5f'(5x)$ **(c)** $f'(5x)$

SOLUTION The correct answer is **(b)**: $5f'(5x)$.

4. Suppose that $f'(4) = g(4) = g'(4) = 1$. Do we have enough information to compute $F'(4)$, where $F(x) = f(g(x))$? If not, what is missing?

SOLUTION If $F(x) = f(g(x))$, then $F'(x) = f'(g(x))g'(x)$ and $F'(4) = f'(g(4))g'(4)$. Thus, we do not have enough information to compute $F'(4)$. We are missing the value of $f'(1)$.

Exercises

In Exercises 1–4, fill in a table of the following type:

$f(g(x))$	$f'(u)$	$f'(g(x))$	$g'(x)$	$(f \circ g)'$

1. $f(u) = u^{3/2}$, $g(x) = x^4 + 1$

SOLUTION

$f(g(x))$	$f'(u)$	$f'(g(x))$	$g'(x)$	$(f \circ g)'$
$(x^4+1)^{3/2}$	$\frac{3}{2}u^{1/2}$	$\frac{3}{2}(x^4+1)^{1/2}$	$4x^3$	$6x^3(x^4+1)^{1/2}$

3. $f(u) = \tan u$, $g(x) = x^4$

SOLUTION

$f(g(x))$	$f'(u)$	$f'(g(x))$	$g'(x)$	$(f \circ g)'$
$\tan(x^4)$	$\sec^2 u$	$\sec^2(x^4)$	$4x^3$	$4x^3 \sec^2(x^4)$

In Exercises 5 and 6, write the function as a composite $f(g(x))$ and compute the derivative using the Chain Rule.

5. $y = (x + \sin x)^4$

SOLUTION Let $f(x) = x^4$, $g(x) = x + \sin x$, and $y = f(g(x)) = (x + \sin x)^4$. Then

$$\frac{dy}{dx} = f'(g(x))g'(x) = 4(x + \sin x)^3(1 + \cos x).$$

7. Calculate $\dfrac{d}{dx}\cos u$ for the following choices of $u(x)$:

(a) $u = 9 - x^2$ **(b)** $u = x^{-1}$ **(c)** $u = \tan x$

SOLUTION

(a) $\cos(u(x)) = \cos(9 - x^2)$.

$$\frac{d}{dx}\cos(u(x)) = -\sin(u(x))u'(x) = -\sin(9 - x^2)(-2x) = 2x\sin(9 - x^2).$$

(b) $\cos(u(x)) = \cos(x^{-1})$.

$$\frac{d}{dx}\cos(u(x)) = -\sin(u(x))u'(x) = -\sin(x^{-1})\left(-\frac{1}{x^2}\right) = \frac{\sin(x^{-1})}{x^2}.$$

(c) $\cos(u(x)) = \cos(\tan x)$.

$$\frac{d}{dx}\cos(u(x)) = -\sin(u(x))u'(x) = -\sin(\tan x)(\sec^2 x) = -\sec^2 x \sin(\tan x).$$

9. Compute $\frac{df}{dx}$ if $\frac{df}{du} = 2$ and $\frac{du}{dx} = 6$.

SOLUTION Assuming f is a function of u, which is in turn a function of x,

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = 2(6) = 12.$$

In Exercises 11–22, use the General Power Rule or the Shifting and Scaling Rule to compute the derivative.

11. $y = (x^4 + 5)^3$

SOLUTION Using the General Power Rule,

$$\frac{d}{dx}(x^4 + 5)^3 = 3(x^4 + 5)^2 \frac{d}{dx}(x^4 + 5) = 3(x^4 + 5)^2(4x^3) = 12x^3(x^4 + 5)^2.$$

13. $y = \sqrt{7x - 3}$

SOLUTION Using the Shifting and Scaling Rule

$$\frac{d}{dx}\sqrt{7x - 3} = \frac{d}{dx}(7x - 3)^{1/2} = \frac{1}{2}(7x - 3)^{-1/2}(7) = \frac{7}{2\sqrt{7x - 3}}.$$

15. $y = (x^2 + 9x)^{-2}$

SOLUTION Using the General Power Rule,

$$\frac{d}{dx}(x^2 + 9x)^{-2} = -2(x^2 + 9x)^{-3} \frac{d}{dx}(x^2 + 9x) = -2(x^2 + 9x)^{-3}(2x + 9).$$

17. $y = \cos^4 \theta$

SOLUTION Using the General Power Rule,

$$\frac{d}{d\theta}\cos^4 \theta = 4\cos^3 \theta \frac{d}{d\theta}\cos\theta = -4\cos^3 \theta \sin\theta.$$

19. $y = (2\cos\theta + 5\sin\theta)^9$

SOLUTION Using the General Power Rule,

$$\frac{d}{d\theta}(2\cos\theta + 5\sin\theta)^9 = 9(2\cos\theta + 5\sin\theta)^8 \frac{d}{d\theta}(2\cos\theta + 5\sin\theta) = 9(2\cos\theta + 5\sin\theta)^8(5\cos\theta - 2\sin\theta).$$

21. $y = e^{x-12}$

SOLUTION Using the Shifting and Scaling Rule,

$$\frac{d}{dx}e^{x-12} = (1)e^{x-12} = e^{x-12}.$$

In Exercises 23–26, compute the derivative of $f \circ g$.

23. $f(u) = \sin u, \quad g(x) = 2x + 1$

SOLUTION Let $h(x) = f(g(x)) = \sin(2x + 1)$. Then, applying the shifting and scaling rule, $h'(x) = 2\cos(2x + 1)$. Alternately,

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) = \cos(2x + 1) \cdot 2 = 2\cos(2x + 1).$$

25. $f(u) = e^u, \quad g(x) = x + x^{-1}$

SOLUTION Let $h(x) = f(g(x)) = e^{x+x^{-1}}$. Then

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) = e^{x+x^{-1}} \left(1 - x^{-2}\right).$$

In Exercises 27 and 28, find the derivatives of $f(g(x))$ and $g(f(x))$.

27. $f(u) = \cos u, \quad u = g(x) = x^2 + 1$

SOLUTION

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) = -\sin(x^2+1)(2x) = -2x\sin(x^2+1).$$

$$\frac{d}{dx} g(f(x)) = g'(f(x))f'(x) = 2(\cos x)(-\sin x) = -2\sin x \cos x.$$

In Exercises 29–42, use the Chain Rule to find the derivative.

29. $y = \sin(x^2)$

SOLUTION Let $y = \sin\left(x^2\right)$. Then $y' = \cos\left(x^2\right) \cdot 2x = 2x\cos\left(x^2\right)$.

31. $y = \sqrt{t^2+9}$

SOLUTION Let $y = \sqrt{t^2+9} = (t^2+9)^{1/2}$. Then

$$y' = \frac{1}{2}(t^2+9)^{-1/2}(2t) = \frac{t}{\sqrt{t^2+9}}.$$

33. $y = (x^4 - x^3 - 1)^{2/3}$

SOLUTION Let $y = \left(x^4 - x^3 - 1\right)^{2/3}$. Then

$$y' = \frac{2}{3}\left(x^4 - x^3 - 1\right)^{-1/3}\left(4x^3 - 3x^2\right).$$

35. $y = \left(\frac{x+1}{x-1}\right)^4$

SOLUTION Let $y = \left(\frac{x+1}{x-1}\right)^4$. Then

$$y' = 4\left(\frac{x+1}{x-1}\right)^3 \cdot \frac{(x-1)\cdot 1 - (x+1)\cdot 1}{(x-1)^2} = -\frac{8\,(x+1)^3}{(x-1)^5} = \frac{8(1+x)^3}{(1-x)^5}.$$

37. $y = \sec\frac{1}{x}$

SOLUTION Let $f(x) = \sec\left(x^{-1}\right)$. Then

$$f'(x) = \sec\left(x^{-1}\right)\tan\left(x^{-1}\right)\cdot\left(-x^{-2}\right) = -\frac{\sec(1/x)\tan(1/x)}{x^2}.$$

39. $y = \tan(\theta + \cos\theta)$

SOLUTION Let $y = \tan(\theta + \cos\theta)$. Then

$$y' = \sec^2(\theta + \cos\theta)\cdot(1 - \sin\theta) = (1 - \sin\theta)\sec^2(\theta + \cos\theta).$$

41. $y = e^{2-9t^2}$

SOLUTION Let $y = e^{2-9t^2}$. Then

$$y' = e^{2-9t^2}(-18t) = -18te^{2-9t^2}.$$

In Exercises 43–72, find the derivative using the appropriate rule or combination of rules.

43. $y = \tan(x^2 + 4x)$

SOLUTION Let $y = \tan(x^2 + 4x)$. By the chain rule,

$$y' = \sec^2(x^2 + 4x) \cdot (2x + 4) = (2x + 4)\sec^2(x^2 + 4x).$$

45. $y = x\cos(1 - 3x)$

SOLUTION Let $y = x\cos(1 - 3x)$. Applying the product rule and then the scaling and shifting rule,

$$y' = x(-\sin(1 - 3x)) \cdot (-3) + \cos(1 - 3x) \cdot 1 = 3x\sin(1 - 3x) + \cos(1 - 3x).$$

47. $y = (4t + 9)^{1/2}$

SOLUTION Let $y = (4t + 9)^{1/2}$. By the shifting and scaling rule,

$$\frac{dy}{dt} = 4\left(\frac{1}{2}\right)(4t + 9)^{-1/2} = 2(4t + 9)^{-1/2}.$$

49. $y = (x^3 + \cos x)^{-4}$

SOLUTION Let $y = (x^3 + \cos x)^{-4}$. By the general power rule,

$$y' = -4(x^3 + \cos x)^{-5}(3x^2 - \sin x) = 4(\sin x - 3x^2)(x^3 + \cos x)^{-5}.$$

51. $y = \sqrt{\sin x \cos x}$

SOLUTION We start by using a trig identity to rewrite

$$y = \sqrt{\sin x \cos x} = \sqrt{\frac{1}{2}\sin 2x} = \frac{1}{\sqrt{2}}(\sin 2x)^{1/2}.$$

Then, after two applications of the chain rule,

$$y' = \frac{1}{\sqrt{2}} \cdot \frac{1}{2}(\sin 2x)^{-1/2} \cdot \cos 2x \cdot 2 = \frac{\cos 2x}{\sqrt{2\sin 2x}}.$$

53. $y = (\cos 6x + \sin x^2)^{1/2}$

SOLUTION Let $y = (\cos 6x + \sin(x^2))^{1/2}$. Applying the general power rule followed by both the scaling and shifting rule and the chain rule,

$$y' = \frac{1}{2}\left(\cos 6x + \sin(x^2)\right)^{-1/2}\left(-\sin 6x \cdot 6 + \cos(x^2) \cdot 2x\right) = \frac{x\cos(x^2) - 3\sin 6x}{\sqrt{\cos 6x + \sin(x^2)}}.$$

55. $y = \tan^3 x + \tan(x^3)$

SOLUTION Let $y = \tan^3 x + \tan(x^3) = (\tan x)^3 + \tan(x^3)$. Applying the general power rule to the first term and the chain rule to the second term,

$$y' = 3(\tan x)^2\sec^2 x + \sec^2(x^3) \cdot 3x^2 = 3\left(x^2\sec^2(x^3) + \sec^2 x\tan^2 x\right).$$

57. $y = \sqrt{\frac{z + 1}{z - 1}}$

SOLUTION Let $y = \left(\frac{z + 1}{z - 1}\right)^{1/2}$. Applying the general power rule followed by the quotient rule,

$$\frac{dy}{dz} = \frac{1}{2}\left(\frac{z + 1}{z - 1}\right)^{-1/2} \cdot \frac{(z - 1) \cdot 1 - (z + 1) \cdot 1}{(z - 1)^2} = \frac{-1}{\sqrt{z + 1}\,(z - 1)^{3/2}}.$$

59. $y = \dfrac{\cos(1+x)}{1+\cos x}$

SOLUTION Let

$$y = \frac{\cos(1+x)}{1+\cos x}.$$

Then, applying the quotient rule and the shifting and scaling rule,

$$\frac{dy}{dx} = \frac{-(1+\cos x)\sin(1+x) + \cos(1+x)\sin x}{(1+\cos x)^2} = \frac{\cos(1+x)\sin x - \cos x \sin(1+x) - \sin(1+x)}{(1+\cos x)^2}$$
$$= \frac{\sin(-1) - \sin(1+x)}{(1+\cos x)^2}.$$

The last line follows from the identity

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

with $A = x$ and $B = 1 + x$.

61. $y = \cot^7(x^5)$

SOLUTION Let $y = \cot^7\left(x^5\right)$. Applying the general power rule followed by the chain rule,

$$\frac{dy}{dx} = 7\cot^6\left(x^5\right) \cdot \left(-\csc^2\left(x^5\right)\right) \cdot 5x^4 = -35x^4 \cot^6\left(x^5\right)\csc^2\left(x^5\right).$$

63. $y = \left(1 + \cot^5(x^4+1)\right)^9$

SOLUTION Let $y = \left(1 + \cot^5\left(x^4+1\right)\right)^9$. Applying the general power rule, the chain rule, and the general power rule in succession,

$$\frac{dy}{dx} = 9\left(1+\cot^5\left(x^4+1\right)\right)^8 \cdot 5\cot^4\left(x^4+1\right) \cdot \left(-\csc^2\left(x^4+1\right)\right) \cdot 4x^3$$
$$= -180x^3 \cot^4\left(x^4+1\right)\csc^2\left(x^4+1\right)\left(1+\cot^5\left(x^4+1\right)\right)^8.$$

65. $y = (2e^{3x} + 3e^{-2x})^4$

SOLUTION Let $y = (2e^{3x} + 3e^{-2x})^4$. Applying the general power rule followed by two applications of the chain rule, one for each exponential function, we find

$$\frac{dy}{dx} = 4(2e^{3x} + 3e^{-2x})^3(6e^{3x} - 6e^{-2x}) = 24(2e^{3x} + 3e^{-2x})^3(e^{3x} - e^{-2x}).$$

67. $y = e^{(x^2+2x+3)^2}$

SOLUTION Let $y = e^{(x^2+2x+3)^2}$. By the chain rule and the general power rule, we obtain

$$\frac{dy}{dx} = e^{(x^2+2x+3)^2} \cdot 2(x^2+2x+3)(2x+2) = 4(x+1)(x^2+2x+3)e^{(x^2+2x+3)^2}.$$

69. $y = \sqrt{1 + \sqrt{1 + \sqrt{x}}}$

SOLUTION Let $y = \left(1 + \left(1 + x^{1/2}\right)^{1/2}\right)^{1/2}$. Applying the general power rule twice,

$$\frac{dy}{dx} = \frac{1}{2}\left(1 + \left(1 + x^{1/2}\right)^{1/2}\right)^{-1/2} \cdot \frac{1}{2}\left(1 + x^{1/2}\right)^{-1/2} \cdot \frac{1}{2}x^{-1/2} = \frac{1}{8\sqrt{x}\sqrt{1+\sqrt{x}}\sqrt{1+\sqrt{1+\sqrt{x}}}}.$$

71. $y = (kx+b)^{-1/3}$; k and b any constants

SOLUTION Let $y = (kx+b)^{-1/3}$, where b and k are constants. By the scaling and shifting rule,

$$y' = -\frac{1}{3}(kx+b)^{-4/3} \cdot k = -\frac{k}{3}(kx+b)^{-4/3}.$$

In Exercises 73–76, compute the higher derivative.

73. $\dfrac{d^2}{dx^2}\sin(x^2)$

SOLUTION Let $f(x) = \sin\left(x^2\right)$. Then, by the chain rule, $f'(x) = 2x\cos\left(x^2\right)$ and, by the product rule and the chain rule,

$$f''(x) = 2x\left(-\sin\left(x^2\right)\cdot 2x\right) + 2\cos\left(x^2\right) = 2\cos\left(x^2\right) - 4x^2\sin\left(x^2\right).$$

75. $\dfrac{d^3}{dx^3}(9-x)^8$

SOLUTION Let $f(x) = (9-x)^8$. Then, by repeated use of the scaling and shifting rule,

$$f'(x) = 8(9-x)^7\cdot(-1) = -8(9-x)^7$$
$$f''(x) = -56(9-x)^6\cdot(-1) = 56(9-x)^6,$$
$$f'''(x) = 336(9-x)^5\cdot(-1) = -336(9-x)^5.$$

77. The average molecular velocity v of a gas in a certain container is given by $v = 29\sqrt{T}$ m/s, where T is the temperature in kelvins. The temperature is related to the pressure (in atmospheres) by $T = 200P$. Find $\left.\dfrac{dv}{dP}\right|_{P=1.5}$.

SOLUTION First note that when $P = 1.5$ atmospheres, $T = 200(1.5) = 300$K. Thus,

$$\left.\frac{dv}{dP}\right|_{P=1.5} = \left.\frac{dv}{dT}\right|_{T=300}\cdot\left.\frac{dT}{dP}\right|_{P=1.5} = \frac{29}{2\sqrt{300}}\cdot 200 = \frac{290\sqrt{3}}{3}\,\frac{\text{m}}{\text{s}\cdot\text{atmospheres}}.$$

Alternately, substituting $T = 200P$ into the equation for v gives $v = 290\sqrt{2P}$. Therefore,

$$\frac{dv}{dP} = \frac{290\sqrt{2}}{2\sqrt{P}} = \frac{290}{\sqrt{2P}},$$

so

$$\left.\frac{dv}{dP}\right|_{P=1.5} = \frac{290}{\sqrt{3}} = \frac{290\sqrt{3}}{3}\,\frac{\text{m}}{\text{s}\cdot\text{atmospheres}}.$$

79. An expanding sphere has radius $r = 0.4t$ cm at time t (in seconds). Let V be the sphere's volume. Find dV/dt when (a) $r = 3$ and (b) $t = 3$.

SOLUTION Let $r = 0.4t$, where t is in seconds (s) and r is in centimeters (cm). With $V = \frac{4}{3}\pi r^3$, we have

$$\frac{dV}{dr} = 4\pi r^2.$$

Thus

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2\cdot(0.4) = 1.6\pi r^2.$$

(a) When $r = 3$, $\dfrac{dV}{dt} = 1.6\pi(3)^2 \approx 45.24$ cm/s.

(b) When $t = 3$, we have $r = 1.2$. Hence $\dfrac{dV}{dt} = 1.6\pi(1.2)^2 \approx 7.24$ cm/s.

81. A 1999 study by Starkey and Scarnecchia developed the following model for the average weight (in kilograms) at age t (in years) of channel catfish in the Lower Yellowstone River (Figure 3):

$$W(t) = (3.46293 - 3.32173e^{-0.03456t})^{3.4026}$$

Find the rate at which average weight is changing at age $t = 10$.

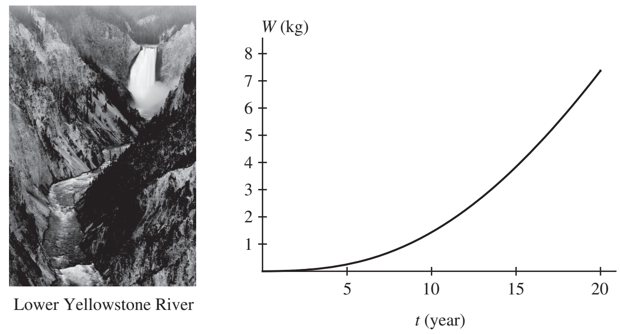


FIGURE 3 Average weight of channel catfish at age t

SOLUTION Let $W(t) = (3.46293 - 3.32173e^{-0.03456t})^{3.4026}$. Then

$$\begin{aligned} W'(t) &= 3.4026(3.46293 - 3.32173e^{-0.03456t})^{2.4026}(3.32173)(0.03456)e^{-0.03456t} \\ &= 0.3906(3.46293 - 3.32173e^{-0.03456t})^{2.4026}e^{-0.03456t}. \end{aligned}$$

At age $t = 10$,

$$W'(10) = 0.3906(1.1118)^{2.4026}(0.7078) \approx 0.3566 \text{ kg/yr}.$$

83. With notation as in Example 7, calculate

(a) $\left.\dfrac{d}{d\theta} \underline{\sin}\,\theta\right|_{\theta=60^\circ}$ **(b)** $\left.\dfrac{d}{d\theta}\left(\theta + \underline{\tan}\,\theta\right)\right|_{\theta=45^\circ}$

SOLUTION

(a) $\left.\dfrac{d}{d\theta} \underline{\sin}\,\theta\right|_{\theta=60^\circ} = \left.\dfrac{d}{d\theta} \sin\left(\dfrac{\pi}{180}\theta\right)\right|_{\theta=60^\circ} = \left(\dfrac{\pi}{180}\right)\cos\left(\dfrac{\pi}{180}(60)\right) = \dfrac{\pi}{180}\dfrac{1}{2} = \dfrac{\pi}{360}.$

(b) $\left.\dfrac{d}{d\theta}\left(\theta + \underline{\tan}\,\theta\right)\right|_{\theta=45^\circ} = \left.\dfrac{d}{d\theta}\left(\theta + \tan\left(\dfrac{\pi}{180}\theta\right)\right)\right|_{\theta=45^\circ} = 1 + \dfrac{\pi}{180}\sec^2\left(\dfrac{\pi}{4}\right) = 1 + \dfrac{\pi}{90}.$

85. Compute the derivative of $h(\sin x)$ at $x = \frac{\pi}{6}$, assuming that $h'(0.5) = 10$.

SOLUTION Let $u = \sin x$ and suppose that $h'(0.5) = 10$. Then

$$\frac{d}{dx}(h(u)) = \frac{dh}{du}\frac{du}{dx} = \frac{dh}{du}\cos x.$$

When $x = \frac{\pi}{6}$, we have $u = .5$. Accordingly, the derivative of $h(\sin x)$ at $x = \frac{\pi}{6}$ is $10\cos\left(\frac{\pi}{6}\right) = 5\sqrt{3}$.

In Exercises 87–90, use the table of values to calculate the derivative of the function at the given point.

x	1	4	6
$f(x)$	4	0	6
$f'(x)$	5	7	4
$g(x)$	4	1	6
$g'(x)$	5	$\frac{1}{2}$	3

87. $f(g(x)), \quad x = 6$

SOLUTION $\left.\dfrac{d}{dx} f(g(x))\right|_{x=6} = f'(g(6))g'(6) = f'(6)g'(6) = 4 \times 3 = 12.$

89. $g(\sqrt{x}), \quad x = 16$

SOLUTION $\left.\dfrac{d}{dx} g(\sqrt{x})\right|_{x=16} = g'(4)\left(\dfrac{1}{2}\right)(1/\sqrt{16}) = \left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)\left(\dfrac{1}{4}\right) = \dfrac{1}{16}.$

91. The price (in dollars) of a computer component is $P = 2C - 18C^{-1}$, where C is the manufacturer's cost to produce it. Assume that cost at time t (in years) is $C = 9 + 3t^{-1}$. Determine the rate of change of price with respect to time at $t = 3$.

SOLUTION $\dfrac{dC}{dt} = -3t^{-2}$. $C(3) = 10$ and $C'(3) = -\frac{1}{3}$, so we compute:

$$\left.\frac{dP}{dt}\right|_{t=3} = 2C'(3) + \frac{18}{(C(3))^2}C'(3) = -\frac{2}{3} + \frac{18}{100}\left(-\frac{1}{3}\right) = -0.727 \ \frac{\text{dollars}}{\text{year}}.$$

93. According to the U.S. standard atmospheric model, developed by the National Oceanic and Atmospheric Administration for use in aircraft and rocket design, atmospheric temperature T (in degrees Celsius), pressure P (kPa = 1,000 pascals), and altitude h (in meters) are related by these formulas (valid in the troposphere $h \le 11{,}000$):

$$T = 15.04 - 0.000649h, \qquad P = 101.29 + \left(\frac{T + 273.1}{288.08}\right)^{5.256}$$

Use the Chain Rule to calculate dP/dh. Then estimate the change in P (in pascals, Pa) per additional meter of altitude when $h = 3{,}000$.

SOLUTION

$$\frac{dP}{dT} = 5.256\left(\frac{T + 273.1}{288.08}\right)^{4.256}\left(\frac{1}{288.08}\right) = 6.21519 \times 10^{-13}\,(273.1 + T)^{4.256}$$

and $\frac{dT}{dh} = -0.000649°\text{C/m}$. $\frac{dP}{dh} = \frac{dP}{dT}\frac{dT}{dh}$, so

$$\frac{dP}{dh} = \left(6.21519 \times 10^{-13}\,(273.1 + T)^{4.256}\right)(-0.000649) = -4.03366 \times 10^{-16}\,(288.14 - 0.000649\,h)^{4.256}.$$

When $h = 3000$,

$$\frac{dP}{dh} = -4.03366 \times 10^{-16}(286.193)^{4.256} = -1.15 \times 10^{-5} \text{ kPa/m};$$

therefore, for each additional meter of altitude,

$$\Delta P \approx -1.15 \times 10^{-5} \text{ kPa} = -1.15 \times 10^{-2} \text{ Pa}.$$

95. In the setting of Exercise 94, calculate the yearly rate of change of T if $T = 283$ K and R increases at a rate of 0.5 $\text{Js}^{-1}\text{m}^{-2}$ per year.

SOLUTION By the Chain Rule,

$$\frac{dR}{dt} = \frac{dR}{dT} \cdot \frac{dT}{dt} = 4\sigma T^3 \frac{dT}{dt}.$$

Assuming $T = 283$ K and $\frac{dR}{dt} = 0.5 \text{ Js}^{-1}\text{m}^{-2}$ per year, it follows that author:

$$0.5 = 4\sigma(283)^3 \frac{dT}{dt} \Rightarrow \frac{dT}{dt} = \frac{0.5}{4\sigma(283)^3} \approx 0.0973 \text{ kelvins/yr}$$

97. Use the Chain Rule to express the second derivative of $f \circ g$ in terms of the first and second derivatives of f and g.

SOLUTION Let $h(x) = f(g(x))$. Then

$$h'(x) = f'(g(x))g'(x)$$

and

$$h''(x) = f'(g(x))g''(x) + g'(x)f''(g(x))g'(x) = f'(g(x))g''(x) + f''(g(x))\left(g'(x)\right)^2.$$

Further Insights and Challenges

99. Show that if f, g, and h are differentiable, then

$$[f(g(h(x)))]' = f'(g(h(x)))g'(h(x))h'(x)$$

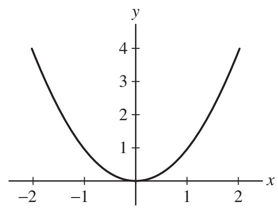
SOLUTION Let f, g, and h be differentiable. Let $u = h(x)$, $v = g(u)$, and $w = f(v)$. Then

$$\frac{dw}{dx} = \frac{df}{dv}\frac{dv}{dx} = \frac{df}{dv}\frac{dg}{du}\frac{du}{dx} = f'(g(h(x))g'(h(x))h'(x)$$

101. **(a)** Sketch a graph of any even function $f(x)$ and explain graphically why $f'(x)$ is odd.
(b) Suppose that $f'(x)$ is even. Is $f(x)$ necessarily odd? *Hint:* Check whether this is true for linear functions.

SOLUTION

(a) The graph of an even function is symmetric with respect to the y-axis. Accordingly, its image in the left half-plane is a mirror reflection of that in the right half-plane through the y-axis. If at $x = a \geq 0$, the slope of f exists and is equal to m, then by reflection its slope at $x = -a \leq 0$ is $-m$. That is, $f'(-a) = -f'(a)$. *Note:* This means that if $f'(0)$ exists, then it equals 0.



(b) Suppose that f' is even. Then f is not necessarily odd. Let $f(x) = 4x + 7$. Then $f'(x) = 4$, an even function. But f is not odd. For example, $f(2) = 15$, $f(-2) = -1$, but $f(-2) \neq -f(2)$.

103. Prove that for all whole numbers $n \geq 1$,

$$\frac{d^n}{dx^n}\sin x = \sin\left(x + \frac{n\pi}{2}\right)$$

Hint: Use the identity $\cos x = \sin\left(x + \frac{\pi}{2}\right)$.

SOLUTION We will proceed by induction on n. For $n = 1$, we find

$$\frac{d}{dx}\sin x = \cos x = \sin\left(x + \frac{\pi}{2}\right),$$

as required. Now, suppose that for some positive integer k,

$$\frac{d^k}{dx^k}\sin x = \sin\left(x + \frac{k\pi}{2}\right).$$

Then

$$\frac{d^{k+1}}{dx^{k+1}}\sin x = \frac{d}{dx}\sin\left(x + \frac{k\pi}{2}\right)$$
$$= \cos\left(x + \frac{k\pi}{2}\right) = \sin\left(x + \frac{(k+1)\pi}{2}\right).$$

105. Chain Rule This exercise proves the Chain Rule without the special assumption made in the text. For any number b, define a new function

$$F(u) = \frac{f(u) - f(b)}{u - b} \quad \text{for all } u \neq b$$

(a) Show that if we define $F(b) = f'(b)$, then $F(u)$ is continuous at $u = b$.
(b) Take $b = g(a)$. Show that if $x \neq a$, then for all u,

$$\frac{f(u) - f(g(a))}{x - a} = F(u)\frac{u - g(a)}{x - a} \qquad \boxed{2}$$

Note that both sides are zero if $u = g(a)$.
(c) Substitute $u = g(x)$ in Eq. (2) to obtain

$$\frac{f(g(x)) - f(g(a))}{x - a} = F(g(x))\,\frac{g(x) - g(a)}{x - a}$$

Derive the Chain Rule by computing the limit of both sides as $x \to a$.

SOLUTION For any differentiable function f and any number b, define

$$F(u) = \frac{f(u) - f(b)}{u - b}$$

for all $u \neq b$.

(a) Define $F(b) = f'(b)$. Then

$$\lim_{u \to b} F(u) = \lim_{u \to b} \frac{f(u) - f(b)}{u - b} = f'(b) = F(b),$$

i.e., $\lim_{u \to b} F(u) = F(b)$. Therefore, F is continuous at $u = b$.

(b) Let g be a differentiable function and take $b = g(a)$. Let x be a number distinct from a. If we substitute $u = g(a)$ into Eq. (2), both sides evaluate to 0, so equality is satisfied. On the other hand, if $u \neq g(a)$, then

$$\frac{f(u) - f(g(a))}{x - a} = \frac{f(u) - f(g(a))}{u - g(a)} \frac{u - g(a)}{x - a} = \frac{f(u) - f(b)}{u - b} \frac{u - g(a)}{x - a} = F(u) \frac{u - g(a)}{x - a}.$$

(c) Hence for all u, we have

$$\frac{f(u) - f(g(a))}{x - a} = F(u) \frac{u - g(a)}{x - a}.$$

(d) Substituting $u = g(x)$ in Eq. (2), we have

$$\frac{f(g(x)) - f(g(a))}{x - a} = F(g(x)) \frac{g(x) - g(a)}{x - a}.$$

Letting $x \to a$ gives

$$\begin{aligned}\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} &= \lim_{x \to a} \left(F(g(x)) \frac{g(x) - g(a)}{x - a} \right) = F(g(a))g'(a) = F(b)g'(a) = f'(b)g'(a) \\ &= f'(g(a))g'(a)\end{aligned}$$

Therefore $(f \circ g)'(a) = f'(g(a))g'(a)$, which is the Chain Rule.

3.8 Derivatives of Inverse Functions

Preliminary Questions

1. What is the slope of the line obtained by reflecting the line $y = \frac{x}{2}$ through the line $y = x$?

SOLUTION The line obtained by reflecting the line $y = x/2$ through the line $y = x$ has slope 2.

2. Suppose that $P = (2, 4)$ lies on the graph of $f(x)$ and that the slope of the tangent line through P is $m = 3$. Assuming that $f^{-1}(x)$ exists, what is the slope of the tangent line to the graph of $f^{-1}(x)$ at the point $Q = (4, 2)$?

SOLUTION The tangent line to the graph of $f^{-1}(x)$ at the point $Q = (4, 2)$ has slope $\frac{1}{3}$.

3. Which inverse trigonometric function $g(x)$ has the derivative $g'(x) = \dfrac{1}{x^2 + 1}$?

SOLUTION $g(x) = \tan^{-1} x$ has the derivative $g'(x) = \dfrac{1}{x^2 + 1}$.

4. What does the following identity tell us about the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$?

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

SOLUTION Angles whose sine and cosine are x are complementary.

Exercises

1. Find the inverse $g(x)$ of $f(x) = \sqrt{x^2 + 9}$ with domain $x \geq 0$ and calculate $g'(x)$ in two ways: using Theorem 1 and by direct calculation.

SOLUTION To find a formula for $g(x) = f^{-1}(x)$, solve $y = \sqrt{x^2 + 9}$ for x. This yields $x = \pm\sqrt{y^2 - 9}$. Because the domain of f was restricted to $x \geq 0$, we must choose the positive sign in front of the radical. Thus

$$g(x) = f^{-1}(x) = \sqrt{x^2 - 9}.$$

Because $x^2 + 9 \geq 9$ for all x, it follows that $f(x) \geq 3$ for all x. Thus, the domain of $g(x) = f^{-1}(x)$ is $x \geq 3$. The range of g is the restricted domain of f: $y \geq 0$.

By Theorem 1,

$$g'(x) = \frac{1}{f'(g(x))}.$$

With

$$f'(x) = \frac{x}{\sqrt{x^2+9}},$$

it follows that

$$f'(g(x)) = \frac{\sqrt{x^2-9}}{\sqrt{(\sqrt{x^2-9})^2+9}} = \frac{\sqrt{x^2-9}}{\sqrt{x^2}} = \frac{\sqrt{x^2-9}}{x}$$

since the domain of g is $x \geq 3$. Thus,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{x}{\sqrt{x^2-9}}.$$

This agrees with the answer we obtain by differentiating directly:

$$g'(x) = \frac{2x}{2\sqrt{x^2-9}} = \frac{x}{\sqrt{x^2-9}}.$$

In Exercises 3–8, use Theorem 1 to calculate $g'(x)$, where $g(x)$ is the inverse of $f(x)$.

3. $f(x) = 7x + 6$

SOLUTION Let $f(x) = 7x + 6$ then $f'(x) = 7$. Solving $y = 7x + 6$ for x and switching variables, we obtain the inverse $g(x) = (x - 6)/7$. Thus,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{7}.$$

5. $f(x) = x^{-5}$

SOLUTION Let $f(x) = x^{-5}$, then $f'(x) = -5x^{-6}$. Solving $y = x^{-5}$ for x and switching variables, we obtain the inverse $g(x) = x^{-1/5}$. Thus,

$$g'(x) = \frac{1}{-5(x^{-1/5})^{-6}} = -\frac{1}{5}x^{-6/5}.$$

7. $f(x) = \dfrac{x}{x+1}$

SOLUTION Let $f(x) = \frac{x}{x+1}$, then

$$f'(x) = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2}.$$

Solving $y = \frac{x}{x+1}$ for x and switching variables, we obtain the inverse $g(x) = \frac{x}{1-x}$. Thus

$$g'(x) = 1 \Big/ \frac{1}{(x/(1-x)+1)^2} = \frac{1}{(1-x)^2}.$$

9. Let $g(x)$ be the inverse of $f(x) = x^3 + 2x + 4$. Calculate $g(7)$ [without finding a formula for $g(x)$], and then calculate $g'(7)$.

SOLUTION Let $g(x)$ be the inverse of $f(x) = x^3 + 2x + 4$. Because

$$f(1) = 1^3 + 2(1) + 4 = 7,$$

it follows that $g(7) = 1$. Moreover, $f'(x) = 3x^2 + 2$, and

$$g'(7) = \frac{1}{f'(g(7))} = \frac{1}{f'(1)} = \frac{1}{5}.$$

In Exercises 11–16, calculate $g(b)$ and $g'(b)$, where g is the inverse of f (in the given domain, if indicated).

11. $f(x) = x + \cos x, \quad b = 1$

SOLUTION $f(0) = 1$, so $g(1) = 0$. $f'(x) = 1 - \sin x$ so $f'(g(1)) = f'(0) = 1 - \sin 0 = 1$. Thus, $g'(1) = 1/1 = 1$.

13. $f(x) = \sqrt{x^2 + 6x}$ for $x \geq 0$, $\quad b = 4$

SOLUTION To determine $g(4)$, we solve $f(x) = \sqrt{x^2 + 6x} = 4$ for x. This yields:

$$x^2 + 6x = 16$$
$$x^2 + 6x - 16 = 0$$
$$(x+8)(x-2) = 0$$

or $x = -8, 2$. Because the domain of f has been restricted to $x \geq 0$, we have $g(4) = 2$. With

$$f'(x) = \frac{x+3}{\sqrt{x^2 + 6x}},$$

it then follows that

$$g'(4) = \frac{1}{f'(g(4))} = \frac{1}{f'(2)} = \frac{4}{5}.$$

15. $f(x) = \dfrac{1}{x+1}$, $\quad b = \dfrac{1}{4}$

SOLUTION $f(3) = 1/4$, so $g(1/4) = 3$. $f'(x) = \frac{-1}{(x+1)^2}$ so $f'(g(1/4)) = f'(3) = \frac{-1}{(3+1)^2} = -1/16$. Thus, $g'(1/4) = -16$.

17. Let $f(x) = x^n$ and $g(x) = x^{1/n}$. Compute $g'(x)$ using Theorem 1 and check your answer using the Power Rule.

SOLUTION Note that $g(x) = f^{-1}(x)$. Therefore,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{n(g(x))^{n-1}} = \frac{1}{n(x^{1/n})^{n-1}} = \frac{1}{n(x^{1-1/n})} = \frac{x^{1/n-1}}{n} = \frac{1}{n}(x^{1/n-1})$$

which agrees with the Power Rule.

In Exercises 19–22, compute the derivative at the point indicated without using a calculator.

19. $y = \sin^{-1} x$, $\quad x = \frac{3}{5}$

SOLUTION Let $y = \sin^{-1} x$. Then $y' = \frac{1}{\sqrt{1-x^2}}$ and

$$y'\left(\frac{3}{5}\right) = \frac{1}{\sqrt{1 - 9/25}} = \frac{1}{4/5} = \frac{5}{4}.$$

21. $y = \sec^{-1} x$, $\quad x = 4$

SOLUTION Let $y = \sec^{-1} x$. Then $y' = \frac{1}{|x|\sqrt{x^2-1}}$ and

$$y'(4) = \frac{1}{4\sqrt{15}}.$$

In Exercises 23–36, find the derivative.

23. $y = \sin^{-1}(7x)$

SOLUTION $\dfrac{d}{dx}\sin^{-1}(7x) = \dfrac{1}{\sqrt{1-(7x)^2}} \cdot \dfrac{d}{dx}7x = \dfrac{7}{\sqrt{1-(7x)^2}}.$

25. $y = \cos^{-1}(x^2)$

SOLUTION $\dfrac{d}{dx}\cos^{-1}(x^2) = \dfrac{-1}{\sqrt{1-x^4}} \cdot \dfrac{d}{dx}x^2 = \dfrac{-2x}{\sqrt{1-x^4}}.$

27. $y = x\tan^{-1} x$

SOLUTION $\dfrac{d}{dx}x\tan^{-1} x = x\left(\dfrac{1}{1+x^2}\right) + \tan^{-1} x.$

29. $y = \arcsin(e^x)$

SOLUTION $\dfrac{d}{dx}\sin^{-1}(e^x) = \dfrac{1}{\sqrt{1-e^{2x}}} \cdot \dfrac{d}{dx}e^x = \dfrac{e^x}{\sqrt{1-e^{2x}}}.$

31. $y = \sqrt{1-t^2} + \sin^{-1} t$

SOLUTION $\dfrac{d}{dt}\left(\sqrt{1-t^2} + \sin^{-1} t\right) = \dfrac{1}{2}(1-t^2)^{-1/2}(-2t) + \dfrac{1}{\sqrt{1-t^2}} = \dfrac{-t}{\sqrt{1-t^2}} + \dfrac{1}{\sqrt{1-t^2}} = \dfrac{1-t}{\sqrt{1-t^2}}.$

33. $y = (\tan^{-1} x)^3$

SOLUTION $\frac{d}{dx}\left((\tan^{-1} x)^3\right) = 3(\tan^{-1} x)^2 \frac{d}{dx} \tan^{-1} x = \frac{3(\tan^{-1} x)^2}{x^2+1}$.

35. $y = \cos^{-1} t^{-1} - \sec^{-1} t$

SOLUTION
$$\frac{d}{dx}(\cos^{-1} t^{-1} - \sec^{-1} t) = \frac{-1}{\sqrt{1-(1/t)^2}}\left(\frac{-1}{t^2}\right) - \frac{1}{|t|\sqrt{t^2-1}}$$
$$= \frac{1}{\sqrt{t^4-t^2}} - \frac{1}{|t|\sqrt{t^2-1}} = \frac{1}{|t|\sqrt{t^2-1}} - \frac{1}{|t|\sqrt{t^2-1}} = 0.$$

Alternately, let $t = \sec\theta$. Then $t^{-1} = \cos\theta$ and $\cos^{-1} t^{-1} - \sec^{-1} t = \theta - \theta = 0$. Consequently,

$$\frac{d}{dx}(\cos^{-1} t^{-1} - \sec^{-1} t) = 0.$$

37. Use Figure 5 to prove that $(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$.

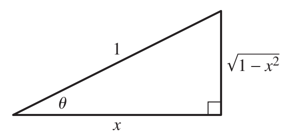


FIGURE 5 Right triangle with $\theta = \cos^{-1} x$.

SOLUTION Let $\theta = \cos^{-1} x$. Then $\cos\theta = x$ and

$$-\sin\theta \frac{d\theta}{dx} = 1 \quad \text{or} \quad \frac{d\theta}{dx} = -\frac{1}{\sin\theta} = -\frac{1}{\sin(\cos^{-1} x)}.$$

From Figure 5, we see that $\sin(\cos^{-1} x) = \sin\theta = \sqrt{1-x^2}$; hence,

$$\frac{d}{dx}\cos^{-1} x = \frac{1}{-\sin(\cos^{-1} x)} = -\frac{1}{\sqrt{1-x^2}}.$$

39. Let $\theta = \sec^{-1} x$. Show that $\tan\theta = \sqrt{x^2-1}$ if $x \ge 1$ and that $\tan\theta = -\sqrt{x^2-1}$ if $x \le -1$. *Hint:* $\tan\theta \ge 0$ on $\left(0, \frac{\pi}{2}\right)$ and $\tan\theta \le 0$ on $\left(\frac{\pi}{2}, \pi\right)$.

SOLUTION In general, $1 + \tan^2\theta = \sec^2\theta$, so $\tan\theta = \pm\sqrt{\sec^2\theta - 1}$. With $\theta = \sec^{-1} x$, it follows that $\sec\theta = x$, so $\tan\theta = \pm\sqrt{x^2-1}$. Finally, if $x \ge 1$ then $\theta = \sec^{-1} x \in [0, \pi/2)$ so $\tan\theta$ is positive; on the other hand, if $x \le 1$ then $\theta = \sec^{-1} x \in (-\pi/2, 0]$ so $\tan\theta$ is negative.

Further Insights and Challenges

41. Let $g(x)$ be the inverse of $f(x)$. Show that if $f'(x) = f(x)$, then $g'(x) = x^{-1}$. We will apply this in the next section to show that the inverse of $f(x) = e^x$ (the natural logarithm) has the derivative $f'(x) = x^{-1}$.

SOLUTION

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f(f^{-1}(x))} = \frac{1}{x}.$$

3.9 Derivatives of General Exponential and Logarithmic Functions

Preliminary Questions

1. What is the slope of the tangent line to $y = 4^x$ at $x = 0$?

SOLUTION The slope of the tangent line to $y = 4^x$ at $x = 0$ is

$$\left.\frac{d}{dx}4^x\right|_{x=0} = \left.4^x \ln 4\right|_{x=0} = \ln 4.$$

2. What is the rate of change of $y = \ln x$ at $x = 10$?

SOLUTION The rate of change of $y = \ln x$ at $x = 10$ is

$$\left.\frac{d}{dx}\ln x\right|_{x=10} = \left.\frac{1}{x}\right|_{x=10} = \frac{1}{10}.$$

3. What is $b > 0$ if the tangent line to $y = b^x$ at $x = 0$ has slope 2?

SOLUTION The tangent line to $y = b^x$ at $x = 0$ has slope

$$\frac{d}{dx}b^x\bigg|_{x=0} = b^x \ln b\bigg|_{x=0} = \ln b.$$

This slope will be equal to 2 when

$$\ln b = 2 \quad \text{or} \quad b = e^2.$$

4. What is b if $(\log_b x)' = \dfrac{1}{3x}$?

SOLUTION $(\log_b x)' = \left(\dfrac{\ln x}{\ln b}\right)' = \dfrac{1}{x \ln b}$. This derivative will equal $\frac{1}{3x}$ when

$$\ln b = 3 \quad \text{or} \quad b = e^3.$$

5. What are $y^{(100)}$ and $y^{(101)}$ for $y = \cosh x$?

SOLUTION Let $y = \cosh x$. Then $y' = \sinh x$, $y'' = \cosh x$, and this pattern repeats indefinitely. Thus, $y^{(100)} = \cosh x$ and $y^{(101)} = \sinh x$.

Exercises

In Exercises 1–20, find the derivative.

1. $y = x \ln x$

SOLUTION $\dfrac{d}{dx} x \ln x = \ln x + \dfrac{x}{x} = \ln x + 1.$

3. $y = (\ln x)^2$

SOLUTION $\dfrac{d}{dx}(\ln x)^2 = (2 \ln x)\,\dfrac{1}{x} = \dfrac{2}{x} \ln x.$

5. $y = \ln(9x^2 - 8)$

SOLUTION $\dfrac{d}{dx} \ln(9x^2 - 8) = \dfrac{18x}{9x^2 - 8}.$

7. $y = \ln(\sin t + 1)$

SOLUTION $\dfrac{d}{dt} \ln(\sin t + 1) = \dfrac{\cos t}{\sin t + 1}.$

9. $y = \dfrac{\ln x}{x}$

SOLUTION $\dfrac{d}{dx}\dfrac{\ln x}{x} = \dfrac{\frac{1}{x}(x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}.$

11. $y = \ln(\ln x)$

SOLUTION $\dfrac{d}{dx} \ln(\ln x) = \dfrac{1}{x \ln x}.$

13. $y = \big(\ln(\ln x)\big)^3$

SOLUTION $\dfrac{d}{dx}(\ln(\ln x))^3 = 3(\ln(\ln x))^2 \left(\dfrac{1}{\ln x}\right)\left(\dfrac{1}{x}\right) = \dfrac{3(\ln(\ln x))^2}{x \ln x}.$

15. $y = \ln\big((x + 1)(2x + 9)\big)$

SOLUTION

$$\frac{d}{dx} \ln\left((x+1)(2x+9)\right) = \frac{1}{(x+1)(2x+9)} \cdot \left((x+1)2 + (2x+9)\right) = \frac{4x+11}{(x+1)(2x+9)}.$$

Alternately, because $\ln((x+1)(2x+9)) = \ln(x+1) + \ln(2x+9)$,

$$\frac{d}{dx} \ln((x+1)(2x+9)) = \frac{1}{x+1} + \frac{2}{2x+9} = \frac{4x+11}{(x+1)(2x+9)}.$$

17. $y = 11^x$

SOLUTION $\dfrac{d}{dx} 11^x = \ln 11 \cdot 11^x.$

19. $y = \dfrac{2^x - 3^{-x}}{x}$

SOLUTION $\dfrac{d}{dx}\dfrac{2^x - 3^{-x}}{x} = \dfrac{x(2^x \ln 2 + 3^{-x}\ln 3) - (2^x - 3^{-x})}{x^2}$.

In Exercises 21–24, compute the derivative.

21. $f'(x)$, $f(x) = \log_2 x$

SOLUTION $f(x) = \log_2 x = \dfrac{\ln x}{\ln 2}$. Thus, $f'(x) = \dfrac{1}{x} \cdot \dfrac{1}{\ln 2}$.

23. $\dfrac{d}{dt}\log_3(\sin t)$

SOLUTION $\dfrac{d}{dt}\log_3(\sin t) = \dfrac{d}{dt}\left(\dfrac{\ln(\sin t)}{\ln 3}\right) = \dfrac{1}{\ln 3} \cdot \dfrac{1}{\sin t} \cdot \cos t = \dfrac{\cot t}{\ln 3}$.

In Exercises 25–36, find an equation of the tangent line at the point indicated.

25. $f(x) = 6^x$, $x = 2$

SOLUTION Let $f(x) = 6^x$. Then $f(2) = 36$, $f'(x) = 6^x \ln 6$ and $f'(2) = 36 \ln 6$. The equation of the tangent line is therefore $y = 36 \ln 6(x - 2) + 36$.

27. $s(t) = 3^{9t}$, $t = 2$

SOLUTION Let $s(t) = 3^{9t}$. Then $s(2) = 3^{18}$, $s'(t) = 3^{9t} 9 \ln 3$, and $s'(2) = 3^{18} \cdot 9 \ln 3 = 3^{20} \ln 3$. The equation of the tangent line is therefore $y = 3^{20} \ln 3(t - 2) + 3^{18}$.

29. $f(x) = 5^{x^2 - 2x}$, $x = 1$

SOLUTION Let $f(x) = 5^{x^2-2x}$. Then $f(1) = 5^{-1}$, $f'(x) = \ln 5 \cdot 5^{x^2-2x}(2x - 2)$, and $f'(1) = \ln 5(0) = 0$. Therefore, the equation of the tangent line is $y = 5^{-1}$.

31. $s(t) = \ln(8 - 4t)$, $t = 1$

SOLUTION Let $s(t) = \ln(8 - 4t)$. Then $s(1) = \ln(8 - 4) = \ln 4$. $s'(t) = \frac{-4}{8-4t}$, so $s'(1) = -4/4 = -1$. Therefore the equation of the tangent line is $y = -1(t - 1) + \ln 4$.

33. $R(z) = \log_5(2z^2 + 7)$, $z = 3$

SOLUTION Let $R(z) = \log_5(2z^2 + 7)$. Then $R(3) = \log_5(25) = 2$,

$$R'(z) = \frac{4z}{(2z^2 + 7)\ln 5}, \quad \text{and} \quad R'(3) = \frac{12}{25 \ln 5}.$$

The equation of the tangent line is therefore

$$y = \frac{12}{25 \ln 5}(z - 3) + 2.$$

35. $f(w) = \log_2 w$, $w = \frac{1}{8}$

SOLUTION Let $f(w) = \log_2 w$. Then

$$f\left(\frac{1}{8}\right) = \log_2 \frac{1}{8} = \log_2 2^{-3} = -3,$$

$f'(w) = \frac{1}{w \ln 2}$, and

$$f'\left(\frac{1}{8}\right) = \frac{8}{\ln 2}.$$

The equation of the tangent line is therefore

$$y = \frac{8}{\ln 2}\left(w - \frac{1}{8}\right) - 3.$$

In Exercises 37–44, find the derivative using logarithmic differentiation as in Example 5.

37. $y = (x + 5)(x + 9)$

SOLUTION Let $y = (x + 5)(x + 9)$. Then $\ln y = \ln((x + 5)(x + 9)) = \ln(x + 5) + \ln(x + 9)$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{1}{x + 5} + \frac{1}{x + 9}$$

or

$$y' = (x+5)(x+9)\left(\frac{1}{x+5} + \frac{1}{x+9}\right) = (x+9) + (x+5) = 2x + 14.$$

39. $y = (x-1)(x-12)(x+7)$

SOLUTION Let $y = (x-1)(x-12)(x+7)$. Then $\ln y = \ln(x-1) + \ln(x-12) + \ln(x+7)$. By logarithmic differentiation,

$$\frac{y'}{y} = \frac{1}{x-1} + \frac{1}{x-12} + \frac{1}{x+7}$$

or

$$y' = (x-12)(x+7) + (x-1)(x+7) + (x-1)(x-12) = 3x^2 - 12x + 79.$$

41. $y = \dfrac{x(x^2+1)}{\sqrt{x+1}}$

SOLUTION Let $y = \frac{x(x^2+1)}{\sqrt{x+1}}$. Then $\ln y = \ln x + \ln(x^2+1) - \frac{1}{2}\ln(x+1)$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{1}{x} + \frac{2x}{x^2+1} - \frac{1}{2(x+1)},$$

so

$$y' = \frac{x(x^2+1)}{\sqrt{x+1}}\left(\frac{1}{x} + \frac{2x}{x^2+1} - \frac{1}{2(x+1)}\right).$$

43. $y = \sqrt{\dfrac{x(x+2)}{(2x+1)(3x+2)}}$

SOLUTION Let $y = \sqrt{\frac{x(x+2)}{(2x+1)(3x+2)}}$. Then $\ln y = \frac{1}{2}[\ln(x) + \ln(x+2) - \ln(2x+1) - \ln(3x+2)]$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x+2} - \frac{2}{2x+1} - \frac{3}{3x+2}\right),$$

so

$$y' = \frac{1}{2}\sqrt{\frac{x(x+2)}{(2x+1)(3x+2)}} \cdot \left(\frac{1}{x} + \frac{1}{x+2} - \frac{2}{2x+1} - \frac{3}{3x+2}\right).$$

In Exercises 45–50, find the derivative using either method of Example 6.

45. $f(x) = x^{3x}$

SOLUTION Method 1: $x^{3x} = e^{3x\ln x}$, so

$$\frac{d}{dx}x^{3x} = e^{3x\ln x}(3 + 3\ln x) = x^{3x}(3 + 3\ln x).$$

Method 2: Let $y = x^{3x}$. Then, $\ln y = 3x\ln x$. By logarithmic differentiation

$$\frac{y'}{y} = 3x \cdot \frac{1}{x} + 3\ln x,$$

so

$$y' = y(3 + 3\ln x) = x^{3x}(3 + 3\ln x).$$

47. $f(x) = x^{e^x}$

SOLUTION Method 1: $x^{e^x} = e^{e^x\ln x}$, so

$$\frac{d}{dx}x^{e^x} = e^{e^x\ln x}\left(\frac{e^x}{x} + e^x\ln x\right) = x^{e^x}\left(\frac{e^x}{x} + e^x\ln x\right).$$

Method 2: Let $y = x^{e^x}$. Then $\ln y = e^x \ln x$. By logarithmic differentiation

$$\frac{y'}{y} = e^x \cdot \frac{1}{x} + e^x \ln x,$$

so

$$y' = y\left(\frac{e^x}{x} + e^x \ln x\right) = x^{e^x}\left(\frac{e^x}{x} + e^x \ln x\right).$$

49. $f(x) = x^{3^x}$

SOLUTION Method 1: $x^{3^x} = e^{3^x \ln x}$, so

$$\frac{d}{dx}x^{3^x} = e^{3^x \ln x}\left(\frac{3^x}{x} + (\ln x)(\ln 3)3^x\right) = x^{3^x}\left(\frac{3^x}{x} + (\ln x)(\ln 3)3^x\right).$$

Method 2: Let $y = x^{3^x}$. Then $\ln y = 3^x \ln x$. By logarithmic differentiation

$$\frac{y'}{y} = 3^x \frac{1}{x} + (\ln x)(\ln 3)3^x,$$

so

$$y' = x^{3^x}\left(\frac{3^x}{x} + (\ln x)(\ln 3)3^x\right).$$

In Exercises 51–74, calculate the derivative.

51. $y = \sinh(9x)$

SOLUTION $\frac{d}{dx}\sinh(9x) = 9\cosh(9x).$

53. $y = \cosh^2(9 - 3t)$

SOLUTION $\frac{d}{dt}\cosh^2(9-3t) = 2\cosh(9-3t)\cdot(-3\sinh(9-3t)) = -6\cosh(9-3t)\sinh(9-3t).$

55. $y = \sqrt{\cosh x + 1}$

SOLUTION $\frac{d}{dx}\sqrt{\cosh x + 1} = \frac{1}{2}(\cosh x + 1)^{-1/2}\sinh x.$

57. $y = \frac{\coth t}{1 + \tanh t}$

SOLUTION $\frac{d}{dt}\frac{\coth t}{1+\tanh t} = \frac{-\operatorname{csch}^2 t(1+\tanh t) - \coth t(\operatorname{sech}^2 t)}{(1+\tanh t)^2} = -\frac{\operatorname{csch}^2 t + 2\operatorname{csch} t \operatorname{sech} t}{(1+\tanh t^2)}$

59. $y = \sinh(\ln x)$

SOLUTION $\frac{d}{dx}\sinh(\ln x) = \frac{\cosh(\ln x)}{x}.$

61. $y = \tanh(e^x)$

SOLUTION $\frac{d}{dx}\tanh(e^x) = e^x \operatorname{sech}^2(e^x).$

63. $y = \operatorname{sech}(\sqrt{x})$

SOLUTION $\frac{d}{dx}\operatorname{sech}(\sqrt{x}) = -\frac{1}{2}x^{-1/2}\operatorname{sech}\sqrt{x}\tanh\sqrt{x}.$

65. $y = \operatorname{sech} x \coth x$

SOLUTION $\frac{d}{dx}\operatorname{sech} x \coth x = \frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x.$

67. $y = \cosh^{-1}(3x)$

SOLUTION $\frac{d}{dx}\cosh^{-1}(3x) = \frac{3}{\sqrt{9x^2 - 1}}.$

69. $y = (\sinh^{-1}(x^2))^3$

SOLUTION $\frac{d}{dx}(\sinh^{-1}(x^2))^3 = 3(\sinh^{-1}(x^2))^2\frac{2x}{\sqrt{x^4+1}}.$

71. $y = e^{\cosh^{-1} x}$

SOLUTION $\frac{d}{dx} e^{\cosh^{-1} x} = e^{\cosh^{-1} x} \left(\frac{1}{\sqrt{x^2 - 1}} \right).$

73. $y = \tanh^{-1}(\ln t)$

SOLUTION $\frac{d}{dt} \tanh^{-1}(\ln t) = \frac{1}{t(1 - (\ln t)^2)}.$

In Exercises 75–77, prove the formula.

75. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

SOLUTION $\frac{d}{dx} \coth x = \frac{d}{dx} \frac{\cosh x}{\sinh x} = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\operatorname{csch}^2 x.$

77. $\frac{d}{dt} \cosh^{-1} t = \frac{1}{\sqrt{t^2 - 1}}$ for $t > 1$

SOLUTION Let $x = \cosh^{-1} t$. Then $x \geq 0$, $t = \cosh x$ and

$$1 = \sinh x \frac{dx}{dt} \quad \text{or} \quad \frac{dx}{dt} = \frac{1}{\sinh x}.$$

Thus, for $t > 1$,

$$\frac{d}{dt} \cosh^{-1} t = \frac{1}{\sinh x},$$

where $\cosh x = t$. Working from the identity $\cosh^2 x - \sinh^2 x = 1$, we find $\sinh x = \pm\sqrt{\cosh^2 x - 1}$. Because $\sinh w \geq 0$ for $w \geq 0$, we know to choose the positive square root. Hence, $\sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{t^2 - 1}$, and

$$\frac{d}{dt} \cosh^{-1} t = \frac{1}{\sinh x} = \frac{1}{\sqrt{t^2 - 1}}.$$

79. According to one simplified model, the purchasing power of a dollar in the year $2000 + t$ is equal to $P(t) = 0.68(1.04)^{-t}$ (in 1983 dollars). Calculate the predicted rate of decline in purchasing power (in cents per year) in the year 2020.

SOLUTION First, note that

$$P'(t) = -0.68(1.04)^{-t} \ln 1.04;$$

thus, the rate of change in the year 2020 is

$$P'(20) = -0.68(1.04)^{-20} \ln 1.04 = -0.0122.$$

That is, the rate of decline is 1.22 cents per year.

81. Show that for any constants M, k, and a, the function

$$y(t) = \frac{1}{2} M \left(1 + \tanh \left(\frac{k(t-a)}{2} \right) \right)$$

satisfies the **logistic equation**: $\frac{y'}{y} = k\left(1 - \frac{y}{M}\right)$.

SOLUTION Let

$$y(t) = \frac{1}{2} M \left(1 + \tanh \left(\frac{k(t-a)}{2} \right) \right).$$

Then

$$1 - \frac{y(t)}{M} = \frac{1}{2} \left(1 - \tanh \left(\frac{k(t-a)}{2} \right) \right),$$

and

$$ky(t) \left(1 - \frac{y(t)}{M} \right) = \frac{1}{4} Mk \left(1 - \tanh^2 \left(\frac{k(t-a)}{2} \right) \right)$$
$$= \frac{1}{4} Mk \operatorname{sech}^2 \left(\frac{k(t-a)}{2} \right).$$

Finally,

$$y'(t) = \frac{1}{4}Mk\,\text{sech}^2\left(\frac{k(t-a)}{2}\right) = ky(t)\left(1 - \frac{y(t)}{M}\right).$$

83. The Palermo Technical Impact Hazard Scale P is used to quantify the risk associated with the impact of an asteroid colliding with the earth:

$$P = \log_{10}\left(\frac{p_i E^{0.8}}{0.03T}\right)$$

where p_i is the probability of impact, T is the number of years until impact, and E is the energy of impact (in megatons of TNT). The risk is greater than a random event of similar magnitude if $P > 0$.
(a) Calculate dP/dT, assuming that $p_i = 2 \times 10^{-5}$ and $E = 2$ megatons.
(b) Use the derivative to estimate the change in P if T increases from 8 to 9 years.

SOLUTION
(a) Observe that

$$P = \log_{10}\left(\frac{p_i E^{0.8}}{0.03T}\right) = \log_{10}\left(\frac{p_i E^{0.8}}{0.03}\right) - \log_{10} T,$$

so

$$\frac{dP}{dT} = -\frac{1}{T \ln 10}.$$

(b) If T increases to 9 years from 8 years, then

$$\Delta P \approx \left.\frac{dP}{dT}\right|_{T=8} \cdot \Delta T = -\frac{1}{(8\text{ yr}) \ln 10} \cdot (1\text{ yr}) = -0.054$$

Further Insights and Challenges

85. Use the formula $\log_b x = \dfrac{\log_a x}{\log_a b}$ for $a, b > 0$ to verify the formula

$$\frac{d}{dx}\log_b x = \frac{1}{(\ln b)x}$$

SOLUTION $\dfrac{d}{dx}\log_b x = \dfrac{d}{dx}\dfrac{\ln x}{\ln b} = \dfrac{1}{(\ln b)x}.$

3.10 Implicit Differentiation

Preliminary Questions

1. Which differentiation rule is used to show $\dfrac{d}{dx}\sin y = \cos y\dfrac{dy}{dx}$?

SOLUTION The chain rule is used to show that $\frac{d}{dx}\sin y = \cos y\frac{dy}{dx}$.

2. One of (a)–(c) is incorrect. Find and correct the mistake.

(a) $\dfrac{d}{dy}\sin(y^2) = 2y\cos(y^2)$ **(b)** $\dfrac{d}{dx}\sin(x^2) = 2x\cos(x^2)$ **(c)** $\dfrac{d}{dx}\sin(y^2) = 2y\cos(y^2)$

SOLUTION
(a) This is correct. Note that the differentiation is with respect to the variable y.
(b) This is correct. Note that the differentiation is with respect to the variable x.
(c) This is incorrect. Because the differentiation is with respect to the variable x, the chain rule is needed to obtain

$$\frac{d}{dx}\sin(y^2) = 2y\cos(y^2)\frac{dy}{dx}.$$

3. On an exam, Jason was asked to differentiate the equation

$$x^2 + 2xy + y^3 = 7$$

Find the errors in Jason's answer: $2x + 2xy' + 3y^2 = 0$

SOLUTION There are two mistakes in Jason's answer. First, Jason should have applied the product rule to the second term to obtain

$$\frac{d}{dx}(2xy) = 2x\frac{dy}{dx} + 2y.$$

Second, he should have applied the general power rule to the third term to obtain

$$\frac{d}{dx}y^3 = 3y^2\frac{dy}{dx}.$$

4. Which of (a) or (b) is equal to $\frac{d}{dx}(x \sin t)$?

(a) $(x\cos t)\frac{dt}{dx}$ **(b)** $(x\cos t)\frac{dt}{dx} + \sin t$

SOLUTION Using the product rule and the chain rule we see that

$$\frac{d}{dx}(x\sin t) = x\cos t\frac{dt}{dx} + \sin t,$$

so the correct answer is **(b)**.

Exercises

1. Show that if you differentiate both sides of $x^2 + 2y^3 = 6$, the result is $2x + 6y^2\frac{dy}{dx} = 0$. Then solve for dy/dx and evaluate it at the point (2, 1).

SOLUTION

$$\begin{aligned}\frac{d}{dx}(x^2 + 2y^3) &= \frac{d}{dx}6\\ 2x + 6y^2\frac{dy}{dx} &= 0\\ 2x + 6y^2\frac{dy}{dx} &= 0\\ 6y^2\frac{dy}{dx} &= -2x\\ \frac{dy}{dx} &= \frac{-2x}{6y^2}.\end{aligned}$$

At (2, 1), $\frac{dy}{dx} = \frac{-4}{6} = -\frac{2}{3}$.

In Exercises 3–8, differentiate the expression with respect to x, assuming that $y = f(x)$.

3. x^2y^3

SOLUTION Assuming that y depends on x, then

$$\frac{d}{dx}\left(x^2y^3\right) = x^2 \cdot 3y^2y' + y^3 \cdot 2x = 3x^2y^2y' + 2xy^3.$$

5. $(x^2 + y^2)^{3/2}$

SOLUTION Assuming that y depends on x, then

$$\frac{d}{dx}\left(\left(x^2 + y^2\right)^{3/2}\right) = \frac{3}{2}\left(x^2 + y^2\right)^{1/2}\left(2x + 2yy'\right) = 3\left(x + yy'\right)\sqrt{x^2 + y^2}.$$

7. $\dfrac{y}{y+1}$

SOLUTION Assuming that y depends on x, then $\frac{d}{dx}\frac{y}{y+1} = \frac{(y+1)y' - yy'}{(y+1)^2} = \frac{y'}{(y+1)^2}$.

In Exercises 9–26, calculate the derivative with respect to x.

9. $3y^3 + x^2 = 5$

SOLUTION Let $3y^3 + x^2 = 5$. Then $9y^2y' + 2x = 0$, and $y' = -\frac{2x}{9y^2}$.

11. $x^2y + 2x^3y = x + y$

SOLUTION Let $x^2y + 2x^3y = x + y$. Then

$$x^2y' + 2xy + 2x^3y' + 6x^2y = 1 + y'$$
$$x^2y' + 2x^3y' - y' = 1 - 2xy - 6x^2y$$
$$y' = \frac{1 - 2xy - 6x^2y}{x^2 + 2x^3 - 1}.$$

13. $x^3R^5 = 1$

SOLUTION Let $x^3R^5 = 1$. Then $x^3 \cdot 5R^4R' + R^5 \cdot 3x^2 = 0$, and $R' = -\dfrac{3x^2R^5}{5x^3R^4} = -\dfrac{3R}{5x}$.

15. $\dfrac{y}{x} + \dfrac{x}{y} = 2y$

SOLUTION Let

$$\frac{y}{x} + \frac{x}{y} = 2y.$$

Then

$$\frac{xy' - y}{x^2} + \frac{y - xy'}{y^2} = 2y'$$
$$\left(\frac{1}{x} - \frac{x}{y^2} - 2\right)y' = \frac{y}{x^2} - \frac{1}{y}$$
$$\frac{y^2 - x^2 - 2xy^2}{xy^2}y' = \frac{y^2 - x^2}{x^2y}$$
$$y' = \frac{y(y^2 - x^2)}{x(y^2 - x^2 - 2xy^2)}.$$

17. $y^{-2/3} + x^{3/2} = 1$

SOLUTION Let $y^{-2/3} + x^{3/2} = 1$. Then

$$-\frac{2}{3}y^{-5/3}y' + \frac{3}{2}x^{1/2} = 0 \quad \text{or} \quad y' = \frac{9}{4}x^{1/2}y^{5/3}.$$

19. $y + \dfrac{1}{y} = x^2 + x$

SOLUTION Let $y + \frac{1}{y} = x^2 + x$. Then

$$y' - \frac{1}{y^2}y' = 2x + 1 \quad \text{or} \quad y' = \frac{2x+1}{1 - y^{-2}} = \frac{(2x+1)y^2}{y^2 - 1}.$$

21. $\sin(x + y) = x + \cos y$

SOLUTION Let $\sin(x + y) = x + \cos y$. Then

$$(1 + y')\cos(x + y) = 1 - y'\sin y$$
$$\cos(x + y) + y'\cos(x + y) = 1 - y'\sin y$$
$$(\cos(x + y) + \sin y)\,y' = 1 - \cos(x + y)$$
$$y' = \frac{1 - \cos(x + y)}{\cos(x + y) + \sin y}.$$

23. $xe^y = 2xy + y^3$

SOLUTION Let $xe^y = 2xy + y^3$. Then $xy'e^y + e^y = 2xy' + 2y + 3y^2y'$, whence

$$y' = \frac{e^y - 2y}{2x + 3y^2 - xe^y}.$$

25. $\ln x + \ln y = x - y$

SOLUTION Let $\ln x + \ln y = x - y$. Then

$$\frac{1}{x} + \frac{y'}{y} = 1 - y' \quad \text{or} \quad y' = \frac{1 - \frac{1}{x}}{1 + \frac{1}{y}} = \frac{xy - y}{xy + x}.$$

27. Show that $x + yx^{-1} = 1$ and $y = x - x^2$ define the same curve (except that $(0, 0)$ is not a solution of the first equation) and that implicit differentiation yields $y' = yx^{-1} - x$ and $y' = 1 - 2x$. Explain why these formulas produce the same values for the derivative.

SOLUTION Multiply the first equation by x and then isolate the y term to obtain

$$x^2 + y = x \quad \Rightarrow \quad y = x - x^2.$$

Implicit differentiation applied to the first equation yields

$$1 - yx^{-2} + x^{-1}y' = 0 \quad \text{or} \quad y' = yx^{-1} - x.$$

From the first equation, we find $yx^{-1} = 1 - x$; upon substituting this expression into the previous derivative, we find

$$y' = 1 - x - x = 1 - 2x,$$

which is the derivative of the second equation.

In Exercises 29 and 30, find dy/dx at the given point.

29. $(x + 2)^2 - 6(2y + 3)^2 = 3, \quad (1, -1)$

SOLUTION By the scaling and shifting rule,

$$2(x + 2) - 24(2y + 3)y' = 0.$$

If $x = 1$ and $y = -1$, then

$$2(3) - 24(1)y' = 0.$$

so that $24y' = 6$, or $y' = \frac{1}{4}$.

In Exercises 31–38, find an equation of the tangent line at the given point.

31. $xy + x^2y^2 = 5, \quad (2, 1)$

SOLUTION Taking the derivative of both sides of $xy + x^2y^2 = 5$ yields

$$xy' + y + 2xy^2 + 2x^2yy' = 0.$$

Substituting $x = 2$, $y = 1$, we find

$$2y' + 1 + 4 + 8y' = 0 \quad \text{or} \quad y' = -\frac{1}{2}.$$

Hence, the equation of the tangent line at $(2, 1)$ is $y - 1 = -\frac{1}{2}(x - 2)$ or $y = -\frac{1}{2}x + 2$.

33. $x^2 + \sin y = xy^2 + 1, \quad (1, 0)$

SOLUTION Taking the derivative of both sides of $x^2 + \sin y = xy^2 + 1$ yields

$$2x + \cos y y' = y^2 + 2xyy'.$$

Substituting $x = 1$, $y = 0$, we find

$$2 + y' = 0 \quad \text{or} \quad y' = -2.$$

Hence, the equation of the tangent line is $y - 0 = -2(x - 1)$ or $y = -2x + 2$.

35. $2x^{1/2} + 4y^{-1/2} = xy$, $(1, 4)$

SOLUTION Taking the derivative of both sides of $2x^{1/2} + 4y^{-1/2} = xy$ yields

$$x^{-1/2} - 2y^{-3/2}y' = xy' + y.$$

Substituting $x = 1$, $y = 4$, we find

$$1 - 2\left(\frac{1}{8}\right)y' = y' + 4 \quad \text{or} \quad y' = -\frac{12}{5}.$$

Hence, the equation of the tangent line is $y - 4 = -\frac{12}{5}(x - 1)$ or $y = -\frac{12}{5}x + \frac{32}{5}$.

37. $e^{2x-y} = \dfrac{x^2}{y}$, $(2, 4)$

SOLUTION taking the derivative of both sides of $e^{2x-y} = \dfrac{x^2}{y}$ yields

$$e^{2x-y}(2 - y') = \frac{2xy - x^2y'}{y^2}.$$

Substituting $x = 2$, $y = 4$, we find

$$e^0(2 - y') = \frac{16 - 4y'}{16} \quad \text{or} \quad y' = \frac{4}{3}.$$

Hence, the equation of the tangent line is $y - 4 = \frac{4}{3}(x - 2)$ or $y = \frac{4}{3}x + \frac{4}{3}$.

39. Find the points on the graph of $y^2 = x^3 - 3x + 1$ (Figure 6) where the tangent line is horizontal.

(a) First show that $2yy' = 3x^2 - 3$, where $y' = dy/dx$.

(b) Do not solve for y'. Rather, set $y' = 0$ and solve for x. This yields two values of x where the slope may be zero.

(c) Show that the positive value of x does not correspond to a point on the graph.

(d) The negative value corresponds to the two points on the graph where the tangent line is horizontal. Find their coordinates.

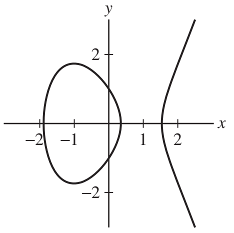


FIGURE 6 Graph of $y^2 = x^3 - 3x + 1$.

SOLUTION

(a) Applying implicit differentiation to $y^2 = x^3 - 3x + 1$, we have

$$2y\frac{dy}{dx} = 3x^2 - 3.$$

(b) Setting $y' = 0$ we have $0 = 3x^2 - 3$, so $x = 1$ or $x = -1$.

(c) If we return to the equation $y^2 = x^3 - 3x + 1$ and substitute $x = 1$, we obtain the equation $y^2 = -1$, which has no real solutions.

(d) Substituting $x = -1$ into $y^2 = x^3 - 3x + 1$ yields

$$y^2 = (-1)^3 - 3(-1) + 1 = -1 + 3 + 1 = 3,$$

so $y = \sqrt{3}$ or $-\sqrt{3}$. The tangent is horizontal at the points $(-1, \sqrt{3})$ and $(-1, -\sqrt{3})$.

41. Find all points on the graph of $3x^2 + 4y^2 + 3xy = 24$ where the tangent line is horizontal (Figure 7).

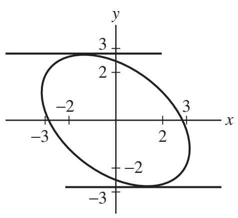


FIGURE 7 Graph of $3x^2 + 4y^2 + 3xy = 24$.

SOLUTION Differentiating the equation $3x^2 + 4y^2 + 3xy = 24$ implicitly yields

$$6x + 8yy' + 3xy' + 3y = 0,$$

so

$$y' = -\frac{6x + 3y}{8y + 3x}.$$

Setting $y' = 0$ leads to $6x + 3y = 0$, or $y = -2x$. Substituting $y = -2x$ into the equation $3x^2 + 4y^2 + 3xy = 24$ yields

$$3x^2 + 4(-2x)^2 + 3x(-2x) = 24,$$

or $13x^2 = 24$. Thus, $x = \pm 2\sqrt{78}/13$, and the coordinates of the two points on the graph of $3x^2 + 4y^2 + 3xy = 24$ where the tangent line is horizontal are

$$\left(\frac{2\sqrt{78}}{13}, -\frac{4\sqrt{78}}{13}\right) \quad \text{and} \quad \left(-\frac{2\sqrt{78}}{13}, \frac{4\sqrt{78}}{13}\right).$$

43. Figure 1 shows the graph of $y^4 + xy = x^3 - x + 2$. Find dy/dx at the two points on the graph with x-coordinate 0 and find an equation of the tangent line at (1, 1).

SOLUTION Consider the equation $y^4 + xy = x^3 - x + 2$. Then $4y^3y' + xy' + y = 3x^2 - 1$, and

$$y' = \frac{3x^2 - y - 1}{x + 4y^3}.$$

- Substituting $x = 0$ into $y^4 + xy = x^3 - x + 2$ gives $y^4 = 2$, which has two real solutions, $y = \pm 2^{1/4}$. When $y = 2^{1/4}$, we have

$$y' = \frac{-2^{1/4} - 1}{4\left(2^{3/4}\right)} = -\frac{\sqrt{2} + \sqrt[4]{2}}{8} \approx -.3254.$$

  When $y = -2^{1/4}$, we have

$$y' = \frac{2^{1/4} - 1}{-4\left(2^{3/4}\right)} = -\frac{\sqrt{2} - \sqrt[4]{2}}{8} \approx -.02813.$$

- At the point (1, 1), we have $y' = \frac{1}{5}$. At this point the tangent line is $y - 1 = \frac{1}{5}(x - 1)$ or $y = \frac{1}{5}x + \frac{4}{5}$.

45. Find a point on the folium $x^3 + y^3 = 3xy$ other than the origin at which the tangent line is horizontal.

SOLUTION Using implicit differentiation, we find

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}(3xy)$$
$$3x^2 + 3y^2y' = 3(xy' + y)$$

Setting $y' = 0$ in this equation yields $3x^2 = 3y$ or $y = x^2$. If we substitute this expression into the original equation $x^3 + y^3 = 3xy$, we obtain:

$$x^3 + x^6 = 3x(x^2) = 3x^3 \quad \text{or} \quad x^3(x^3 - 2) = 0.$$

One solution of this equation is $x = 0$ and the other is $x = 2^{1/3}$. Thus, the two points on the folium $x^3 + y^3 = 3xy$ at which the tangent line is horizontal are (0, 0) and $(2^{1/3}, 2^{2/3})$.

47. Find the x-coordinates of the points where the tangent line is horizontal on the *trident curve* $xy = x^3 - 5x^2 + 2x - 1$, so named by Isaac Newton in his treatise on curves published in 1710 (Figure 9).
Hint: $2x^3 - 5x^2 + 1 = (2x - 1)(x^2 - 2x - 1)$.

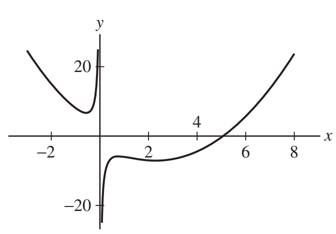


FIGURE 9 Trident curve: $xy = x^3 - 5x^2 + 2x - 1$.

SOLUTION Take the derivative of the equation of a trident curve:

$$xy = x^3 - 5x^2 + 2x - 1$$

to obtain

$$xy' + y = 3x^2 - 10x + 2.$$

Setting $y' = 0$ gives $y = 3x^2 - 10x + 2$. Substituting this into the equation of the trident, we have

$$xy = x(3x^2 - 10x + 2) = x^3 - 5x^2 + 2x - 1$$

or

$$3x^3 - 10x^2 + 2x = x^3 - 5x^2 + 2x - 1$$

Collecting like terms and setting to zero, we have

$$0 = 2x^3 - 5x^2 + 1 = (2x - 1)(x^2 - 2x - 1).$$

Hence, $x = \frac{1}{2}, 1 \pm \sqrt{2}$.

49. Find the derivative at the points where $x = 1$ on the folium $(x^2 + y^2)^2 = \frac{25}{4}xy^2$. See Figure 11.

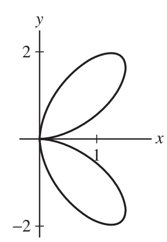


FIGURE 11 Folium curve: $(x^2 + y^2)^2 = \dfrac{25}{4}xy^2$

SOLUTION First, find the points $(1, y)$ on the curve. Setting $x = 1$ in the equation $(x^2 + y^2)^2 = \frac{25}{4}xy^2$ yields

$$\begin{aligned}(1 + y^2)^2 &= \frac{25}{4}y^2\\ y^4 + 2y^2 + 1 &= \frac{25}{4}y^2\\ 4y^4 + 8y^2 + 4 &= 25y^2\\ 4y^4 - 17y^2 + 4 &= 0\\ (4y^2 - 1)(y^2 - 4) &= 0\\ y^2 &= \frac{1}{4} \text{ or } y^2 = 4\end{aligned}$$

Hence $y = \pm\frac{1}{2}$ or $y = \pm 2$. Taking $\frac{d}{dx}$ of both sides of the original equation yields

$$\begin{aligned}2(x^2 + y^2)(2x + 2yy') &= \frac{25}{4}y^2 + \frac{25}{2}xyy'\\ 4(x^2 + y^2)x + 4(x^2 + y^2)yy' &= \frac{25}{4}y^2 + \frac{25}{2}xyy'\\ (4(x^2 + y^2) - \frac{25}{2}x)yy' &= \frac{25}{4}y^2 - 4(x^2 + y^2)x\\ y' &= \frac{\frac{25}{4}y^2 - 4(x^2 + y^2)x}{y(4(x^2 + y^2) - \frac{25}{2}x)}\end{aligned}$$

- At $(1, 2)$, $x^2 + y^2 = 5$, and

$$y' = \frac{\frac{25}{4}2^2 - 4(5)(1)}{2(4(5) - \frac{25}{2}(1))} = \frac{1}{3}.$$

- At $(1, -2)$, $x^2 + y^2 = 5$ as well, and

$$y' = \frac{\frac{25}{4}(-2)^2 - 4(5)(1)}{-2(4(5) - \frac{25}{2}(1))} = -\frac{1}{3}.$$

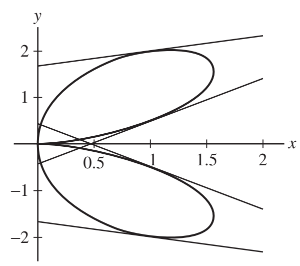
- At $(1, \frac{1}{2})$, $x^2 + y^2 = \frac{5}{4}$, and

$$y' = \frac{\frac{25}{4}\left(\frac{1}{2}\right)^2 - 4\left(\frac{5}{4}\right)(1)}{\frac{1}{2}\left(4\left(\frac{5}{4}\right) - \frac{25}{2}(1)\right)} = \frac{11}{12}.$$

- At $(1, -\frac{1}{2})$, $x^2 + y^2 = \frac{5}{4}$, and

$$y' = \frac{\frac{25}{4}\left(-\frac{1}{2}\right)^2 - 4\left(\frac{5}{4}\right)(1)}{-\frac{1}{2}\left(4\left(\frac{5}{4}\right) - \frac{25}{2}(1)\right)} = -\frac{11}{12}.$$

The folium and its tangent lines are plotted below:



Exercises 51–53: If the derivative dx/dy (instead of $dy/dx = 0$) exists at a point and $dx/dy = 0$, then the tangent line at that point is vertical.

51. Calculate dx/dy for the equation $y^4 + 1 = y^2 + x^2$ and find the points on the graph where the tangent line is vertical.

SOLUTION Let $y^4 + 1 = y^2 + x^2$. Differentiating this equation with respect to y yields

$$4y^3 = 2y + 2x\frac{dx}{dy},$$

so

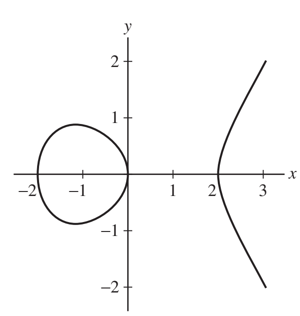
$$\frac{dx}{dy} = \frac{4y^3 - 2y}{2x} = \frac{y(2y^2 - 1)}{x}.$$

Thus, $\frac{dx}{dy} = 0$ when $y = 0$ and when $y = \pm\frac{\sqrt{2}}{2}$. Substituting $y = 0$ into the equation $y^4 + 1 = y^2 + x^2$ gives $1 = x^2$, so $x = \pm 1$. Substituting $y = \pm\frac{\sqrt{2}}{2}$, gives $x^2 = 3/4$, so $x = \pm\frac{\sqrt{3}}{2}$. Thus, there are six points on the graph of $y^4 + 1 = y^2 + x^2$ where the tangent line is vertical:

$$(1, 0), (-1, 0), \left(\frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{3}}{2}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{\sqrt{2}}{2}\right).$$

53. CAS Use a computer algebra system to plot $y^2 = x^3 - 4x$ for $-4 \le x \le 4$, $4 \le y \le 4$. Show that if $dx/dy = 0$, then $y = 0$. Conclude that the tangent line is vertical at the points where the curve intersects the x-axis. Does your plot confirm this conclusion?

SOLUTION A plot of the curve $y^2 = x^3 - 4x$ is shown below.



Differentiating the equation $y^2 = x^3 - 4x$ with respect to y yields

$$2y = 3x^2 \frac{dx}{dy} - 4\frac{dx}{dy},$$

or

$$\frac{dx}{dy} = \frac{2y}{3x^2 - 4}.$$

From here, it follows that $\frac{dx}{dy} = 0$ when $y = 0$, so the tangent line to this curve is vertical at the points where the curve intersects the x-axis. This conclusion is confirmed by the plot of the curve shown above.

In Exercises 55–58, use implicit differentiation to calculate higher derivatives.

55. Consider the equation $y^3 - \frac{3}{2}x^2 = 1$.

(a) Show that $y' = x/y^2$ and differentiate again to show that

$$y'' = \frac{y^2 - 2xyy'}{y^4}$$

(b) Express y'' in terms of x and y using part (a).

SOLUTION

(a) Let $y^3 - \frac{3}{2}x^2 = 1$. Then $3y^2y' - 3x = 0$, and $y' = x/y^2$. Therefore,

$$y'' = \frac{y^2 \cdot 1 - x \cdot 2yy'}{y^4} = \frac{y^2 - 2xyy'}{y^4}.$$

(b) Substituting the expression for y' into the result for y'' gives

$$y'' = \frac{y^2 - 2xy\left(x/y^2\right)}{y^4} = \frac{y^3 - 2x^2}{y^5}.$$

57. Calculate y'' at the point $(1, 1)$ on the curve $xy^2 + y - 2 = 0$ by the following steps:

(a) Find y' by implicit differentiation and calculate y' at the point $(1, 1)$.

(b) Differentiate the expression for y' found in (a). Then compute y'' at $(1, 1)$ by substituting $x = 1$, $y = 1$, and the value of y' found in (a).

SOLUTION Let $xy^2 + y - 2 = 0$.

(a) Then $x \cdot 2yy' + y^2 \cdot 1 + y' = 0$, and $y' = -\frac{y^2}{2xy + 1}$. At $(x, y) = (1, 1)$, we have $y' = -\frac{1}{3}$.

(b) Therefore,

$$y'' = -\frac{(2xy + 1)\left(2yy'\right) - y^2\left(2xy' + 2y\right)}{(2xy + 1)^2} = -\frac{(3)\left(-\frac{2}{3}\right) - (1)\left(-\frac{2}{3} + 2\right)}{3^2} = -\frac{-6 + 2 - 6}{27} = \frac{10}{27}$$

given that $(x, y) = (1, 1)$ and $y' = -\frac{1}{3}$.

In Exercises 59–61, x and y are functions of a variable t and use implicit differentiation to relate dy/dt and dx/dt.

59. Differentiate $xy = 1$ with respect to t and derive the relation $\frac{dy}{dt} = -\frac{y}{x}\frac{dx}{dt}$.

SOLUTION Let $xy = 1$. Then $x\frac{dy}{dt} + y\frac{dx}{dt} = 0$, and $\frac{dy}{dt} = -\frac{y}{x}\frac{dx}{dt}$.

61. Calculate dy/dt in terms of dx/dt.

(a) $x^3 - y^3 = 1$

(b) $y^4 + 2xy + x^2 = 0$

SOLUTION

(a) Taking the derivative of both sides of the equation $x^3 - y^3 = 1$ with respect to t yields

$$3x^2\frac{dx}{dt} - 3y^2\frac{dy}{dt} = 0 \quad \text{or} \quad \frac{dy}{dt} = \frac{x^2}{y^2}\frac{dx}{dt}.$$

(b) Taking the derivative of both sides of the equation $y^4 + 2xy + x^2 = 0$ with respect to t yields

$$4y^3 \frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} + 2x\frac{dx}{dt} = 0,$$

or

$$\frac{dy}{dt} = -\frac{x+y}{2y^3+x}\frac{dx}{dt}.$$

Further Insights and Challenges

63. Show that if P lies on the intersection of the two curves $x^2 - y^2 = c$ and $xy = d$ (c, d constants), then the tangents to the curves at P are perpendicular.

SOLUTION Let $C1$ be the curve described by $x^2 - y^2 = c$, and let $C2$ be the curve described by $xy = d$. Suppose that $P = (x_0, y_0)$ lies on the intersection of the two curves $x^2 - y^2 = c$ and $xy = d$. Since $x^2 - y^2 = c$, the chain rule gives us $2x - 2yy' = 0$, so that $y' = \frac{2x}{2y} = \frac{x}{y}$. The slope to the tangent line to $C1$ is $\frac{x_0}{y_0}$. On the curve $C2$, since $xy = d$, the product rule yields that $xy' + y = 0$, so that $y' = -\frac{y}{x}$. Therefore the slope to the tangent line to $C2$ is $-\frac{y_0}{x_0}$. The two slopes are negative reciprocals of one another, hence the tangents to the two curves are perpendicular.

65. Divide the curve in Figure 15 into five branches, each of which is the graph of a function. Sketch the branches.

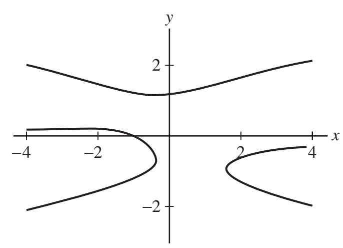
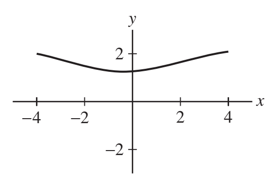


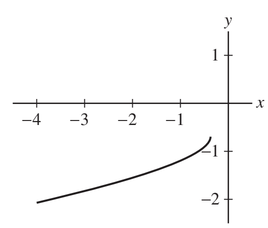
FIGURE 15 Graph of $y^5 - y = x^2y + x + 1$.

SOLUTION The branches are:

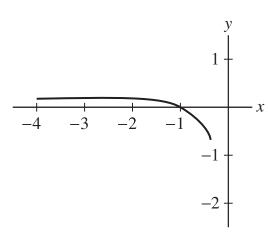
- Upper branch:



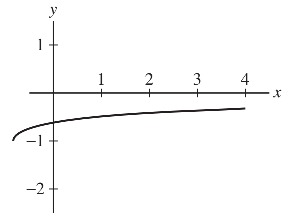
- Lower part of lower left curve:



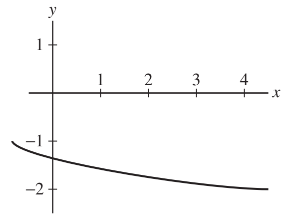
- Upper part of lower left curve:



- Upper part of lower right curve:



- Lower part of lower right curve:



3.11 Related Rates

Preliminary Questions

1. Assign variables and restate the following problem in terms of known and unknown derivatives (but do not solve it): How fast is the volume of a cube increasing if its side increases at a rate of 0.5 cm/s?

SOLUTION Let s and V denote the length of the side and the corresponding volume of a cube, respectively. Determine $\frac{dV}{dt}$ if $\frac{ds}{dt} = 0.5$ cm/s.

2. What is the relation between dV/dt and dr/dt if $V = \left(\frac{4}{3}\right)\pi r^3$?

SOLUTION Applying the general power rule, we find $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. Therefore, the ratio is $4\pi r^2$.

In Questions 3 and 4, water pours into a cylindrical glass of radius 4 cm. Let V and h denote the volume and water level respectively, at time t.

3. Restate this question in terms of dV/dt and dh/dt: How fast is the water level rising if water pours in at a rate of 2 cm^3/min?

SOLUTION Determine $\frac{dh}{dt}$ if $\frac{dV}{dt} = 2\ \text{cm}^3$/min.

4. Restate this question in terms of dV/dt and dh/dt: At what rate is water pouring in if the water level rises at a rate of 1 cm/min?

SOLUTION Determine $\frac{dV}{dt}$ if $\frac{dh}{dt} = 1$ cm/min.

Exercises

In Exercises 1 and 2, consider a rectangular bathtub whose base is 18 ft^2.

1. How fast is the water level rising if water is filling the tub at a rate of 0.7 ft^3/min?

SOLUTION Let h be the height of the water in the tub and V be the volume of the water. Then $V = 18h$ and $\frac{dV}{dt} = 18\frac{dh}{dt}$. Thus

$$\frac{dh}{dt} = \frac{1}{18}\frac{dV}{dt} = \frac{1}{18}(0.7) \approx 0.039 \text{ ft/min}.$$

3. The radius of a circular oil slick expands at a rate of 2 m/min.
(a) How fast is the area of the oil slick increasing when the radius is 25 m?
(b) If the radius is 0 at time $t = 0$, how fast is the area increasing after 3 min?

SOLUTION Let r be the radius of the oil slick and A its area.

(a) Then $A = \pi r^2$ and $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$. Substituting $r = 25$ and $\frac{dr}{dt} = 2$, we find

$$\frac{dA}{dt} = 2\pi\,(25)\,(2) = 100\pi \approx 314.16\ \text{m}^2/\text{min}.$$

(b) Since $\frac{dr}{dt} = 2$ and $r(0) = 0$, it follows that $r(t) = 2t$. Thus, $r(3) = 6$ and

$$\frac{dA}{dt} = 2\pi\,(6)\,(2) = 24\pi \approx 75.40\ \text{m}^2/\text{min}.$$

In Exercises 5–8, assume that the radius r of a sphere is expanding at a rate of 30 cm/min. *The volume of a sphere is* $V = \frac{4}{3}\pi r^3$ *and its surface area is* $4\pi r^2$. *Determine the given rate.*

5. Volume with respect to time when $r = 15$ cm.

SOLUTION As the radius is expanding at 30 centimeters per minute, we know that $\frac{dr}{dt} = 30$ cm/min. Taking $\frac{d}{dt}$ of the equation $V = \frac{4}{3}\pi r^3$ yields

$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right) = 4\pi r^2\frac{dr}{dt}.$$

Substituting $r = 15$ and $\frac{dr}{dt} = 30$ yields

$$\frac{dV}{dt} = 4\pi(15)^2(30) = 27000\pi\ \text{cm}^3/\text{min}.$$

7. Surface area with respect to time when $r = 40$ cm.

SOLUTION Taking the derivative of both sides of $A = 4\pi r^2$ with respect to t yields $\frac{dA}{dt} = 8\pi r\frac{dr}{dt}$. $\frac{dr}{dt} = 30$, so

$$\frac{dA}{dt} = 8\pi(40)(30) = 9600\pi\ \text{cm}^2/\text{min}.$$

In Exercises 9–12, refer to a 5-meter ladder sliding down a wall, as in Figures 1 and 2. The variable h is the height of the ladder's top at time t, and x is the distance from the wall to the ladder's bottom.

9. Assume the bottom slides away from the wall at a rate of 0.8 m/s. Find the velocity of the top of the ladder at $t = 2$ s if the bottom is 1.5 m from the wall at $t = 0$ s.

SOLUTION Let x denote the distance from the base of the ladder to the wall, and h denote the height of the top of the ladder from the floor. The ladder is 5 m long, so $h^2 + x^2 = 5^2$. At any time t, $x = 1.5 + 0.8t$. Therefore, at time $t = 2$, the base is $x = 1.5 + 0.8(2) = 3.1$ m from the wall. Furthermore, we have

$$2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0 \quad \text{so} \quad \frac{dh}{dt} = -\frac{x}{h}\frac{dx}{dt}.$$

Substituting $x = 3.1$, $h = \sqrt{5^2 - 3.1^2}$ and $\frac{dx}{dt} = 0.8$, we obtain

$$\frac{dh}{dt} = -\frac{3.1}{\sqrt{5^2 - 3.1^2}}(0.8) \approx -0.632\ \text{m/s}.$$

11. Suppose that $h(0) = 4$ and the top slides down the wall at a rate of 1.2 m/s. Calculate x and dx/dt at $t = 2$ s.

SOLUTION Let h and x be the height of the ladder's top and the distance from the wall of the ladder's bottom, respectively. After 2 seconds, $h = 4 + 2\,(-1.2) = 1.6$ m. Since $h^2 + x^2 = 5^2$,

$$x = \sqrt{5^2 - 1.6^2} = 4.737\ \text{m}.$$

Furthermore, we have $2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0$, so that $\frac{dx}{dt} = -\frac{h}{x}\frac{dh}{dt}$. Substituting $h = 1.6$, $x = 4.737$, and $\frac{dh}{dt} = -1.2$, we find

$$\frac{dx}{dt} = -\frac{1.6}{4.737}\,(-1.2) \approx 0.405\ \text{m/s}.$$

13. A conical tank has height 3 m and radius 2 m at the top. Water flows in at a rate of 2 m^3/min. How fast is the water level rising when it is 2 m?

SOLUTION Consider the cone of water in the tank at a certain instant. Let r be the radius of its (inverted) base, h its height, and V its volume. By similar triangles, $\frac{r}{h} = \frac{2}{3}$ or $r = \frac{2}{3}h$ and thus $V = \frac{1}{3}\pi r^2 h = \frac{4}{27}\pi h^3$. Therefore,

$$\frac{dV}{dt} = \tfrac{4}{9}\pi h^2\frac{dh}{dt},$$

and

$$\frac{dh}{dt} = \frac{9}{4\pi h^2}\frac{dV}{dt}.$$

Substituting $h = 2$ and $\frac{dV}{dt} = 2$ yields

$$\frac{dh}{dt} = \frac{9}{4\pi\,(2)^2} \times 2 = \frac{9}{8\pi} \approx -0.36 \text{ m/min}.$$

15. The radius r and height h of a circular cone change at a rate of 2 cm/s. How fast is the volume of the cone increasing when $r = 10$ and $h = 20$?

SOLUTION Let r be the radius, h be the height, and V be the volume of a right circular cone. Then $V = \frac{1}{3}\pi r^2 h$, and

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(r^2\frac{dh}{dt} + 2hr\frac{dr}{dt}\right).$$

When $r = 10$, $h = 20$, and $\frac{dr}{dt} = \frac{dh}{dt} = 2$, we find

$$\frac{dV}{dt} = \frac{\pi}{3}\left(10^2 \cdot 2 + 2 \cdot 20 \cdot 10 \cdot 2\right) = \frac{1000\pi}{3} \approx 1047.20 \text{ cm}^3/\text{s}.$$

17. A man of height 1.8 meters walks away from a 5-meter lamppost at a speed of 1.2 m/s (Figure 9). Find the rate at which his shadow is increasing in length.

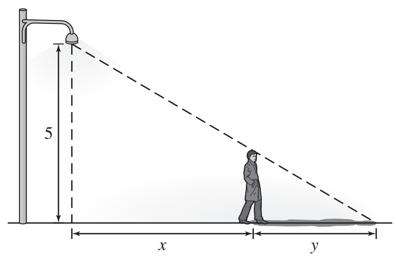


FIGURE 9

SOLUTION Since the man is moving at a rate of 1.2 m/s, his distance from the light post at any given time is $x = 1.2t$. Knowing the man is 1.8 meters tall and that the length of his shadow is denoted by y, we set up a proportion of similar triangles from the diagram:

$$\frac{y}{1.8} = \frac{1.2t + y}{5}.$$

Clearing fractions and solving for y yields

$$y = 0.675t.$$

Thus, $dy/dt = 0.675$ meters per second is the rate at which the length of the shadow is increasing.

19. At a given moment, a plane passes directly above a radar station at an altitude of 6 km.

(a) The plane's speed is 800 km/h. How fast is the distance between the plane and the station changing half a minute later?

(b) How fast is the distance between the plane and the station changing when the plane passes directly above the station?

SOLUTION Let x be the distance of the plane from the station along the ground and h the distance through the air.

(a) By the Pythagorean Theorem, we have

$$h^2 = x^2 + 6^2 = x^2 + 36.$$

Thus $2h\frac{dh}{dt} = 2x\frac{dx}{dt}$, and $\frac{dh}{dt} = \frac{x}{h}\frac{dx}{dt}$. After half a minute, $x = \frac{1}{2} \times \frac{1}{60} \times 800 = \frac{20}{3}$ kilometers. With $x = \frac{20}{3}$,

$$h = \sqrt{\left(\frac{20}{3}\right)^2 + 36} = \frac{1}{3}\sqrt{724} = \frac{2}{3}\sqrt{181} \approx 8.969 \text{ km},$$

and $\frac{dx}{dt} = 800$,

$$\frac{dh}{dt} = \frac{20}{3}\frac{3}{2\sqrt{181}} \times 800 = \frac{8000}{\sqrt{181}} \approx 594.64 \text{ km/h}.$$

(b) When the plane is directly above the station, $x = 0$, so the distance between the plane and the station is not changing, for at this instant we have

$$\frac{dh}{dt} = \frac{0}{6} \times 800 = 0 \text{ km/h}.$$

21. A hot air balloon rising vertically is tracked by an observer located 4 km from the lift-off point. At a certain moment, the angle between the observer's line of sight and the horizontal is $\frac{\pi}{5}$, and it is changing at a rate of 0.2 rad/min. How fast is the balloon rising at this moment?

SOLUTION Let y be the height of the balloon (in miles) and θ the angle between the line-of-sight and the horizontal. Via trigonometry, we have $\tan\theta = \frac{y}{4}$. Therefore,

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{4}\frac{dy}{dt},$$

and

$$\frac{dy}{dt} = 4\frac{d\theta}{dt}\sec^2\theta.$$

Using $\frac{d\theta}{dt} = 0.2$ and $\theta = \frac{\pi}{5}$ yields

$$\frac{dy}{dt} = 4\,(0.2)\,\frac{1}{\cos^2(\pi/5)} \approx 1.22 \text{ km/min}.$$

23. A rocket travels vertically at a speed of 1200 km/h. The rocket is tracked through a telescope by an observer located 16 km from the launching pad. Find the rate at which the angle between the telescope and the ground is increasing 3 min after lift-off.

SOLUTION Let y be the height of the rocket and θ the angle between the telescope and the ground. Using trigonometry, we have $\tan\theta = \frac{y}{16}$. Therefore,

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{16}\frac{dy}{dt},$$

and

$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{16}\frac{dy}{dt}.$$

After the rocket has traveled for 3 minutes (or $\frac{1}{20}$ hour), its height is $\frac{1}{20} \times 1200 = 60$ km. At this instant, $\tan\theta = 60/16 = 15/4$ and thus

$$\cos\theta = \frac{4}{\sqrt{15^2 + 4^2}} = \frac{4}{\sqrt{241}}.$$

Finally,

$$\frac{d\theta}{dt} = \frac{16/241}{16}(1200) = \frac{1200}{241} \approx 4.98 \text{ rad/hr}.$$

25. A police car traveling south toward Sioux Falls at 160 km/h pursues a truck traveling east away from Sioux Falls, Iowa, at 140 km/h (Figure 11). At time $t = 0$, the police car is 20 km north and the truck is 30 km east of Sioux Falls. Calculate the rate at which the distance between the vehicles is changing:

(a) At time $t = 0$

(b) 5 minutes later

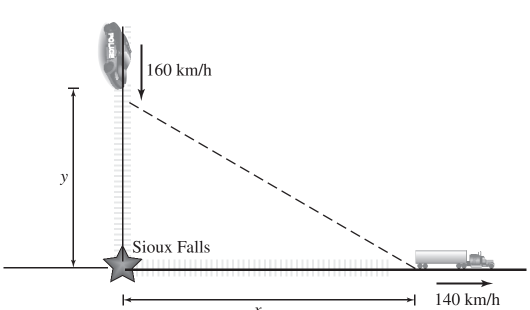


FIGURE 11

SOLUTION Let y denote the distance the police car is north of Sioux Falls and x the distance the truck is east of Sioux Falls. Then $y = 20 - 160t$ and $x = 30 + 140t$. If ℓ denotes the distance between the police car and the truck, then

$$\ell^2 = x^2 + y^2 = (30 + 140t)^2 + (20 - 160t)^2$$

and

$$\ell\frac{d\ell}{dt} = 140(30 + 140t) - 160(20 - 160t) = 1000 + 45200t.$$

(a) At $t = 0$, $\ell = \sqrt{30^2 + 20^2} = 10\sqrt{13}$, so

$$\frac{d\ell}{dt} = \frac{1000}{10\sqrt{13}} = \frac{100\sqrt{13}}{13} \approx 27.735 \text{ km/h}.$$

(b) At $t = 5$ minutes $= \frac{1}{12}$ hour,

$$\ell = \sqrt{\left(30 + 140 \cdot \frac{1}{12}\right)^2 + \left(20 - 160 \cdot \frac{1}{12}\right)^2} \approx 42.197 \text{ km},$$

and

$$\frac{d\ell}{dt} = \frac{1000 + 45200 \cdot \frac{1}{12}}{42.197} \approx 112.962 \text{ km/h}.$$

27. In the setting of Example 5, at a certain moment, the tractor's speed is 3 m/s and the bale is rising at 2 m/s. How far is the tractor from the bale at this moment?

SOLUTION From Example 5, we have the equation

$$\frac{x\frac{dx}{dt}}{\sqrt{x^2 + 4.5^2}} = \frac{dh}{dt},$$

where x denote the distance from the tractor to the bale and h denotes the height of the bale. Given

$$\frac{dx}{dt} = 3 \quad \text{and} \quad \frac{dh}{dt} = 2,$$

it follows that

$$\frac{3x}{\sqrt{4.5^2 + x^2}} = 2,$$

which yields $x = \sqrt{16.2} \approx 4.025$ m.

29. Julian is jogging around a circular track of radius 50 m. In a coordinate system with origin at the center of the track, Julian's x-coordinate is changing at a rate of -1.25 m/s when his coordinates are (40, 30). Find dy/dt at this moment.

SOLUTION We have $x^2 + y^2 = 50^2$, so

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \text{or} \quad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

Given $x = 40$, $y = 30$ and $dx/dt = -1.25$, we find

$$\frac{dy}{dt} = -\frac{40}{30}(-1.25) = \frac{5}{3} \text{ m/s}.$$

In Exercises 31 and 32, assume that the pressure P (in kilopascals) and volume V (in cubic centimeters) of an expanding gas are related by $PV^b = C$, where b and C are constants (this holds in an adiabatic *expansion, without heat gain or loss).*

31. Find dP/dt if $b = 1.2$, $P = 8$ kPa, $V = 100$ cm^2, and $dV/dt = 20$ cm^3/min.

SOLUTION Let $PV^b = C$. Then

$$PbV^{b-1}\frac{dV}{dt} + V^b\frac{dP}{dt} = 0,$$

and

$$\frac{dP}{dt} = -\frac{Pb}{V}\frac{dV}{dt}.$$

Substituting $b = 1.2$, $P = 8$, $V = 100$, and $\frac{dV}{dt} = 20$, we find

$$\frac{dP}{dt} = -\frac{(8)\,(1.2)}{100}\,(20) = -1.92 \text{ kPa/min}.$$

33. The base x of the right triangle in Figure 14 increases at a rate of 5 cm/s, while the height remains constant at $h = 20$. How fast is the angle θ changing when $x = 20$?

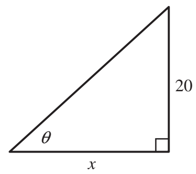


FIGURE 14

SOLUTION We have $\cot\theta = \frac{x}{20}$, from which

$$-\csc^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt}$$

and thus

$$\frac{d\theta}{dt} = -\frac{\sin^2\theta}{20}\frac{dx}{dt}.$$

We are given $\frac{dx}{dt} = 5$ and when $x = h = 20$, $\theta = \frac{\pi}{4}$. Hence,

$$\frac{d\theta}{dt} = -\frac{\sin^2\left(\frac{\pi}{4}\right)}{20}(5) = -\frac{1}{8}\text{ rad/s}.$$

35. A particle travels along a curve $y = f(x)$ as in Figure 15. Let $L(t)$ be the particle's distance from the origin.

(a) Show that $\frac{dL}{dt} = \left(\frac{x + f(x)f'(x)}{\sqrt{x^2 + f(x)^2}}\right)\frac{dx}{dt}$ if the particle's location at time t is $P = (x, f(x))$.

(b) Calculate $L'(t)$ when $x = 1$ and $x = 2$ if $f(x) = \sqrt{3x^2 - 8x + 9}$ and $dx/dt = 4$.

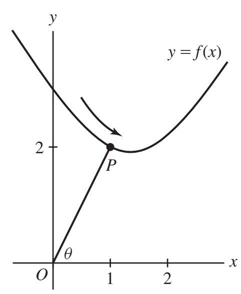


FIGURE 15

SOLUTION

(a) If the particle's location at time t is $P = (x, f(x))$, then

$$L(t) = \sqrt{x^2 + f(x)^2}.$$

Thus,

$$\frac{dL}{dt} = \frac{1}{2}(x^2 + f(x)^2)^{-1/2}\left(2x\frac{dx}{dt} + 2f(x)f'(x)\frac{dx}{dt}\right) = \left(\frac{x + f(x)f'(x)}{\sqrt{x^2 + f(x)^2}}\right)\frac{dx}{dt}.$$

(b) Given $f(x) = \sqrt{3x^2 - 8x + 9}$, it follows that

$$f'(x) = \frac{3x - 4}{\sqrt{3x^2 - 8x + 9}}.$$

Let's start with $x = 1$. Then $f(1) = 2$, $f'(1) = -\frac{1}{2}$ and

$$\frac{dL}{dt} = \left(\frac{1 - 1}{\sqrt{1^2 + 2^2}}\right)(4) = 0.$$

With $x = 2$, $f(2) = \sqrt{5}$, $f'(2) = 2/\sqrt{5}$ and

$$\frac{dL}{dt} = \frac{2 + 2}{\sqrt{2^2 + \sqrt{5}^2}}(4) = \frac{16}{3}.$$

Exercises 37 and 38 refer to the baseball diamond (a square of side 90 ft*) in Figure 16.*

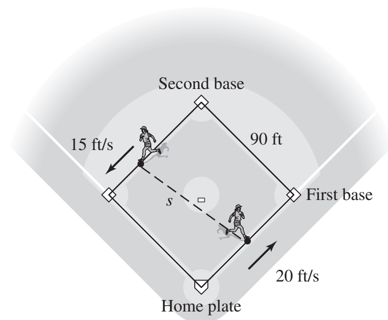


FIGURE 16

37. A baseball player runs from home plate toward first base at 20 ft/s. How fast is the player's distance from second base changing when the player is halfway to first base?

SOLUTION Let x be the distance of the player from home plate and h the player's distance from second base. Using the Pythagorean theorem, we have $h^2 = 90^2 + (90 - x)^2$. Therefore,

$$2h\frac{dh}{dt} = 2\,(90 - x)\left(-\frac{dx}{dt}\right),$$

and

$$\frac{dh}{dt} = -\frac{90 - x}{h}\frac{dx}{dt}.$$

We are given $\frac{dx}{dt} = 20$. When the player is halfway to first base, $x = 45$ and $h = \sqrt{90^2 + 45^2}$, so

$$\frac{dh}{dt} = -\frac{45}{\sqrt{90^2 + 45^2}}\,(20) = -4\sqrt{5} \approx -8.94 \text{ ft/s}.$$

39. The conical watering pail in Figure 17 has a grid of holes. Water flows out through the holes at a rate of kA m^3/min, where k is a constant and A is the surface area of the part of the cone in contact with the water. This surface area is $A = \pi r\sqrt{h^2 + r^2}$ and the volume is $V = \frac{1}{3}\pi r^2 h$. Calculate the rate dh/dt at which the water level changes at $h = 0.3$ m, assuming that $k = 0.25$ m.

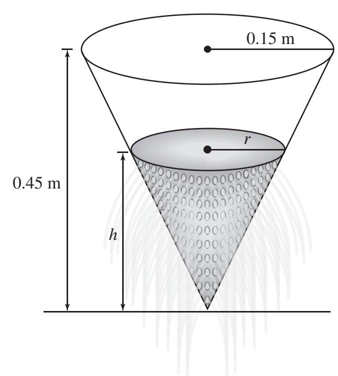


FIGURE 17

SOLUTION By similar triangles, we have

$$\frac{r}{h} = \frac{0.15}{0.45} = \frac{1}{3} \quad \text{so} \quad r = \frac{1}{3}h.$$

Substituting this expression for r into the formula for V yields

$$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{1}{27}\pi h^3.$$

From here and the problem statement, it follows that

$$\frac{dV}{dt} = \frac{1}{9}\pi h^2\frac{dh}{dt} = -kA = -0.25\pi r\sqrt{h^2 + r^2}.$$

Solving for dh/dt gives

$$\frac{dh}{dt} = -\frac{9}{4}\frac{r}{h^2}\sqrt{h^2 + r^2}.$$

When $h = 0.3$, $r = 0.1$ and

$$\frac{dh}{dt} = -\frac{9}{4}\frac{0.1}{0.3^2}\sqrt{0.3^2 + 0.1^2} = -0.79 \text{ m/min}.$$

Further Insights and Challenges

41. A roller coaster has the shape of the graph in Figure 19. Show that when the roller coaster passes the point $(x, f(x))$, the vertical velocity of the roller coaster is equal to $f'(x)$ times its horizontal velocity.

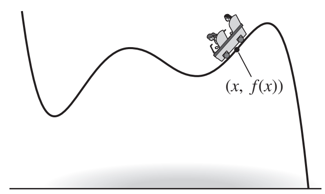


FIGURE 19 Graph of $f(x)$ as a roller coaster track.

SOLUTION Let the equation $y = f(x)$ describe the shape of the roller coaster track. Taking $\frac{d}{dt}$ of both sides of this equation yields $\frac{dy}{dt} = f'(x)\frac{dx}{dt}$. In other words, the vertical velocity of a car moving along the track, $\frac{dy}{dt}$, is equal to $f'(x)$ times the horizontal velocity, $\frac{dx}{dt}$.

43. As the wheel of radius r cm in Figure 20 rotates, the rod of length L attached at point P drives a piston back and forth in a straight line. Let x be the distance from the origin to point Q at the end of the rod, as shown in the figure.

(a) Use the Pythagorean Theorem to show that

$$L^2 = (x - r\cos\theta)^2 + r^2\sin^2\theta \qquad \boxed{6}$$

(b) Differentiate Eq. (6) with respect to t to prove that

$$2(x - r\cos\theta)\left(\frac{dx}{dt} + r\sin\theta\frac{d\theta}{dt}\right) + 2r^2\sin\theta\cos\theta\frac{d\theta}{dt} = 0$$

(c) Calculate the speed of the piston when $\theta = \frac{\pi}{2}$, assuming that $r = 10$ cm, $L = 30$ cm, and the wheel rotates at 4 revolutions per minute.

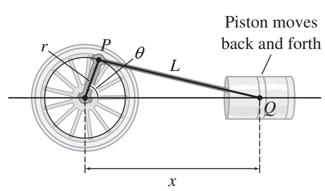


FIGURE 20

SOLUTION From the diagram, the coordinates of P are $(r\cos\theta, r\sin\theta)$ and those of Q are $(x, 0)$.

(a) The distance formula gives

$$L = \sqrt{(x - r\cos\theta)^2 + (-r\sin\theta)^2}.$$

Thus,

$$L^2 = (x - r\cos\theta)^2 + r^2\sin^2\theta.$$

Note that L (the length of the fixed rod) and r (the radius of the wheel) are constants.

(b) From (a) we have

$$0 = 2(x - r\cos\theta)\left(\frac{dx}{dt} + r\sin\theta\frac{d\theta}{dt}\right) + 2r^2\sin\theta\cos\theta\frac{d\theta}{dt}.$$

(c) Solving for dx/dt in (b) gives

$$\frac{dx}{dt} = \frac{r^2\sin\theta\cos\theta\frac{d\theta}{dt}}{r\cos\theta - x} - r\sin\theta\frac{d\theta}{dt} = \frac{rx\sin\theta\frac{d\theta}{dt}}{r\cos\theta - x}.$$

With $\theta = \frac{\pi}{2}$, $r = 10$, $L = 30$, and $\frac{d\theta}{dt} = 8\pi$,

$$\frac{dx}{dt} = \frac{(10)\,(x)\left(\sin\frac{\pi}{2}\right)(8\pi)}{(10)\,(0) - x} = -80\pi \approx -251.33 \text{ cm/min}$$

45. A cylindrical tank of radius R and length L lying horizontally as in Figure 21 is filled with oil to height h.

(a) Show that the volume $V(h)$ of oil in the tank is

$$V(h) = L\left(R^2\cos^{-1}\left(1 - \frac{h}{R}\right) - (R-h)\sqrt{2hR - h^2}\right)$$

(b) Show that $\frac{dV}{dh} = 2L\sqrt{h(2R-h)}$.

(c) Suppose that $R = 1.5$ m and $L = 10$ m and that the tank is filled at a constant rate of $0.6\ \text{m}^3$/min. How fast is the height h increasing when $h = 0.5$?

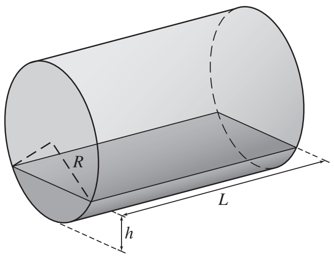


FIGURE 21 Oil in the tank has level h.

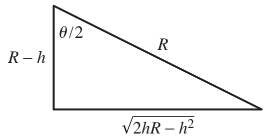
SOLUTION

(a) From Figure 21, we see that the volume of oil in the tank, $V(h)$, is equal to L times $A(h)$, the area of that portion of the circular cross section occupied by the oil. Now,

$$A(h) = \text{area of sector} - \text{area of triangle} = \frac{R^2\theta}{2} - \frac{R^2\sin\theta}{2},$$

where θ is the central angle of the sector. Referring to the diagram below,

$$\cos\frac{\theta}{2} = \frac{R-h}{R} \quad \text{and} \quad \sin\frac{\theta}{2} = \frac{\sqrt{2hR - h^2}}{R}.$$



Thus,

$$\theta = 2\cos^{-1}\left(1 - \frac{h}{R}\right),$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\frac{(R-h)\sqrt{2hR-h^2}}{R^2},$$

and

$$V(h) = L\left(R^2\cos^{-1}\left(1 - \frac{h}{R}\right) - (R-h)\sqrt{2hR - h^2}\right).$$

(b) Recalling that $\frac{d}{dx}\cos^{-1} u = -\frac{1}{\sqrt{1-x^2}}\frac{du}{dx}$,

$$\frac{dV}{dh} = L\left(\frac{d}{dh}\left(R^2\cos^{-1}\left(1-\frac{h}{R}\right)\right) - \frac{d}{dh}\left((R-h)\sqrt{2hR-h^2}\right)\right)$$

$$= L\left(-R\frac{-1}{\sqrt{1-(1-(h/R))^2}} + \sqrt{2hR-h^2} - \frac{(R-h)^2}{\sqrt{2hR-h^2}}\right)$$

$$= L\left(\frac{R^2}{\sqrt{2hR-h^2}} + \sqrt{2hR-h^2} - \frac{R^2-2Rh+h^2}{\sqrt{2hR-h^2}}\right)$$

$$= L\left(\frac{R^2+(2hR-h^2)-(R^2-2Rh+h^2)}{\sqrt{2hR-h^2}}\right)$$

$$= L\left(\frac{4hR-2h^2}{\sqrt{2hR-h^2}}\right) = L\left(\frac{2(2hR-h^2)}{\sqrt{2hR-h^2}}\right) = 2L\sqrt{2hR-h^2}.$$

(c) $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{1}{dV/dh}\frac{dV}{dt}$. From part (b) with $R = 1.5$, $L = 10$ and $h = 0.5$,

$$\frac{dV}{dh} = 2(10)\sqrt{2(0.5)(1.5)-0.5^2} = 10\sqrt{5}\text{ m}^2.$$

Thus,

$$\frac{dh}{dt} = \frac{1}{10\sqrt{5}}(0.6) = \frac{3\sqrt{5}}{2500} \approx 0.0027\text{ m/min}.$$

CHAPTER REVIEW EXERCISES

In Exercises 1–4, refer to the function $f(x)$ whose graph is shown in Figure 1.

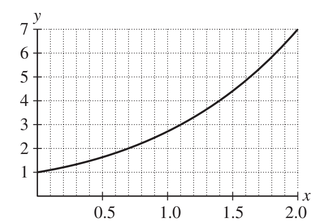


FIGURE 1

1. Compute the average rate of change of $f(x)$ over [0, 2]. What is the graphical interpretation of this average rate?

SOLUTION The average rate of change of $f(x)$ over [0, 2] is

$$\frac{f(2)-f(0)}{2-0} = \frac{7-1}{2-0} = 3.$$

Graphically, this average rate of change represents the slope of the secant line through the points (2, 7) and (0, 1) on the graph of $f(x)$.

3. Estimate $\frac{f(0.7+h)-f(0.7)}{h}$ for $h = 0.3$. Is this number larger or smaller than $f'(0.7)$?

SOLUTION For $h = 0.3$,

$$\frac{f(0.7+h)-f(0.7)}{h} = \frac{f(1)-f(0.7)}{0.3} \approx \frac{2.8-2}{0.3} = \frac{8}{3}.$$

Because the curve is concave up, the slope of the secant line is larger than the slope of the tangent line, so the value of the difference quotient should be larger than the value of the derivative.

In Exercises 5–8, compute $f'(a)$ using the limit definition and find an equation of the tangent line to the graph of $f(x)$ at $x = a$.

5. $f(x) = x^2 - x$, $a = 1$

SOLUTION Let $f(x) = x^2 - x$ and $a = 1$. Then

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{(1+h)^2-(1+h)-(1^2-1)}{h}$$

$$= \lim_{h\to 0}\frac{1+2h+h^2-1-h}{h} = \lim_{h\to 0}(1+h) = 1$$

and the equation of the tangent line to the graph of $f(x)$ at $x = a$ is

$$y = f'(a)(x-a)+f(a) = 1(x-1)+0 = x-1.$$

7. $f(x) = x^{-1}, \quad a = 4$

SOLUTION Let $f(x) = x^{-1}$ and $a = 4$. Then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{4+h} - \frac{1}{4}}{h} = \lim_{h\to 0} \frac{4-(4+h)}{4h(4+h)}$$
$$= \lim_{h\to 0} \frac{-1}{4(4+h)} = -\frac{1}{4(4+0)} = -\frac{1}{16}$$

and the equation of the tangent line to the graph of $f(x)$ at $x = a$ is

$$y = f'(a)(x-a) + f(a) = -\frac{1}{16}(x-4) + \frac{1}{4} = -\frac{1}{16}x + \frac{1}{2}.$$

In Exercises 9–12, compute dy/dx using the limit definition.

9. $y = 4 - x^2$

SOLUTION Let $y = 4 - x^2$. Then

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{4-(x+h)^2-(4-x^2)}{h} = \lim_{h\to 0} \frac{4-x^2-2xh-h^2-4+x^2}{h} = \lim_{h\to 0}(-2x-h) = -2x-0 = -2x.$$

11. $y = \dfrac{1}{2-x}$

SOLUTION Let $y = \dfrac{1}{2-x}$. Then

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\frac{1}{2-(x+h)} - \frac{1}{2-x}}{h} = \lim_{h\to 0} \frac{(2-x)-(2-x-h)}{h(2-x-h)(2-x)} = \lim_{h\to 0} \frac{1}{(2-x-h)(2-x)} = \frac{1}{(2-x)^2}.$$

In Exercises 13–16, express the limit as a derivative.

13. $\displaystyle\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}$

SOLUTION Let $f(x) = \sqrt{x}$. Then

$$\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = f'(1).$$

15. $\displaystyle\lim_{t\to \pi} \frac{\sin t \cos t}{t-\pi}$

SOLUTION Let $f(t) = \sin t \cos t$ and note that $f(\pi) = \sin\pi \cos\pi = 0$. Then

$$\lim_{t\to\pi} \frac{\sin t\cos t}{t-\pi} = \lim_{t\to\pi} \frac{f(t)-f(\pi)}{t-\pi} = f'(\pi).$$

17. Find $f(4)$ and $f'(4)$ if the tangent line to the graph of $f(x)$ at $x = 4$ has equation $y = 3x - 14$.

SOLUTION The equation of the tangent line to the graph of $f(x)$ at $x = 4$ is $y = f'(4)(x-4) + f(4) = f'(4)x + (f(4) - 4f'(4))$. Matching this to $y = 3x - 14$, we see that $f'(4) = 3$ and $f(4) - 4(3) = -14$, so $f(4) = -2$.

19. Is (A), (B), or (C) the graph of the derivative of the function $f(x)$ shown in Figure 3?

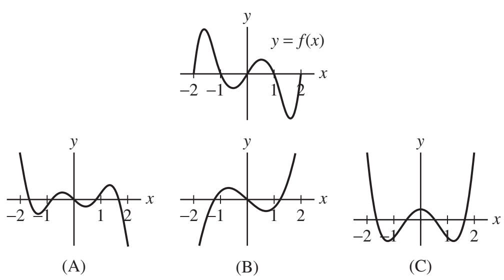


FIGURE 3

SOLUTION The graph of $f(x)$ has four horizontal tangent lines on $[-2, 2]$, so the graph of its derivative must have four x-intercepts on $[-2, 2]$. This eliminates (B). Moreover, $f(x)$ is increasing at both ends of the interval, so its derivative must be positive at both ends. This eliminates (A) and identifies (C) as the graph of $f'(x)$.

21. A girl's height $h(t)$ (in centimeters) is measured at time t (in years) for $0 \le t \le 14$:

52, 75.1, 87.5, 96.7, 104.5, 111.8, 118.7, 125.2,
131.5, 137.5, 143.3, 149.2, 155.3, 160.8, 164.7

(a) What is the average growth rate over the 14-year period?
(b) Is the average growth rate larger over the first half or the second half of this period?
(c) Estimate $h'(t)$ (in centimeters per year) for $t = 3, 8$.

SOLUTION
(a) The average growth rate over the 14-year period is

$$\frac{164.7 - 52}{14} = 8.05 \text{ cm/year}.$$

(b) Over the first half of the 14-year period, the average growth rate is

$$\frac{125.2 - 52}{7} \approx 10.46 \text{ cm/year},$$

which is larger than the average growth rate over the second half of the 14-year period:

$$\frac{164.7 - 125.2}{7} \approx 5.64 \text{ cm/year}.$$

(c) For $t = 3$,

$$h'(3) \approx \frac{h(4) - h(3)}{4 - 3} = \frac{104.5 - 96.7}{1} = 7.8 \text{ cm/year};$$

for $t = 8$,

$$h'(8) \approx \frac{h(9) - h(8)}{9 - 8} = \frac{137.5 - 131.5}{1} = 6.0 \text{ cm/year}.$$

In Exercises 23 and 24, use the following table of values for the number $A(t)$ of automobiles (in millions) manufactured in the United States in year t.

t	1970	1971	1972	1973	1974	1975	1976
$A(t)$	6.55	8.58	8.83	9.67	7.32	6.72	8.50

23. What is the interpretation of $A'(t)$? Estimate $A'(1971)$. Does $A'(1974)$ appear to be positive or negative?

SOLUTION Because $A(t)$ measures the number of automobiles manufactured in the United States in year t, $A'(t)$ measures the rate of change in automobile production in the United States. For $t = 1971$,

$$A'(1971) \approx \frac{A(1972) - A(1971)}{1972 - 1971} = \frac{8.83 - 8.58}{1} = 0.25 \text{ million automobiles/year}.$$

Because $A(t)$ decreases from 1973 to 1974 and from 1974 to 1975, it appears that $A'(1974)$ would be negative.

25. Which of the following is equal to $\frac{d}{dx}2^x$?

(a) 2^x **(b)** $(\ln 2)2^x$ **(c)** $x2^{x-1}$ **(d)** $\frac{1}{\ln 2}2^x$

SOLUTION The derivative of $f(x) = 2^x$ is

$$\frac{d}{dx}2^x = 2^x \ln 2.$$

Hence, the correct answer is **(b)**.

27. Show that if $f(x)$ is a function satisfying $f'(x) = f(x)^2$, then its inverse $g(x)$ satisfies $g'(x) = x^{-2}$.

SOLUTION

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f(g(x))^2} = \frac{1}{x^2} = x^{-2}.$$

In Exercises 29–80, compute the derivative.

29. $y = 3x^5 - 7x^2 + 4$

SOLUTION Let $y = 3x^5 - 7x^2 + 4$. Then

$$\frac{dy}{dx} = 15x^4 - 14x.$$

31. $y = t^{-7.3}$

SOLUTION Let $y = t^{-7.3}$. Then

$$\frac{dy}{dt} = -7.3t^{-8.3}.$$

33. $y = \dfrac{x+1}{x^2+1}$

SOLUTION Let $y = \dfrac{x+1}{x^2+1}$. Then

$$\frac{dy}{dx} = \frac{(x^2+1)(1) - (x+1)(2x)}{(x^2+1)^2} = \frac{1 - 2x - x^2}{(x^2+1)^2}.$$

35. $y = (x^4 - 9x)^6$

SOLUTION Let $y = (x^4 - 9x)^6$. Then

$$\frac{dy}{dx} = 6(x^4 - 9x)^5 \frac{d}{dx}(x^4 - 9x) = 6(4x^3 - 9)(x^4 - 9x)^5.$$

37. $y = (2 + 9x^2)^{3/2}$

SOLUTION Let $y = (2 + 9x^2)^{3/2}$. Then

$$\frac{dy}{dx} = \frac{3}{2}(2 + 9x^2)^{1/2} \frac{d}{dx}(2 + 9x^2) = 27x(2 + 9x^2)^{1/2}.$$

39. $y = \dfrac{z}{\sqrt{1-z}}$

SOLUTION Let $y = \dfrac{z}{\sqrt{1-z}}$. Then

$$\frac{dy}{dz} = \frac{\sqrt{1-z} - (-\frac{z}{2})\frac{1}{\sqrt{1-z}}}{1-z} = \frac{1 - z + \frac{z}{2}}{(1-z)^{3/2}} = \frac{2-z}{2(1-z)^{3/2}}.$$

41. $y = \dfrac{x^4 + \sqrt{x}}{x^2}$

SOLUTION Let

$$y = \frac{x^4 + \sqrt{x}}{x^2} = x^2 + x^{-3/2}.$$

Then

$$\frac{dy}{dx} = 2x - \frac{3}{2}x^{-5/2}.$$

43. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

SOLUTION Let $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$. Then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2} \frac{d}{dx}\left(x + \sqrt{x + \sqrt{x}}\right) \\
&= \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2} \left(1 + \frac{1}{2}\left(x + \sqrt{x}\right)^{-1/2} \frac{d}{dx}\left(x + \sqrt{x}\right)\right) \\
&= \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2} \left(1 + \frac{1}{2}\left(x + \sqrt{x}\right)^{-1/2} \left(1 + \frac{1}{2}x^{-1/2}\right)\right).
\end{aligned}$$

45. $y = \tan(t^{-3})$

SOLUTION Let $y = \tan(t^{-3})$. Then

$$\frac{dy}{dt} = \sec^2(t^{-3})\frac{d}{dt}t^{-3} = -3t^{-4}\sec^2(t^{-3}).$$

47. $y = \sin(2x)\cos^2 x$

SOLUTION Let $y = \sin(2x)\cos^2 x = 2\sin x \cos^3 x$. Then

$$\frac{dy}{dx} = -6\sin^2 x \cos^2 x + 2\cos^4 x.$$

49. $y = \dfrac{t}{1 + \sec t}$

SOLUTION Let $y = \dfrac{t}{1 + \sec t}$. Then

$$\frac{dy}{dt} = \frac{1 + \sec t - t \sec t \tan t}{(1 + \sec t)^2}.$$

51. $y = \dfrac{8}{1 + \cot\theta}$

SOLUTION Let $y = \dfrac{8}{1 + \cot\theta} = 8(1 + \cot\theta)^{-1}$. Then

$$\frac{dy}{d\theta} = -8(1 + \cot\theta)^{-2}\frac{d}{d\theta}(1 + \cot\theta) = \frac{8\csc^2\theta}{(1 + \cot\theta)^2}.$$

53. $y = \tan(\sqrt{1 + \csc\theta})$

SOLUTION

$$\begin{aligned}\frac{dy}{dx} &= \sec^2(\sqrt{1 + \csc\theta})\frac{d}{dx}\sqrt{1 + \csc\theta} \\ &= \sec^2(\sqrt{1 + \csc\theta}) \cdot \frac{1}{2}(1 + \csc\theta)^{-1/2}\frac{d}{dx}(1 + \csc\theta) \\ &= -\frac{\sec^2(\sqrt{1 + \csc\theta})\csc\theta\cot\theta}{2(\sqrt{1 + \csc\theta})}.\end{aligned}$$

55. $f(x) = 9e^{-4x}$

SOLUTION $\dfrac{d}{dx}9e^{-4x} = -36e^{-4x}$.

57. $g(t) = e^{4t-t^2}$

SOLUTION $\dfrac{d}{dt}e^{4t-t^2} = (4 - 2t)e^{4t-t^2}$.

59. $f(x) = \ln(4x^2 + 1)$

SOLUTION $\dfrac{d}{dx}\ln(4x^2 + 1) = \dfrac{8x}{4x^2 + 1}$.

61. $G(s) = (\ln(s))^2$

SOLUTION $\dfrac{d}{ds}(\ln s)^2 = \dfrac{2\ln s}{s}$.

63. $f(\theta) = \ln(\sin\theta)$

SOLUTION $\dfrac{d}{d\theta}\ln(\sin\theta) = \dfrac{\cos\theta}{\sin\theta} = \cot\theta$.

65. $h(z) = \sec(z + \ln z)$

SOLUTION $\dfrac{d}{dz}\sec(z + \ln z) = \sec(z + \ln z)\tan(z + \ln z)\left(1 + \dfrac{1}{z}\right)$.

67. $f(x) = 7^{-2x}$

SOLUTION $\dfrac{d}{dx}7^{-2x} = \big(-2\ln 7\big)\big(7^{-2x}\big)$.

69. $g(x) = \tan^{-1}(\ln x)$

SOLUTION $\frac{d}{dx}\tan^{-1}(\ln x) = \frac{1}{1+(\ln x)^2}\cdot\frac{1}{x}$.

71. $f(x) = \ln(\csc^{-1} x)$

SOLUTION $\frac{d}{dx}\ln(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}\csc^{-1} x}$.

73. $R(s) = s^{\ln s}$

SOLUTION Rewrite

$$R(s) = \left(e^{\ln s}\right)^{\ln s} = e^{(\ln s)^2}.$$

Then

$$\frac{dR}{ds} = e^{(\ln s)^2}\cdot 2\ln s\cdot\frac{1}{s} = \frac{2\ln s}{s}s^{\ln s}.$$

Alternately, $R(s) = s^{\ln s}$ implies that $\ln R = \ln\left(s^{\ln s}\right) = (\ln s)^2$. Thus,

$$\frac{1}{R}\frac{dR}{ds} = 2\ln s\cdot\frac{1}{s} \quad\text{or}\quad \frac{dR}{ds} = \frac{2\ln s}{s}s^{\ln s}.$$

75. $G(t) = (\sin^2 t)^t$

SOLUTION Rewrite

$$G(t) = \left(e^{\ln\sin^2 t}\right)^t = e^{2t\ln\sin t}.$$

Then

$$\frac{dG}{dt} = e^{2t\ln\sin t}\left(2t\cdot\frac{\cos t}{\sin t} + 2\ln\sin t\right) = 2(\sin^2 t)^t(t\cot t + \ln\sin t).$$

Alternately, $G(t) = (\sin^2 t)^t$ implies that $\ln G = t\ln\sin^2 t = 2t\ln\sin t$. Thus,

$$\frac{1}{G}\frac{dG}{dt} = 2t\cdot\frac{\cos t}{\sin t} + 2\ln\sin t,$$

and

$$\frac{dG}{dt} = 2(\sin^2 t)^t(t\cot t + \ln\sin t).$$

77. $g(t) = \sinh(t^2)$

SOLUTION $\frac{d}{dt}\sinh(t^2) = 2t\cosh(t^2)$.

79. $g(x) = \tanh^{-1}(e^x)$

SOLUTION $\frac{d}{dx}\tanh^{-1}(e^x) = \frac{1}{1-(e^x)^2}e^x = \frac{e^x}{1-e^{2x}}$.

81. For which values of α is $f(x) = |x|^\alpha$ differentiable at $x = 0$?

SOLUTION Let $f(x) = |x|^\alpha$. If $\alpha < 0$, then $f(x)$ is not continuous at $x = 0$ and therefore cannot be differentiable at $x = 0$. If $\alpha = 0$, then the function reduces to $f(x) = 1$, which is differentiable at $x = 0$. Now, suppose $\alpha > 0$ and consider the limit

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{|x|^\alpha}{x}.$$

If $0 < \alpha < 1$, then

$$\lim_{x\to 0-}\frac{|x|^\alpha}{x} = -\infty \qquad\text{while}\qquad \lim_{x\to 0+}\frac{|x|^\alpha}{x} = \infty$$

and $f'(0)$ does not exist. If $\alpha = 1$, then

$$\lim_{x\to 0-} \frac{|x|}{x} = -1 \qquad \text{while} \qquad \lim_{x\to 0+} \frac{|x|}{x} = 1$$

and $f'(0)$ again does not exist. Finally, if $\alpha > 1$, then

$$\lim_{x\to 0} \frac{|x|^\alpha}{x} = 0,$$

so $f'(0)$ does exist.

In summary, $f(x) = |x|^\alpha$ is differentiable at $x = 0$ when $\alpha = 0$ and when $\alpha > 1$.

In Exercises 83 and 84, let $f(x) = xe^{-x}$.

83. Show that $f(x)$ has an inverse on $[1, \infty)$. Let $g(x)$ be this inverse. Find the domain and range of $g(x)$ and compute $g'(2e^{-2})$.

SOLUTION Let $f(x) = xe^{-x}$. Then $f'(x) = e^{-x}(1-x)$. On $[1, \infty)$, $f'(x) < 0$, so $f(x)$ is decreasing and therefore one-to-one. It follows that $f(x)$ has an inverse on $[1, \infty)$. Let $g(x)$ denote this inverse. Because $f(1) = e^{-1}$ and $f(x) \to 0$ as $x \to \infty$, the domain of $g(x)$ is $(0, e^{-1}]$, and the range is $[1, \infty)$.

To determine $g'(2e^{-2})$, we use the formula $g'(x) = 1/f'(g(x))$. Because $f(2) = 2e^{-2}$, it follows that $g(2e^{-2}) = 2$. Then,

$$g'(2e^{-2}) = \frac{1}{f'(g(2e^{-2}))} = \frac{1}{f'(2)} = \frac{1}{-e^{-2}} = -e^2.$$

In Exercises 85–90, use the following table of values to calculate the derivative of the given function at $x = 2$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	5	4	−3	9
4	3	2	−2	3

85. $S(x) = 3f(x) - 2g(x)$

SOLUTION Let $S(x) = 3f(x) - 2g(x)$. Then $S'(x) = 3f'(x) - 2g'(x)$ and

$$S'(2) = 3f'(2) - 2g'(2) = 3(-3) - 2(9) = -27.$$

87. $R(x) = \dfrac{f(x)}{g(x)}$

SOLUTION Let $R(x) = f(x)/g(x)$. Then

$$R'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

and

$$R'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{g(2)^2} = \frac{4(-3) - 5(9)}{4^2} = -\frac{57}{16}.$$

89. $F(x) = f(g(2x))$

SOLUTION Let $F(x) = f(g(2x))$. Then $F'(x) = 2f'(g(2x))g'(2x)$ and

$$F'(2) = 2f'(g(4))g'(4) = 2f'(2)g'(4) = 2(-3)(3) = -18.$$

91. Find the points on the graph of $f(x) = x^3 - 3x^2 + x + 4$ where the tangent line has slope 10.

SOLUTION Let $f(x) = x^3 - 3x^2 + x + 4$. Then $f'(x) = 3x^2 - 6x + 1$. The tangent line to the graph of $f(x)$ will have slope 10 when $f'(x) = 10$. Solving the quadratic equation $3x^2 - 6x + 1 = 10$ yields $x = -1$ and $x = 3$. Thus, the points on the graph of $f(x)$ where the tangent line has slope 10 are $(-1, -1)$ and $(3, 7)$.

93. Find a such that the tangent lines $y = x^3 - 2x^2 + x + 1$ at $x = a$ and $x = a + 1$ are parallel.

SOLUTION Let $f(x) = x^3 - 2x^2 + x + 1$. Then $f'(x) = 3x^2 - 4x + 1$ and the slope of the tangent line at $x = a$ is $f'(a) = 3a^2 - 4a + 1$, while the slope of the tangent line at $x = a + 1$ is

$$f'(a+1) = 3(a+1)^2 - 4(a+1) + 1 = 3(a^2 + 2a + 1) - 4a - 4 + 1 = 3a^2 + 2a.$$

In order for the tangent lines at $x = a$ and $x = a + 1$ to have the same slope, we must have $f'(a) = f'(a+1)$, or

$$3a^2 - 4a + 1 = 3a^2 + 2a.$$

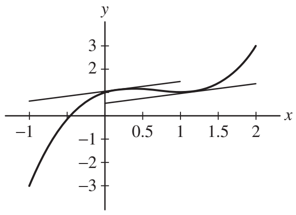
The only solution to this equation is $a = \frac{1}{6}$. The equation of the tangent line at $x = \frac{1}{6}$ is

$$y = f'\left(\frac{1}{6}\right)\left(x - \frac{1}{6}\right) + f\left(\frac{1}{6}\right) = \frac{5}{12}\left(x - \frac{1}{6}\right) + \frac{241}{216} = \frac{5}{12}x + \frac{113}{108},$$

and the equation of the tangent line at $x = \frac{7}{6}$ is

$$y = f'\left(\frac{7}{6}\right)\left(x - \frac{7}{6}\right) + f\left(\frac{7}{6}\right) = \frac{5}{12}\left(x - \frac{7}{6}\right) + \frac{223}{216} = \frac{5}{12}x + \frac{59}{108}.$$

The graphs of $f(x)$ and the two tangent lines appear below.



In Exercises 95–100, calculate y''.

95. $y = 12x^3 - 5x^2 + 3x$

SOLUTION Let $y = 12x^3 - 5x^2 + 3x$. Then

$$y' = 36x^2 - 10x + 3 \quad \text{and} \quad y'' = 72x - 10.$$

97. $y = \sqrt{2x+3}$

SOLUTION Let $y = \sqrt{2x+3} = (2x+3)^{1/2}$. Then

$$y' = \frac{1}{2}(2x+3)^{-1/2}\frac{d}{dx}(2x+3) = (2x+3)^{-1/2} \quad \text{and} \quad y'' = -\frac{1}{2}(2x+3)^{-3/2}\frac{d}{dx}(2x+3) = -(2x+3)^{-3/2}.$$

99. $y = \tan(x^2)$

SOLUTION Let $y = \tan(x^2)$. Then

$$y' = 2x\sec^2(x^2) \quad \text{and}$$
$$y'' = 2x\left(2\sec(x^2)\frac{d}{dx}\sec(x^2)\right) + 2\sec^2(x^2) = 8x^2\sec^2(x^2)\tan(x^2) + 2\sec^2(x^2).$$

In Exercises 101–106, compute $\frac{dy}{dx}$.

101. $x^3 - y^3 = 4$

SOLUTION Consider the equation $x^3 - y^3 = 4$. Differentiating with respect to x yields

$$3x^2 - 3y^2\frac{dy}{dx} = 0.$$

Therefore,

$$\frac{dy}{dx} = \frac{x^2}{y^2}.$$

103. $y = xy^2 + 2x^2$

SOLUTION Consider the equation $y = xy^2 + 2x^2$. Differentiating with respect to x yields

$$\frac{dy}{dx} = 2xy\frac{dy}{dx} + y^2 + 4x.$$

Therefore,

$$\frac{dy}{dx} = \frac{y^2 + 4x}{1 - 2xy}.$$

105. $y = \sin(x + y)$

SOLUTION Consider the equation $y = \sin(x + y)$. Differentiating with respect to x yields

$$\frac{dy}{dx} = \cos(x + y)\left(1 + \frac{dy}{dx}\right).$$

Therefore,

$$\frac{dy}{dx} = \frac{\cos(x + y)}{1 - \cos(x + y)}.$$

107. In Figure 5, label the graphs f, f', and f''.

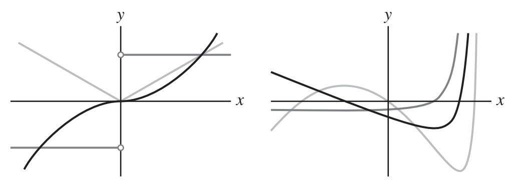


FIGURE 5

SOLUTION First consider the plot on the left. Observe that the green curve is nonnegative whereas the red curve is increasing, suggesting that the green curve is the derivative of the red curve. Moreover, the green curve is linear with negative slope for $x < 0$ and linear with positive slope for $x > 0$ while the blue curve is a negative constant for $x < 0$ and a positive constant for $x > 0$, suggesting the blue curve is the derivative of the green curve. Thus, the red, green and blue curves, respectively, are the graphs of f, f' and f''.

Now consider the plot on the right. Because the red curve is decreasing when the blue curve is negative and increasing when the blue curve is positive and the green curve is decreasing when the red curve is negative and increasing when the red curve is positive, it follows that the green, red and blue curves, respectively, are the graphs of f, f' and f''.

In Exercises 109–114, use logarithmic differentiation to find the derivative.

109. $y = \dfrac{(x + 1)^3}{(4x - 2)^2}$

SOLUTION Let $y = \dfrac{(x + 1)^3}{(4x - 2)^2}$. Then

$$\ln y = \ln\left(\frac{(x + 1)^3}{(4x - 2)^2}\right) = \ln\,(x + 1)^3 - \ln\,(4x - 2)^2 = 3\ln(x + 1) - 2\ln(4x - 2).$$

By logarithmic differentiation,

$$\frac{y'}{y} = \frac{3}{x + 1} - \frac{2}{4x - 2}\cdot 4 = \frac{3}{x + 1} - \frac{4}{2x - 1},$$

so

$$y' = \frac{(x + 1)^3}{(4x - 2)^2}\left(\frac{3}{x + 1} - \frac{4}{2x - 1}\right).$$

111. $y = e^{(x-1)^2}e^{(x-3)^2}$

SOLUTION Let $y = e^{(x-1)^2}e^{(x-3)^2}$. Then

$$\ln y = \ln\left(e^{(x-1)^2}e^{(x-3)^2}\right) = \ln\left(e^{(x-1)^2+(x-3)^2}\right) = (x - 1)^2 + (x - 3)^2.$$

By logarithmic differentiation,

$$\frac{y'}{y} = 2(x - 1) + 2(x - 3) = 4x - 8,$$

so

$$y' = 4e^{(x-1)^2}e^{(x-3)^2}(x - 2).$$

113. $y = \dfrac{e^{3x}(x-2)^2}{(x+1)^2}$

SOLUTION Let $y = \dfrac{e^{3x}(x-2)^2}{(x+1)^2}$. Then

$$\ln y = \ln\left(\frac{e^{3x}(x-2)^2}{(x+1)^2}\right) = \ln e^{3x} + \ln(x-2)^2 - \ln(x+1)^2$$
$$= 3x + 2\ln(x-2) - 2\ln(x+1).$$

By logarithmic differentiation,

$$\frac{y'}{y} = 3 + \frac{2}{x-2} - \frac{2}{x+1},$$

so

$$y = \frac{e^{3x}(x-2)^2}{(x+1)^2}\left(3 + \frac{2}{x-2} - \frac{2}{x+1}\right).$$

Exercises 115–117: Let q be the number of units of a product (cell phones, barrels of oil, etc.) that can be sold at the price p. The ***price elasticity of demand*** *E is defined as the percentage rate of change of q with respect to p. In terms of derivatives,*

$$E = \frac{p}{q}\frac{dq}{dp} = \lim_{\Delta p \to 0} \frac{(100\Delta q)/q}{(100\Delta p)/p}$$

115. Show that the total revenue $R = pq$ satisfies $\dfrac{dR}{dp} = q(1+E)$.

SOLUTION Let $R = pq$. Then

$$\frac{dR}{dp} = p\frac{dq}{dp} + q = q\frac{p}{q}\frac{dq}{dp} + q = q(E+1).$$

117. The monthly demand (in thousands) for flights between Chicago and St. Louis at the price p is $q = 40 - 0.2p$. Calculate the price elasticity of demand when $p = \$150$ and estimate the percentage increase in number of additional passengers if the ticket price is lowered by 1%.

SOLUTION Let $q = 40 - 0.2p$. Then $q'(p) = -0.2$ and

$$E(p) = \left(\frac{p}{q}\right)\frac{dq}{dp} = \frac{0.2p}{0.2p - 40}.$$

For $p = 150$,

$$E(150) = \frac{0.2(150)}{0.2(150) - 40} = -3,$$

so a 1% decrease in price increases demand by 3%. The demand when $p = 150$ is $q = 40 - 0.2(150) = 10$, or 10000 passengers. Therefore, a 1% increase in demand translates to 300 additional passengers.

119. The minute hand of a clock is 8 cm long, and the hour hand is 5 cm long. How fast is the distance between the tips of the hands changing at 3 o'clock?

SOLUTION Let S be the distance between the tips of the two hands. By the law of cosines

$$S^2 = 8^2 + 5^2 - 2 \cdot 8 \cdot 5\cos(\theta),$$

where θ is the angle between the hands. Thus

$$2S\frac{dS}{dt} = 80\sin(\theta)\frac{d\theta}{dt}.$$

At three o'clock $\theta = \pi/2$, $S = \sqrt{89}$, and

$$\frac{d\theta}{dt} = \left(\frac{\pi}{360} - \frac{\pi}{30}\right) \text{ rad/min} = -\frac{11\pi}{360} \text{ rad/min},$$

so

$$\frac{dS}{dt} = \frac{1}{2\sqrt{89}}(80)(1)\frac{-11\pi}{360} \approx -0.407 \text{ cm/min}.$$

121. A bead slides down the curve $xy = 10$. Find the bead's horizontal velocity at time $t = 2$ s if its height at time t seconds is $y = 400 - 16t^2$ cm.

SOLUTION Let $xy = 10$. Then $x = 10/y$ and

$$\frac{dx}{dt} = -\frac{10}{y^2}\frac{dy}{dt}.$$

If $y = 400 - 16t^2$, then $\frac{dy}{dt} = -32t$ and

$$\frac{dx}{dt} = -\frac{10}{(400 - 16t^2)^2}(-32t) = \frac{320t}{(400 - 16t^2)^2}.$$

Thus, at $t = 2$,

$$\frac{dx}{dt} = \frac{640}{(336)^2} \approx 0.00567 \text{ cm/s}.$$

123. A light moving at 0.8 m/s approaches a man standing 4 m from a wall (Figure 9). The light is 1 m above the ground. How fast is the tip P of the man's shadow moving when the light is 7 m from the wall?

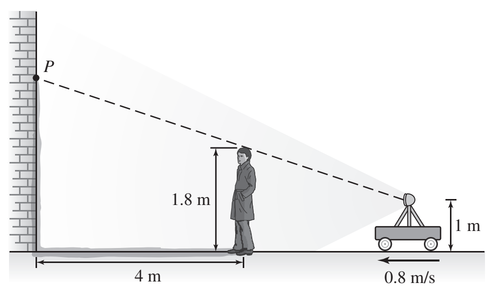


FIGURE 9

SOLUTION Let x denote the distance between the man and the light. Using similar triangles, we find

$$\frac{0.8}{x} = \frac{P-1}{4+x} \qquad \text{or} \qquad P = \frac{3.2}{x} + 1.8.$$

Therefore,

$$\frac{dP}{dt} = -\frac{3.2}{x^2}\frac{dx}{dt}.$$

When the light is 7 feet from the wall, $x = 3$. With $\frac{dx}{dt} = -0.8$, we have

$$\frac{dP}{dt} = -\frac{3.2}{3^2}(-0.8) = 0.284 \text{ m/s}.$$