4 APPLICATIONS OF THE DERIVATIVE

4.1 Linear Approximation and Applications

Preliminary Questions
1. True or False? The Linear Approximation says that the vertical change in the graph is approximately equal to the vertical change in the tangent line.

SOLUTION This statement is true. The linear approximation does say that the vertical change in the graph is approximately equal to the vertical change in the tangent line.

2. Estimate \( g(1.2) - g(1) \) if \( g'(1) = 4 \).

SOLUTION Using the Linear Approximation,
\[
g(1.2) - g(1) \approx g'(1)(1.2 - 1) = 4(0.2) = 0.8.
\]

3. Estimate \( f(2.1) \) if \( f(2) = 1 \) and \( f'(2) = 3 \).

SOLUTION Using the Linearization,
\[
f(2.1) \approx f(2) + f'(2)(2.1 - 2) = 1 + 3(0.1) = 1.3
\]

4. Complete the sentence: The Linear Approximation tells us that up to a small error, the change in output \( \Delta f \) is directly proportional to …. 

SOLUTION The Linear Approximation shows that up to a small error, the change in output \( \Delta f \) is directly proportional to the change in input \( \Delta x \) when \( \Delta x \) is small.

Exercises

In Exercises 1–6, use Eq. (1) to estimate \( \Delta f = f(3.02) - f(3) \).

1. \( f(x) = x^2 \)

SOLUTION Let \( f(x) = x^2 \). Then \( f'(x) = 2x \) and \( \Delta f \approx f'(3)\Delta x = 6(0.02) = 0.12 \).

3. \( f(x) = x^{-1} \)

SOLUTION Let \( f(x) = x^{-1} \). Then \( f'(x) = -x^{-2} \) and \( \Delta f \approx f'(3)\Delta x = -\frac{1}{9}(0.02) = -0.00222 \).

5. \( f(x) = \sqrt{x + 6} \)

SOLUTION Let \( f(x) = \sqrt{x + 6} \). Then \( f'(x) = \frac{1}{2}(x + 6)^{-1/2} \) and
\[
\Delta f \approx f'(3)\Delta x = \frac{1}{2}g^{-1/2}(0.02) = 0.00333.
\]

7. The cube root of 27 is 3. How much larger is the cube root of 27.2? Estimate using the Linear Approximation.

SOLUTION Let \( f(x) = x^{1/3}, a = 27 \), and \( \Delta x = 0.2 \). Then \( f'(x) = \frac{1}{3}x^{-2/3} \) and \( f'(a) = f'(27) = \frac{1}{27} \). The Linear Approximation is
\[
\Delta f \approx f'(a)\Delta x = \frac{1}{27}(0.2) = 0.0074074
\]

In Exercises 9–12, use Eq. (1) to estimate \( \Delta f \). Use a calculator to compute both the error and the percentage error.

9. \( f(x) = \sqrt{1 + x}, \ a = 3, \ \Delta x = 0.2 \)

SOLUTION Let \( f(x) = (1 + x)^{1/2}, a = 3, \) and \( \Delta x = 0.2 \). Then \( f'(x) = \frac{1}{2}(1 + x)^{-1/2} \), \( f'(a) = f'(3) = \frac{1}{2} \) and \( \Delta f \approx f'(a)\Delta x = \frac{1}{2}(0.2) = 0.05 \). The actual change is
\[
\Delta f = f(a + \Delta x) - f(a) = f(3.2) - f(3) = \sqrt{4.2} - 2 \approx 0.049390.
\]
The error in the Linear Approximation is therefore \(|0.049390 - 0.05| = 0.000610\); in percentage terms, the error is 
\[
\frac{0.000610}{0.049390} \times 100\% \approx 1.24\%.
\]

11. \(f(x) = \frac{1}{1 + x^2}, \quad a = 3, \quad \Delta x = 0.5\)

**SOLUTION** Let \(f(x) = \frac{1}{1 + x^2}, \quad a = 3, \quad \Delta x = 0.5\). Then \(f'(x) = -\frac{2x}{(1 + x^2)^2}\), \(f'(a) = f'(3) = -0.06\) and \(\Delta f \approx f'(a)\Delta x = -0.06(0.5) = -0.03\). The actual change is 
\[
\Delta f = f(a + \Delta x) - f(a) = f(3.5) - f(3) \approx -0.0245283.
\]

The error in this estimate is 
\[
\frac{0.0054717}{-0.0245283} \times 100\% \approx 22.31\%.
\]

In Exercises 13–16, estimate \(\Delta y\) using differentials [Eq. (3)].

13. \(y = \cos x, \quad a = \frac{\pi}{6}, \quad dx = 0.014\)

**SOLUTION** Let \(f(x) = \cos x\). Then \(f'(x) = -\sin x\) and 
\[
\Delta y \approx dy = f'(a)dx = -\sin \left( \frac{\pi}{6} \right)(0.014) = -0.007.
\]

15. \(y = \frac{10 - x^2}{2 + x^2}, \quad a = 1, \quad dx = 0.01\)

**SOLUTION** Let \(f(x) = \frac{10 - x^2}{2 + x^2}\). Then 
\[
f'(x) = \frac{(2 + x^2)(-2x) - (10 - x^2)(2x)}{(2 + x^2)^2} = -\frac{24x}{(2 + x^2)^2}
\]
and 
\[
\Delta y \approx dy = f'(a)dx = -\frac{24}{9}(0.01) = -0.026667.
\]

In Exercises 17–24, estimate using the Linear Approximation and find the error using a calculator:

17. \(\sqrt{26} - \sqrt{25}\)

**SOLUTION** Let \(f(x) = \sqrt{x}, \quad a = 25, \quad \Delta x = 1\). Then \(f'(x) = \frac{1}{2}x^{-1/2}\) and \(f'(a) = f'(25) = \frac{1}{10}\).

- The Linear Approximation is \(\Delta f \approx f'(a)\Delta x = \frac{1}{10}(1) = 0.1\).
- The actual change is \(\Delta f = f(a + \Delta x) - f(a) = f(26) - f(25) \approx 0.0990195\).
- The error in this estimate is \(|0.0990195 - 0.1| = 0.000980486\).

19. \(\frac{1}{\sqrt{10}} - \frac{1}{10}\)

**SOLUTION** Let \(f(x) = \frac{1}{\sqrt{x}}, \quad a = 100, \quad \Delta x = 1\). Then \(f'(x) = \frac{d}{dx}(x^{-1/2}) = -\frac{1}{2}x^{-3/2}\) and \(f'(a) = -\frac{1}{2}\left(\frac{1}{100}\right) = -0.0005\).

- The Linear Approximation is \(\Delta f \approx f'(a)\Delta x = -0.0005(1) = -0.0005\).
- The actual change is 
\[
\Delta f = f(a + \Delta x) - f(a) = \frac{1}{\sqrt{101}} - \frac{1}{10} = -0.000496281.
\]
- The error in this estimate is \(|-0.0005 - (-0.000496281)| = 3.71902 \times 10^{-6}\).

21. \(9^{1/3} - 2\)

**SOLUTION** Let \(f(x) = x^{1/3}, \quad a = 8, \quad \Delta x = 1\). Then \(f'(x) = \frac{1}{3}x^{-2/3}\) and \(f'(a) = f'(8) = \frac{1}{12}\).

- The Linear Approximation is \(\Delta f \approx f'(a)\Delta x = \frac{1}{12}(1) = 0.083333\).
- The actual change is \(\Delta f = f(a + \Delta x) - f(a) = f(9) - f(8) = 0.080084\).
- The error in this estimate is \(|0.080084 - 0.083333| \approx 3.25 \times 10^{-3}\).
23. \(e^{-0.1} - 1\)

**SOLUTION** Let \(f(x) = e^x, a = 0,\) and \(\Delta x = -0.1.\) Then \(f'(x) = e^x\) and \(f'(a) = f'(0) = 1.\)

- The Linear Approximation is \(\Delta f \approx f'(a)\Delta x = 1(-0.1) = -0.1.\)
- The actual change is \(\Delta f = f(a + \Delta x) - f(a) = f(-0.1) - f(0) = -0.095163.\)
- The error in this estimate is \(| - 0.095163 - (-0.1)| \approx 4.84 \times 10^{-3}.\)

25. Estimate \(f(4.03)\) for \(f(x)\) as in Figure 8.

**FIGURE 8**

![Graph](image)

**SOLUTION** Using the Linear Approximation, \(f(4.03) \approx f(4) + f'(4)(0.03).\) From the figure, we find that \(f(4) = 2\) and

\[
f'(4) = \frac{4 - 2}{10 - 4} = \frac{1}{3}
\]

Thus,

\[
f(4.03) \approx 2 + \frac{1}{3}(0.03) = 2.01.
\]

27. Which is larger: \(\sqrt{2.1} - \sqrt{2}\) or \(\sqrt{9.1} - \sqrt{9}?\) Explain using the Linear Approximation.

**SOLUTION** Let \(f(x) = \sqrt{x},\) and \(\Delta x = 0.1.\) Then \(f'(x) = \frac{1}{2}x^{-1/2}\) and the Linear Approximation at \(x = a\) gives

\[
\Delta f = \sqrt{a + 0.1} - \sqrt{a} \approx f'(a)(0.1) = \frac{1}{2}a^{-1/2}(0.1) = \frac{0.05}{\sqrt{a}}
\]

We see that \(\Delta f\) decreases as \(a\) increases. In particular

\[
\sqrt{2.1} - \sqrt{2} = \frac{0.05}{\sqrt{2}} \text{ is larger than } \sqrt{9.1} - \sqrt{9} = \frac{0.05}{3}
\]

29. Box office revenue at a multiplex cinema in Paris is \(R(p) = 3600p - 10p^3\) euros per showing when the ticket price is \(p\) euros. Calculate \(R(p)\) for \(p = 9\) and use the Linear Approximation to estimate \(\Delta R\) if \(p\) is raised or lowered by 0.5 euros.

**SOLUTION** Let \(R(p) = 3600p - 10p^3,\) Then \(R(9) = 3600(9) - 10(9)^3 = 25110\) euros. Moreover, \(R'(p) = 3600 - 30p^2,\) so by the Linear Approximation,

\[
\Delta R \approx R'(9)\Delta p = 1170\Delta p.
\]

If \(p\) is raised by 0.5 euros, then \(\Delta R \approx 585\) euros; on the other hand, if \(p\) is lowered by 0.5 euros, then \(\Delta R \approx -585\) euros.

31. A thin silver wire has length \(L = 18\text{ cm}\) when the temperature is \(T = 30^\circ\text{C}\). Estimate \(\Delta L\) when \(T\) decreases to \(25^\circ\text{C}\) if the coefficient of thermal expansion is \(k = 1.9 \times 10^{-5}\text{C}^{-1}\) (see Example 3).

**SOLUTION** We have

\[
\frac{dL}{dT} = kL = (1.9 \times 10^{-5})(18) = 3.42 \times 10^{-4}\text{ cm}/^\circ\text{C}
\]

The change in temperature is \(\Delta T = -5^\circ\text{C},\) so by the Linear Approximation, the change in length is approximately

\[
\Delta L \approx 3.42 \times 10^{-4}\Delta T = (3.42 \times 10^{-4})(-5) = -0.00171\text{ cm}
\]

At \(T = 25^\circ\text{C},\) the length of the wire is approximately 17.99829 cm.

33. The atmospheric pressure at altitude \(h\) (kilometers) for \(11 \leq h \leq 25\) is approximately

\[
P(h) = 128e^{-0.157h}\text{ kilopascals.}
\]

(a) Estimate \(\Delta P\) at \(h = 20\) when \(\Delta h = 0.5.\)

(b) Compute the actual change, and compute the percentage error in the Linear Approximation.
Linear Approximation and Applications

Accordingly, there is a bigger effect at higher velocities. Thus, \( h \)

The percentage error in the Linear Approximation is

\[ \frac{434906 - (-0.418274)}{-0.418274} \times 100\% \approx 3.98\%. \]

35. Newton’s Law of Gravitation shows that if a person weighs \( w \) pounds on the surface of the earth, then his or her weight at distance \( x \) from the center of the earth is

\[ W(x) = \frac{wR^2}{x^2} \quad \text{(for } x \geq R) \]

where \( R = 3960 \) miles is the radius of the earth (Figure 9).

(a) Show that the weight lost at altitude \( h \) miles above the earth’s surface is approximately \( \Delta W \approx -(0.0005u)h \). Hint: Use the Linear Approximation with \( dx = h \).

(b) Estimate the weight lost by a 200-lb football player flying in a jet at an altitude of 7 miles.

![FIGURE 9 The distance to the center of the earth is 3960 + h miles.](image)

37. A stone tossed vertically into the air with initial velocity \( v \) cm/s reaches a maximum height of \( h = v^2/1960 \) cm.

(a) Estimate \( \Delta h \) if \( v = 700 \) cm/s and \( \Delta v = 1 \) cm/s.

(b) Estimate \( \Delta h \) if \( v = 1000 \) cm/s and \( \Delta v = 1 \) cm/s.

(c) In general, does a 1 cm/s increase in \( v \) lead to a greater change in \( h \) at low or high initial velocities? Explain.

SOLUTION A stone tossed vertically with initial velocity \( v \) cm/s attains a maximum height of \( h(v) = v^2/1960 \) cm. Thus, \( h'(v) = v/980 \).

(a) If \( v = 700 \) and \( \Delta v = 1 \), then \( \Delta h \approx h'(v)\Delta v = \frac{1}{980}(700)(1) \approx 0.71 \) cm.

(b) If \( v = 1000 \) and \( \Delta v = 1 \), then \( \Delta h \approx h'(v)\Delta v = \frac{1}{980}(1000)(1) = 1.02 \) cm.

(c) A one centimeter per second increase in initial velocity \( v \) increases the maximum height by approximately \( v/980 \) cm. Accordingly, there is a bigger effect at higher velocities.

In Exercises 39 and 40, use the following fact derived from Newton’s Laws: An object released at an angle \( \theta \) with initial velocity \( v \) ft/s travels a horizontal distance

\[ s = \frac{1}{32} v^2 \sin 2\theta \text{ ft} \] (Figure 10)

May 23, 2011
39. A player located 18.1 ft from the basket launches a successful jump shot from a height of 10 ft (level with the rim of the basket), at an angle \( \theta = 34^\circ \) and initial velocity \( v = 25 \) ft/s.

(a) Show that \( \Delta s \approx 0.255\Delta \theta \) ft for a small change of \( \Delta \theta \).

(b) Is it likely that the shot would have been successful if the angle had been off by 2°?

**SOLUTION** Using Newton’s laws and the given initial velocity of \( v = 25 \) ft/s, the shot travels \( s = \frac{1}{2}v^2 \sin 2t = \frac{625}{16} \sin 2t \) ft, where \( t \) is in radians.

(a) If \( \theta = 34^\circ \) (i.e., \( t = \frac{17}{45} \pi \)), then

\[
\Delta s \approx s'(t)\Delta t = \frac{625}{16} \cos \left( \frac{17}{45} \pi \right) \Delta t \approx \frac{625}{16} \cos \left( \frac{17}{45} \pi \right) \Delta \theta \cdot \frac{\pi}{180} \approx 0.255\Delta \theta.
\]

(b) If \( \Delta \theta = 2^\circ \), this gives \( \Delta s \approx 0.51 \) ft, in which case the shot would not have been successful, having been off half a foot.

41. The radius of a spherical ball is measured at \( r = 25 \) cm. Estimate the maximum error in the volume and surface area if \( r \) is accurate to within 0.5 cm.

**SOLUTION** The volume and surface area of the sphere are given by \( V = \frac{4}{3} \pi r^3 \) and \( S = 4\pi r^2 \), respectively. If \( r = 25 \) and \( \Delta r = \pm 0.5 \), then

\[
\Delta V \approx V'(25)\Delta r = 4\pi(25)^2(0.5) \approx 3927 \text{ cm}^3,
\]

and

\[
\Delta S \approx S'(25)\Delta r = 8\pi(25)(0.5) \approx 314.2 \text{ cm}^2.
\]

43. The volume (in liters) and pressure \( P \) (in atmospheres) of a certain gas satisfy \( PV = 24 \). A measurement yields \( V = 4 \) with a possible error of \( \pm 0.3 \) L. Compute \( P \) and estimate the maximum error in this computation.

**SOLUTION** Given \( PV = 24 \) and \( V = 4 \), it follows that \( P = 6 \) atmospheres. Solving \( PV = 24 \) for \( P \) yields \( P = 24V^{-1} \).

Thus, \( P' = -24V^{-2} \) and

\[
\Delta P \approx P'(4)\Delta V = -24(4)^{-2}(\pm 0.3) = \pm 0.45 \text{ atmospheres}.
\]

In Exercises 45–54, find the linearization at \( x = a \).

45. \( f(x) = x^4, \quad a = 1 \)

**SOLUTION** Let \( f(x) = x^4 \). Then \( f'(x) = 4x^3 \). The linearization at \( a = 1 \) is

\[
L(x) = f'(a)(x-a) + f(a) = 4(x-1) + 1 = 4x - 3.
\]

47. \( f(\theta) = \sin^2 \theta, \quad a = \frac{\pi}{4} \)

**SOLUTION** Let \( f(\theta) = \sin^2 \theta \). Then \( f'(\theta) = 2\sin \theta \cos \theta = \sin 2\theta \). The linearization at \( a = \frac{\pi}{4} \) is

\[
L(\theta) = f'(a)(\theta-a) + f(a) = 1 \left( \theta - \frac{\pi}{4} \right) + \frac{1}{2} = \theta - \frac{\pi}{4} + \frac{1}{2}.
\]

49. \( y = (1 + x)^{-1/2}, \quad a = 0 \)

**SOLUTION** Let \( f(x) = (1 + x)^{-1/2} \). Then \( f'(x) = -\frac{1}{2}(1 + x)^{-3/2} \). The linearization at \( a = 0 \) is

\[
L(x) = f'(a)(x-a) + f(a) = -\frac{1}{2}x + 1.
\]

51. \( y = (1 + x^2)^{-1/2}, \quad a = 0 \)

**SOLUTION** Let \( f(x) = (1 + x^2)^{-1/2} \). Then \( f'(x) = -x(1 + x^2)^{-3/2} \). If \( a = 1 \) and \( f'(a) = 0 \), so the linearization at \( a \) is

\[
L(x) = f'(a)(x-a) + f(a) = 1.
\]
Thus, we have

\[ f'(x) = \frac{1}{2\sqrt{x}} e^{\sqrt{x}} \quad \text{and} \quad f'(a) = \frac{1}{2} e. \]

so the linearization of \( f(x) \) at \( a \) is

\[ L(x) = f'(a)(x - a) + f(a) = \frac{1}{2} e(x - 1) + e = \frac{1}{2} e(x + 1). \]

55. What is \( f(2) \) if the linearization of \( f(x) \) at \( a = 2 \) is \( L(x) = 2x + 4 \)?

**Solution**

\[ f(2) = L(2) = 2(2) + 4 = 8. \]

57. Estimate \( \sqrt{16.2} \) using the linearization \( L(x) \) of \( f(x) = \sqrt{x} \) at \( a = 16 \). Plot \( f(x) \) and \( L(x) \) on the same set of axes and determine whether the estimate is too large or too small.

**Solution**

Let \( f(x) = x^{1/2}, a = 16, \) and \( \Delta x = 0.2 \). Then \( f'(x) = \frac{1}{2} x^{-1/2} \) and \( f'(a) = f'(16) = \frac{1}{8} \). The linearization to \( f(x) \) is

\[ L(x) = f'(a)(x - a) + f(a) = \frac{1}{8}(x - 16) + 4 = \frac{1}{8} x + 2. \]

Thus, we have \( \sqrt{16.2} \approx L(16.2) = 4.025 \). Graphs of \( f(x) \) and \( L(x) \) are shown below. Because the graph of \( L(x) \) lies above the graph of \( f(x) \), we expect that the estimate from the Linear Approximation is too large.

---

**In Exercises 59–67, approximate using linearization and use a calculator to compute the percentage error.**

59. \( \frac{1}{\sqrt{17}} \)

**Solution**

Let \( f(x) = x^{-1/2}, a = 16, \) and \( \Delta x = 1 \). Then \( f'(x) = -\frac{1}{2} x^{-3/2}, f'(a) = f'(16) = -\frac{1}{128} \) and the linearization to \( f(x) \) is

\[ L(x) = f'(a)(x - a) + f(a) = -\frac{1}{128}(x - 16) + \frac{1}{4} = -\frac{1}{128} x + \frac{3}{8}. \]

Thus, we have \( \frac{1}{\sqrt{17}} \approx L(17) \approx 0.24219 \). The percentage error in this estimate is

\[ \left| \frac{\frac{1}{\sqrt{17}} - 0.24219}{\frac{1}{\sqrt{17}}} \right| \times 100\% \approx 0.14\% \]

61. \( \frac{1}{(10.03)^2} \)

**Solution**

Let \( f(x) = x^{-2}, a = 10 \) and \( \Delta x = 0.03 \). Then \( f'(x) = -2x^{-3}, f'(a) = f'(10) = -0.002 \) and the linearization to \( f(x) \) is

\[ L(x) = f'(a)(x - a) + f(a) = -0.002(x - 10) + 0.01 = -0.002x + 0.03. \]

Thus, we have

\[ \frac{1}{(10.03)^2} \approx L(10.03) = -0.002(10.03) + 0.03 = 0.00994. \]

The percentage error in this estimate is

\[ \left| \frac{\frac{1}{(10.03)^2} - 0.00994}{\frac{1}{(10.03)^2}} \right| \times 100\% \approx 0.0027\% \]
63. \((64.1)^{1/3}\)

**Solution** Let \(f(x) = x^{1/3}, a = 64,\) and \(\Delta x = 0.1.\) Then \(f'(x) = \frac{1}{3}x^{-2/3},\ \ f'(a) = f'(64) = \frac{1}{48}\) and the linearization to \(f(x)\) is

\[
L(x) = f'(a)(x - a) + f(a) = \frac{1}{48}(x - 64) + 4 = \frac{1}{48}x + \frac{8}{3}.
\]

Thus, we have \((64.1)^{1/3} \approx L(64.1) \approx 4.002083.\) The percentage error in this estimate is

\[
\left| \frac{(64.1)^{1/3} - 4.002083}{(64.1)^{1/3}} \right| \times 100% \approx 0.000019%.
\]

65. \(\cos^{-1}(0.52)\)

**Solution** Let \(f(x) = \cos^{-1} x\) and \(a = 0.5.\) Then

\[
f'(x) = -\frac{1}{\sqrt{1 - x^2}}, \quad f'(a) = f'(0.5) = -\frac{2\sqrt{3}}{3},
\]

and the linearization to \(f(x)\) is

\[
L(x) = f'(a)(x - a) + f(a) = -\frac{2\sqrt{3}}{3}(x - 0.5) + \frac{\pi}{3}.
\]

Thus, we have \(\cos^{-1}(0.52) \approx L(0.02) = 1.024104.\) The percentage error in this estimate is

\[
\left| \frac{\cos^{-1}(0.52) - 1.024104}{\cos^{-1}(0.52)} \right| \times 100% \approx 0.015%.
\]

67. \(e^{-0.012}\)

**Solution** Let \(f(x) = e^x\) and \(a = 0.\) Then \(f'(x) = e^x,\ \ f'(a) = f'(0) = 1\) and the linearization to \(f(x)\) is

\[
L(x) = f'(a)(x - a) + f(a) = 1(x - 0) + 1 = x + 1.
\]

Thus, we have \(e^{-0.012} \approx L(-0.012) = 1 - 0.012 = 0.988.\) The percentage error in this estimate is

\[
\left| \frac{e^{-0.012} - 0.988}{e^{-0.012}} \right| \times 100% \approx 0.0073%.
\]

69. Show that the Linear Approximation to \(f(x) = \sqrt{x}\) at \(x = 9\) yields the estimate \(\sqrt{9 + h} - 3 \approx \frac{1}{6} h.\) Set \(K = 0.01\) and show that \(|f^{(n)}(x)| \leq K\) for \(x \geq 9.\) Then verify numerically that the error \(E\) satisfies Eq. (5) for \(h = 10^{-n},\) for \(1 \leq n \leq 4.\)

**Solution** Let \(f(x) = \sqrt{x} .\) Then \(f(9) = 3,\ \ f'(x) = \frac{1}{2}x^{-1/2}\) and \(f'(9) = \frac{1}{6}.\) Therefore, by the Linear Approximation,

\[
f(9 + h) - f(9) = \sqrt{9 + h} - 3 \approx \frac{1}{6} h.
\]

Moreover, \(f''(x) = -\frac{1}{4}x^{-3/2},\) so \(|f''(x)| = \frac{1}{4} x^{-3/2}.\) Because this is a decreasing function, it follows that for \(x \geq 9,\)

\[
K = \max |f''(x)| \leq |f''(9)| = \frac{1}{108} < 0.01.
\]

From the following table, we see that for \(h = 10^{-n}, 1 \leq n \leq 4,\ E \leq \frac{1}{2} Kh^2.\)

| \(h\) | \(E = |\sqrt{9 + h} - 3 - \frac{1}{6} h|\) | \(\frac{1}{2} Kh^2\) |
|------|-----------------|----------|
| 10^{-1} | 4.604 \times 10^{-5} | 5.00 \times 10^{-5} |
| 10^{-2} | 4.627 \times 10^{-7} | 5.00 \times 10^{-7} |
| 10^{-3} | 4.629 \times 10^{-9} | 5.00 \times 10^{-9} |
| 10^{-4} | 4.627 \times 10^{-11} | 5.00 \times 10^{-11} |
Further Insights and Challenges

71. Compute $\frac{dy}{dx}$ at the point $P = (2, 1)$ on the curve $y^3 + 3xy = 7$ and show that the linearization at $P$ is $L(x) = -\frac{1}{3}x + \frac{2}{3}$. Use $L(x)$ to estimate the $y$-coordinate of the point on the curve where $x = 2.1$.

**SOLUTION** Differentiating both sides of the equation $y^3 + 3xy = 7$ with respect to $x$ yields

$$3y^2 \frac{dy}{dx} + 3x \frac{dy}{dx} + 3y = 0,$$

so

$$\frac{dy}{dx} = -\frac{y}{y^2 + x}.$$

Thus,

$$\left. \frac{dy}{dx} \right|_{(2,1)} = -\frac{1}{2^2 + 2} = -\frac{1}{3},$$

and the linearization at $P = (2, 1)$ is

$$L(x) = 1 - \frac{1}{3}(x - 2) = -\frac{1}{3}x + \frac{5}{3}.$$

Finally, when $x = 2.1$, we estimate that the $y$-coordinate of the point on the curve is

$$y \approx L(2.1) = -\frac{1}{3}(2.1) + \frac{5}{3} = 0.967.$$ 

73. Apply the method of Exercise 71 to $P = (-1, 2)$ on $y^4 + 7xy = 2$ to estimate the solution of $y^4 - 7.7y = 2$ near $y = 2$.

**SOLUTION** Differentiating both sides of the equation $y^4 + 7xy = 2$ with respect to $x$ yields

$$4y^3 \frac{dy}{dx} + 7x \frac{dy}{dx} + 7y = 0,$$

so

$$\frac{dy}{dx} = -\frac{7y}{4y^3 + 7x}.$$

Thus,

$$\left. \frac{dy}{dx} \right|_{(-1,2)} = -\frac{7(-2)}{4(-1)^3 + 7(-1)} = -\frac{14}{25},$$

and the linearization at $P = (-1, 2)$ is

$$L(x) = 2 - \frac{14}{25}(x + 1) = -\frac{14}{25}x + \frac{36}{25}.$$

Finally, the equation $y^4 - 7.7y = 2$ corresponds to $x = -1.1$, so we estimate the solution of this equation near $y = 2$ is

$$y \approx L(-1.1) = -\frac{14}{25}(-1.1) + \frac{36}{25} = 2.056.$$ 

75. Let $\Delta f = f(5 + h) - f(5)$, where $f(x) = x^2$. Verify directly that $E = |\Delta f - f'(5)h|$ satisfies (5) with $K = 2$.

**SOLUTION** Let $f(x) = x^2$. Then

$$\Delta f = f(5 + h) - f(5) = (5 + h)^2 - 5^2 = h^2 + 10h$$

and

$$E = |\Delta f - f'(5)h| = |h^2 + 10h - 10h| = h^2 = \frac{1}{2}h^2 = \frac{1}{2}K^2.$$
4.2 Extreme Values

Preliminary Questions

1. What is the definition of a critical point?

**Solution** A critical point is a value of the independent variable x in the domain of a function f at which either f'(x) = 0 or f'(x) does not exist.

In Questions 2 and 3, choose the correct conclusion.

2. If f(x) is not continuous on [0, 1], then
   (a) f(x) has no extreme values on [0, 1].
   (b) f(x) might not have any extreme values on [0, 1].

**Solution** The correct response is (b): f(x) might not have any extreme values on [0, 1]. Although [0, 1] is closed, because f is not continuous, the function is not guaranteed to have any extreme values on [0, 1].

3. If f(x) is continuous but has no critical points in [0, 1], then
   (a) f(x) has no min or max on [0, 1].
   (b) Either f(0) or f(1) is the minimum value on [0, 1].

**Solution** The correct response is (b): either f(0) or f(1) is the minimum value on [0, 1]. Remember that extreme values occur either at critical points or endpoints. If a continuous function on a closed interval has no critical points, the extreme values must occur at the endpoints.

4. Fermat’s Theorem does not claim that if f'(c) = 0, then f(c) is a local extreme value (this is false). What does Fermat’s Theorem assert?

**Solution** Fermat’s Theorem claims: If f(c) is a local extreme value, then either f'(c) = 0 or f'(c) does not exist.

Exercises

1. The following questions refer to Figure 15.
   (a) How many critical points does f(x) have on [0, 8]?
   (b) What is the maximum value of f(x) on [0, 8]?
   (c) What are the local maximum values of f(x)?
   (d) Find a closed interval on which both the minimum and maximum values of f(x) occur at critical points.
   (e) Find an interval on which the minimum value occurs at an endpoint.

**Solution**

(a) f(x) has three critical points on the interval [0, 8]: at x = 3, x = 5 and x = 7. Two of these, x = 3 and x = 5, are where the derivative is zero and one, x = 7, is where the derivative does not exist.

(b) The maximum value of f(x) on [0, 8] is 6; the function takes this value at x = 0.

(c) f(x) achieves a local maximum of 5 at x = 5.

(d) Answers may vary. One example is the interval [4, 8]. Another is [2, 6].

(e) Answers may vary. The easiest way to ensure this is to choose an interval on which the graph takes no local minimum. One example is [0, 2].

In Exercises 3–20, find all critical points of the function.

3. f(x) = x^2 - 2x + 4

**Solution** Let f(x) = x^2 - 2x + 4. Then f'(x) = 2x - 2 = 0 implies that x = 1 is the lone critical point of f.

5. f(x) = x^3 - \frac{9}{2}x^2 - 54x + 2

**Solution** Let f(x) = x^3 - \frac{9}{2}x^2 - 54x + 2. Then f'(x) = 3x^2 - 9x - 54 = 3(x + 3)(x - 6) = 0 implies that x = -3 and x = 6 are the critical points of f.
7. \( f(x) = x^{-1} - x^{-2} \)
SOLUTION Let \( f(x) = x^{-1} - x^{-2} \). Then

\[
    f'(x) = -x^{-2} + 2x^{-3} = \frac{2-2x}{x^3} = 0
\]

implies that \( x = 2 \) is the only critical point of \( f \). Though \( f'(x) \) does not exist at \( x = 0 \), this is not a critical point of \( f \) because \( x = 0 \) is not in the domain of \( f \).

9. \( f(x) = \frac{x}{x^2 + 1} \)
SOLUTION Let \( f(x) = \frac{x}{x^2 + 1} \). Then \( f'(x) = \frac{1-x^2}{(x^2 + 1)^2} = 0 \) implies that \( x = \pm 1 \) are the critical points of \( f \).

11. \( f(t) = t - 4\sqrt{t} + 1 \)
SOLUTION Let \( f(t) = t - 4\sqrt{t} + 1 \). Then

\[
    f'(t) = 1 - \frac{2}{\sqrt{t} + 1} = 0
\]

implies that \( t = 3 \) is a critical point of \( f \). Because \( f'(t) \) does not exist at \( t = -1 \), this is another critical point of \( f \).

13. \( f(x) = x^2\sqrt{1 - x^2} \)
SOLUTION Let \( f(x) = x^2\sqrt{1 - x^2} \). Then

\[
    f'(x) = -\frac{x^3}{\sqrt{1-x^2}} + 2x\sqrt{1-x^2} = \frac{2x - 3x^3}{\sqrt{1-x^2}}
\]

This derivative is 0 when \( x = 0 \) and when \( x = \pm \sqrt{2}/3 \); the derivative does not exist when \( x = \pm 1 \). All five of these values are critical points of \( f \).

15. \( g(\theta) = \sin^2 \theta \)
SOLUTION Let \( g(\theta) = \sin^2 \theta \). Then \( g'(\theta) = 2\sin \theta \cos \theta = \sin 2\theta = 0 \) implies that

\[
    \theta = \frac{n\pi}{2}
\]

is a critical value of \( g \) for all integer values of \( n \).

17. \( f(x) = x \ln x \)
SOLUTION Let \( f(x) = x \ln x \). Then \( f'(x) = 1 + \ln x = 0 \) implies that \( x = e^{-1} = \frac{1}{e} \) is the only critical point of \( f \).

19. \( f(x) = \sin^{-1} x - 2x \)
SOLUTION Let \( f(x) = \sin^{-1} x - 2x \). Then

\[
    f'(x) = \frac{1}{\sqrt{1-x^2}} - 2 = 0
\]

implies that \( x = \pm \sqrt{3}/2 \) are the critical points of \( f \).

21. Let \( f(x) = x^2 - 4x + 1 \).
(a) Find the critical point \( c \) of \( f(x) \) and compute \( f(c) \).
(b) Compute the value of \( f(x) \) at the endpoints of the interval \([0, 4]\).
(c) Determine the min and max of \( f(x) \) on \([0, 4]\).
(d) Find the extreme values of \( f(x) \) on \([0, 1]\).

SOLUTION Let \( f(x) = x^2 - 4x + 1 \).
(a) Then \( f'(x) = 2c - 4 = 0 \) implies that \( c = 2 \) is the only critical point of \( f \). We have \( f(2) = -3 \).
(b) \( f(0) = f(4) = 1 \).
(c) Using the results from (a) and (b), we find the maximum value of \( f \) on \([0, 4]\) is 1 and the minimum value is \(-3 \).
(d) We have \( f(1) = -2 \). Hence the maximum value of \( f \) on \([0, 1]\) is 1 and the minimum value is \(-2 \).

23. Find the critical points of \( f(x) = \sin x + \cos x \) and determine the extreme values on \([0, \frac{\pi}{2}] \).

SOLUTION

- Let \( f(x) = \sin x + \cos x \). Then on the interval \([0, \frac{\pi}{2}] \), we have \( f'(x) = \cos x - \sin x = 0 \) at \( x = \frac{\pi}{4} \), the only critical point of \( f \) in this interval.
- Since \( f\left(\frac{\pi}{2}\right) = \sqrt{2} \) and \( f(0) = f\left(\frac{\pi}{2}\right) = 1 \), the maximum value of \( f \) on \([0, \frac{\pi}{2}] \) is \( \sqrt{2} \), while the minimum value is \( 1 \).
25. **GU** Plot \( f(x) = 2\sqrt{x} - x \) on \([0, 4]\) and determine the maximum value graphically. Then verify your answer using calculus.

**SOLUTION** The graph of \( y = 2\sqrt{x} - x \) over the interval \([0, 4]\) is shown below. From the graph, we see that at \( x = 1 \), the function achieves its maximum value of 1.

![Graph of y = 2sqrt(x) - x](image)

To verify the information obtained from the plot, let \( f(x) = 2\sqrt{x} - x \). Then \( f'(x) = x^{-1/2} - 1 \). Solving \( f'(x) = 0 \) yields the critical points \( x = 0 \) and \( x = 1 \). Because \( f(0) = f(4) = 0 \) and \( f(1) = 1 \), we see that the maximum value of \( f \) on \([0, 4]\) is 1.

27. **CAS** Approximate the critical points of \( g(x) = x \cos^{-1} x \) and estimate the maximum value of \( g(x) \).

**SOLUTION** \( g'(x) = -\frac{x}{\sqrt{1-x^2}} + \cos^{-1} x \), so \( g'(x) = 0 \) when \( x \approx 0.652185 \). Evaluating \( g \) at the endpoints of its domain, \( x = \pm 1 \), and at the critical point \( x \approx 0.652185 \), we find \( g(-1) = -\pi \), \( g(0.652185) \approx 0.561096 \), and \( g(1) = 0 \). Hence, the maximum value of \( g(x) \) is approximately 0.561096.

In Exercises 29–38, find the min and max of the function on the given interval by comparing values at the critical points and endpoints.

29. \( y = 2x^2 + 4x + 5 \), \([-2, 2]\)

**SOLUTION** Let \( f(x) = 2x^2 + 4x + 5 \). Then \( f'(x) = 4x + 4 = 0 \) implies that \( x = -1 \) is the only critical point of \( f \). The minimum of \( f \) on the interval \([-2, 2]\) is \( f(-1) = 3 \), whereas its maximum is \( f(2) = 21 \). (Note: \( f(-2) = 5 \).)

31. \( y = 6t - t^2 \), \([0, 5]\)

**SOLUTION** Let \( f(t) = 6t - t^2 \). Then \( f'(t) = 6 - 2t = 0 \) implies that \( t = 3 \) is the only critical point of \( f \). The minimum of \( f \) on the interval \([0, 5]\) is \( f(0) = 0 \), whereas the maximum is \( f(3) = 9 \). (Note: \( f(5) = 5 \).)

33. \( y = x^3 - 6x^2 + 8 \), \([1, 6]\)

**SOLUTION** Let \( f(x) = x^3 - 6x^2 + 8 \). Then \( f'(x) = 3x^2 - 12x = 3x(x - 4) = 0 \) implies that \( x = 0 \) and \( x = 4 \) are the critical points of \( f \). The minimum of \( f \) on the interval \([1, 6]\) is \( f(4) = -24 \), whereas the maximum is \( f(6) = 8 \). (Note: \( f(1) = 3 \) and the critical point \( x = 0 \) is not in the interval \([1, 6]\).)

35. \( y = 2t^3 + 3t^2 \), \([1, 2]\)

**SOLUTION** Let \( f(t) = 2t^3 + 3t^2 \). Then \( f'(t) = 6t^2 + 6t = 6t(t + 1) = 0 \) implies that \( t = 0 \) and \( t = -1 \) are the critical points of \( f \). The minimum of \( f \) on the interval \([1, 2]\) is \( f(1) = 5 \), whereas the maximum is \( f(2) = 28 \). (Note: Neither critical points are in the interval \([1, 2]\).)

37. \( y = z^5 - 80z \), \([-3, 3]\)

**SOLUTION** Let \( f(z) = z^5 - 80z \). Then \( f'(z) = 5z^4 - 80 = 5(z^4 - 16) = 5(z^2 + 4)(z + 2)(z - 2) = 0 \) implies that \( z = \pm 2 \) are the critical points of \( f \). The minimum value of \( f \) on the interval \([-3, 3]\) is \( f(2) = -128 \), whereas the maximum is \( f(-2) = 128 \). (Note: \( f(-3) = 3 \) and \( f(3) = -3 \).)

39. \( y = \frac{x^2 + 1}{x - 4} \), \([5, 6]\)

**SOLUTION** Let \( f(x) = \frac{x^2 + 1}{x - 4} \). Then

\[
f'(x) = \frac{(x - 4) \cdot 2x - (x^2 + 1) \cdot 1}{(x - 4)^2} = \frac{x^2 - 8x - 1}{(x - 4)^2} = 0
\]

implies \( x = 4 \pm \sqrt{17} \) are critical points of \( f \). \( x = 4 \) is not a critical point because \( x = 4 \) is not in the domain of \( f \). On the interval \([5, 6]\), the minimum of \( f \) is \( f(6) = \frac{37}{2} = 18.5 \), whereas the maximum of \( f \) is \( f(5) = 26 \). (Note: The critical points \( x = 4 \pm \sqrt{17} \) are not in the interval \([5, 6]\).)
41. \( y = x - \frac{4x}{x + 1} \), \([0, 3]\)

**SOLUTION**

Let \( f(x) = x - \frac{4x}{x + 1} \). Then

\[
    f'(x) = 1 - \frac{4}{(x + 1)^2} = \frac{(x - 1)(x + 3)}{(x + 1)^2} = 0
\]

implies that \( x = 1 \) and \( x = -3 \) are critical points of \( f \). \( x = -1 \) is not a critical point because \( x = -1 \) is not in the domain of \( f \). The minimum of \( f \) on the interval \([0, 3]\) is \( f(1) = -1 \), whereas the maximum is \( f(0) = f(3) = 0 \). \( \text{(Note: The critical point } x = -3 \text{ is not in the interval } [0, 3] \text{.)} \)

43. \( y = (2 + x)\sqrt{2 + (2 - x)^2} \), \([0, 2]\)

**SOLUTION**

Let \( f(x) = (2 + x)\sqrt{2 + (2 - x)^2} \). Then

\[
    f'(x) = \frac{\sqrt{2 + (2 - x)^2} - (2 + x)(2 + (2 - x)^2)^{-1/2}(2 - x)}{2\sqrt{2 + (2 - x)^2}} = 0
\]

implies that \( x = 1 \) is the critical point of \( f \). On the interval \([0, 2]\), the minimum is \( f(0) = 2\sqrt{6} \approx 4.9 \) and the maximum is \( f(2) = 4\sqrt{2} \approx 5.66 \). \( \text{(Note: } f(1) = 3\sqrt{3} \approx 5.2 \text{.)} \)

45. \( y = \sqrt{x + x^2} - 2\sqrt{x} \), \([0, 4]\)

**SOLUTION**

Let \( f(x) = \sqrt{x + x^2} - 2\sqrt{x} \). Then

\[
    f'(x) = \frac{1}{2}(x + x^2)^{-1/2}(1 + 2x) - x^{-1/2} = \frac{1 + 2x - 2\sqrt{1 + x}}{2\sqrt{x}\sqrt{1 + x}} = 0
\]

implies that \( x = 0 \) and \( x = \frac{\sqrt{3}}{2} \) are the critical points of \( f \). Neither \( x = -1 \) nor \( x = -\frac{\sqrt{3}}{2} \) is a critical point because neither is in the domain of \( f \). On the interval \([0, 4]\), the minimum of \( f \) is \( f\left(\frac{\sqrt{3}}{2}\right) \approx -0.589980 \) and the maximum is \( f(4) \approx 0.472136 \). \( \text{(Note: } f(0) = 0 \text{.)} \)

47. \( y = \sin x \cos x \), \([0, \frac{\pi}{2}]\)

**SOLUTION**

Let \( f(x) = \sin x \cos x = \frac{1}{2}\sin 2x \). On the interval \([0, \frac{\pi}{2}]\), \( f'(x) = \cos 2x = 0 \) when \( x = \frac{\pi}{4} \). The minimum of \( f \) on this interval is \( f(0) = f\left(\frac{\pi}{4}\right) = 0 \), whereas the maximum is \( f\left(\frac{\pi}{4}\right) = \frac{1}{2} \).

49. \( y = \sqrt{2} \theta - \sec \theta \), \([0, \frac{\pi}{4}]\)

**SOLUTION**

Let \( f(\theta) = \sqrt{2} \theta - \sec \theta \). On the interval \([0, \frac{\pi}{4}]\), \( f'(\theta) = \sqrt{2} - \sec \theta \tan \theta = 0 \) at \( \theta = \frac{\pi}{4} \). The minimum value of \( f \) on this interval is \( f(0) = -1 \), whereas the maximum value over this interval is \( f\left(\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{\pi}{4} - 1\right) \approx -0.303493 \). \( \text{(Note: } f\left(\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{\pi}{4} - 2\right) \approx -0.519309 \text{.)} \)

51. \( y = \theta - 2 \sin \theta \), \([0, 2\pi]\)

**SOLUTION**

Let \( g(\theta) = \theta - 2 \sin \theta \). On the interval \([0, 2\pi]\), \( g'(\theta) = 1 - 2 \cos \theta = 0 \) at \( \theta = \frac{\pi}{3} \) and \( \theta = \frac{5}{3} \pi \). The minimum of \( g \) on this interval is \( g\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3} \approx -0.685 \) and the maximum is \( g\left(\frac{5}{3}\pi\right) = \frac{5}{3}\pi + \sqrt{3} \approx 6.968 \). \( \text{(Note: } g(0) = 0 \text{ and } g(2\pi) = 2\pi \approx 6.283 \text{.)} \)

53. \( y = \tan x - 2x \), \([0, 1]\)

**SOLUTION**

Let \( f(x) = \tan x - 2x \). Then on the interval \([0, 1]\), \( f'(x) = \sec^2 x - 2 = 0 \) at \( x = \frac{\pi}{4} \). The minimum of \( f \) is \( f\left(\frac{\pi}{4}\right) = 1 - \frac{\pi}{4} \approx -0.570796 \) and the maximum is \( f(0) = 0 \). \( \text{(Note: } f(1) = \tan 1 - 2 \approx -0.442592 \text{.)} \)

55. \( y = \frac{\ln x}{x} \), \([1, 3]\)

**SOLUTION**

Let \( f(x) = \frac{\ln x}{x} \). Then, on the interval \([1, 3]\),

\[
    f'(x) = \frac{1 - \ln x}{x^2} = 0
\]

at \( x = e \). The minimum of \( f \) on this interval is \( f(1) = 0 \) and the maximum is \( f(e) = e^{-1} \approx 0.367879 \). \( \text{(Note: } f(3) = \frac{1}{3} \ln 3 \approx 0.366204 \text{.)} \)
57. \( y = 5 \tan^{-1} x - x \), \([1, 5]\)

**SOLUTION**  Let \( f(x) = 5 \tan^{-1} x - x \). Then, on the interval \([1, 5]\),

\[
 f'(x) = 5 \frac{1}{1+x^2} - 1 = 0
\]

at \( x = 2 \). The minimum of \( f \) on this interval is \( f(5) = 5 \tan^{-1} 5 - 5 \approx 1.867004 \) and the maximum is \( f(2) = 5 \tan^{-1} 2 - 2 \approx 3.535744 \). (Note: \( f(1) = \frac{5\pi}{4} - 1 \approx 2.926991 \).)

59. Let \( f(\theta) = 2 \sin 2\theta + \sin 4\theta \).

(a) Show that \( \theta \) is a critical point if \( \cos 4\theta = -\cos 2\theta \).

(b) Show, using a unit circle, that \( \cos \theta_1 = -\cos \theta_2 \) if and only if \( \theta_1 = \pi + \theta_2 + 2\pi k \) for an integer \( k \).

(c) Show that \( \cos 4\theta = -\cos 2\theta \) if and only if \( \theta = \frac{\pi}{2} + \pi k \) or \( \theta = \frac{3\pi}{2} + \pi k \).

(d) Find the six critical points of \( f(\theta) \) on \([0, 2\pi]\) and find the extreme values of \( f(\theta) \) on this interval.

(e) Check your results against a graph of \( f(\theta) \).

**SOLUTION**  \( f(\theta) = 2 \sin 2\theta + \sin 4\theta \) is differentiable at all \( \theta \), so the way to find the critical points is to find all points such that \( f'(\theta) = 0 \).

(a) \( f'(\theta) = 4 \cos 2\theta + 4 \cos 4\theta \). If \( f'(\theta) = 0 \), then \( 4 \cos 4\theta = -4 \cos 2\theta \), so \( \cos 4\theta = -\cos 2\theta \).

(b) Given the point \((\cos \theta, \sin \theta)\) at angle \( \theta \) on the unit circle, there are two points with \( x \) coordinate \( -\cos \theta \). The graphic shows these two points, which are:

- The point \((\cos(\theta + \pi), \sin(\theta + \pi))\) on the opposite end of the unit circle.
- The point \((\cos(\pi - \theta), \sin(\pi - \theta))\) obtained by reflecting through the \( y \) axis.

If we include all angles representing these points on the circle, we find that \( \cos \theta_1 = -\cos \theta_2 \) if and only if \( \theta_1 = (\pi + \theta_2) + 2\pi k \) or \( \theta_1 = (\pi - \theta_2) + 2\pi k \) for integers \( k \).

(c) Using (b), we recognize that \( \cos 4\theta = -\cos 2\theta \) if \( 4\theta = 2\theta + \pi + 2\pi k \) or \( 4\theta = \pi - 2\theta + 2\pi k \). Solving for \( \theta \), we obtain \( \theta = \frac{\pi}{2} + k \pi \) or \( \theta = \frac{3\pi}{2} + k \pi \).

(d) To find all \( \theta, 0 \leq \theta < 2\pi \) indicated by (c), we use the following table:

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\pi}{2} + k \pi )</td>
<td>( \frac{\pi}{2} )</td>
<td>( \frac{3\pi}{2} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \frac{3\pi}{2} + k \pi )</td>
<td>( \frac{3\pi}{2} )</td>
<td>( \frac{5\pi}{2} )</td>
<td>( \frac{7\pi}{2} )</td>
<td>( \frac{9\pi}{2} )</td>
<td>( \frac{11\pi}{2} )</td>
<td></td>
</tr>
</tbody>
</table>

The critical points in the range \([0, 2\pi]\) are \( \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \) and \( \frac{11\pi}{2} \). On this interval, the maximum value is \( f(\frac{7\pi}{2}) = f(\frac{11\pi}{2}) = \frac{3\sqrt{3}}{2} \) and the minimum value is \( f(\frac{5\pi}{2}) = f(\frac{11\pi}{2}) = -\frac{3\sqrt{3}}{2} \).

(e) The graph of \( f(\theta) = 2 \sin 2\theta + \sin 4\theta \) is shown here:

We can see that there are six flat points on the graph between 0 and 2\( \pi \), as predicted. There are 4 local extrema, and two points at \( \left( \frac{\pi}{2}, 0 \right) \) and \( \left( \frac{3\pi}{2}, 0 \right) \) where the graph has neither a local maximum nor a local minimum.
In Exercises 61–64, find the critical points and the extreme values on [0, 4]. In Exercises 63 and 64, refer to Figure 18.

61. \( y = |x - 2| \)

**Solution** Let \( f(x) = |x - 2| \). For \( x < 2 \), we have \( f'(x) = -1 \). For \( x > 2 \), we have \( f'(x) = 1 \). Now as \( x \to 2^- \), we have \( \frac{f(x) - f(2)}{x - 2} = \frac{2 - 2 - 0}{x - 2} \to -1 \); whereas as \( x \to 2^+ \), we have \( \frac{f(x) - f(2)}{x - 2} = \frac{(x - 2) - 0}{x - 2} \to 1 \). Therefore, \( f'(2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} \) does not exist and the lone critical point of \( f \) is \( x = 2 \). Alternately, we examine the graph of \( f(x) = |x - 2| \) shown below.

To find the extremum, we check the values of \( f(x) \) at the critical point and the endpoints. \( f(0) = 2 \), \( f(4) = 2 \), and \( f(2) = 0 \). \( f(x) \) takes its minimum value of \( 0 \) at \( x = 2 \), and its maximum value of \( 2 \) at \( x = 0 \) and \( x = 4 \).

63. \( y = |x^2 + 4x - 12| \)

**Solution** Let \( f(x) = |x^2 + 4x - 12| = |(x + 6)(x - 2)| \). From the graph of \( f \) in Figure 18, we see that \( f'(x) \) does not exist at \( x = -6 \) and \( x = 2 \), so these are critical points of \( f \). There is also a critical point between \( x = -6 \) and \( x = 2 \) at which \( f'(x) = 0 \). For \( -6 < x < 2 \), \( f(x) = -(x^2 + 4x - 12) \), so \( f'(x) = -2x - 4 = 0 \) when \( x = -2 \). On the interval \([0, 4]\) the minimum value of \( f \) is \( f(2) = 0 \) and the maximum value is \( f(4) = 20 \). (Note: \( f(0) = 12 \) and the critical points \( x = -6 \) and \( x = -2 \) are not in the interval.)

In Exercises 65–68, verify Rolle’s Theorem for the given interval.

65. \( f(x) = x + x^{-1}, \quad \left[\frac{1}{2}, 2\right] \)

**Solution** Because \( f \) is continuous on \( \left[\frac{1}{2}, 2\right] \), differentiable on \( \left(\frac{1}{2}, 2\right) \) and

\[
 f\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2} = 1
\]

we may conclude from Rolle’s Theorem that there exists a \( c \in \left(\frac{1}{2}, 2\right) \) at which \( f'(c) = 0 \). Here, \( f'(x) = 1 - x^{-2} = \frac{x^2 - 1}{x^2} \), so we may take \( c = 1 \).

67. \( f(x) = \frac{x^2}{8x - 15}, \quad [3, 5] \)

**Solution** Because \( f \) is continuous on \([3, 5]\), differentiable on \([3, 5)\) and \( f(3) = f(5) = 1 \), we may conclude from Rolle’s Theorem that there exists a \( c \in (3, 5) \) at which \( f'(c) = 0 \). Here,

\[
 f'(x) = \frac{(8x - 15)(2x) - 8x^2}{(8x - 15)^2} = \frac{2x(4x - 15)}{(8x - 15)^2}
\]

so we may take \( c = \frac{15}{4} \).

69. Prove that \( f(x) = x^5 + 2x^3 + 4x - 12 \) has precisely one real root.

**Solution** Let’s first establish the \( f(x) = x^5 + 2x^3 + 4x - 12 \) has at least one root. Because \( f \) is a polynomial, it is continuous for all \( x \). Moreover, \( f(0) = -12 < 0 \) and \( f(2) = 44 > 0 \). Therefore, by the Intermediate Value Theorem, there exists a \( c \in (0, 2) \) such that \( f(c) = 0 \).
Next, we prove that this is the only root. We will use proof by contradiction. Suppose \( f(x) = x^5 + 2x^3 + 4x - 12 \) has two real roots, \( x = a \) and \( x = b \). Then \( f(a) = f(b) = 0 \) and Rolle’s Theorem guarantees that there exists a \( c \in (a, b) \) at which \( f'(c) = 0 \). However, \( f'(x) = 5x^4 + 6x^2 + 4 \geq 4 \) for all \( x \), so there is no \( c \in (a, b) \) at which \( f'(c) = 0 \). Based on this contradiction, we conclude that \( f(x) = x^5 + 2x^3 + 4x - 12 \) cannot have more than one real root. Finally, \( f \) must have precisely one real root.

71. Prove that \( f(x) = x^4 + 5x^3 + 4x \) has no root \( c \) satisfying \( c > 0 \). \textbf{Hint:} Note that \( x = 0 \) is a root and apply Rolle’s Theorem.

\textbf{SOLUTION} \quad \text{We will proceed by contradiction. Note that } f(0) = 0 \text{ and suppose that there exists a } c > 0 \text{ such that } f(c) = 0 \text{. Then } f(0) = f(c) = 0 \text{ and Rolle’s Theorem guarantees that there exists a } d \in (0, c) \text{ such that } f'(d) = 0 \text{. However, } f'(x) = 4x^3 + 15x^2 + 4 > 0 \text{ for all } x > 0, \text{ so there is no } d \in (0, c) \text{ such that } f'(d) = 0 \text{. Based on this contradiction, we conclude that } f(x) = x^4 + 5x^3 + 4x \text{ has no root } c \text{ satisfying } c > 0.\]

73. The position of a mass oscillating at the end of a spring is \( s(t) = A \sin \omega t \), where \( A \) is the amplitude and \( \omega \) is the angular frequency. Show that the speed \( |v(t)| \) is at a maximum when the acceleration \( a(t) \) is zero and that \( |a(t)| \) is at a maximum when \( v(t) \) is zero.

\textbf{SOLUTION} \quad \text{Let } s(t) = A \sin \omega t \text{. Then }

\begin{align*}
v(t) &= \frac{ds}{dt} = A\omega \cos \omega t \\
a(t) &= \frac{dv}{dt} = -A\omega^2 \sin \omega t.
\end{align*}

Thus, the speed

\[|v(t)| = |A\omega \cos \omega t|\]

is a maximum when \( |\cos \omega t| = 1 \), which is precisely when \( \sin \omega t = 0 \); that is, the speed \( |v(t)| \) is at a maximum when the acceleration \( a(t) \) is zero. Similarly,

\[|a(t)| = |A\omega^2 \sin \omega t|\]

is a maximum when \( |\sin \omega t| = 1 \), which is precisely when \( \cos \omega t = 0 \); that is, \( |a(t)| \) is at a maximum when \( v(t) \) is zero.

\textbf{75. \textit{CAS} Antibiotic Levels} \quad \text{A study shows that the concentration } C(t) \text{ (in micrograms per milliliter) of antibiotic in a patient’s blood serum after } t \text{ hours is } C(t) = 120(e^{-0.2t} - e^{-bt}), \text{ where } b \geq 1 \text{ is a constant that depends on the particular combination of antibiotic agents used. Solve numerically for the value of } b \text{ (to two decimal places) for which maximum concentration occurs at } t = 1 \text{ h. You may assume that the maximum occurs at a critical point as suggested by Figure 19.}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure19.png}
\caption{Graph of } C(t) = 120(e^{-0.2t} - e^{-bt}) \text{ with } b \text{ chosen so that the maximum occurs at } t = 1 \text{ h.}
\end{figure}

\textbf{SOLUTION} \quad \text{Answer is } b = 2.86. \text{ The max of } C(t) \text{ occurs at } t = \ln(5b)/(b - 0.2) \text{ so we solve } \ln(5b)/(b - 0.1) = 1 \text{ numerically.}

Let \( C(t) = 120(e^{-0.2t} - e^{-bt}) \). Then \( C'(t) = 120(-0.2e^{-0.2t} + be^{-bt}) = 0 \) when

\[t = \frac{\ln 5b}{b - 0.2}.
\]

Substituting \( t = 1 \) and solving for \( b \) numerically yields \( b \approx 2.86.\]

77. In 1919, physicist Alfred Betz argued that the maximum efficiency of a wind turbine is around 59%. If wind enters a turbine with speed \( v_1 \) and exits with speed \( v_2 \), then the power extracted is the difference in kinetic energy per unit time:
where \( m \) is the mass of wind flowing through the rotor per unit time (Figure 20). Betz assumed that \( m = \rho A(v_1 + v_2)/2 \), where \( \rho \) is the density of air and \( A \) is the area swept out by the rotor. Wind flowing undisturbed through the same area \( A \) would have mass per unit time \( \rho Av_1 \) and power \( P_0 = \frac{1}{2} \rho Av_1^3 \). The fraction of power extracted by the turbine is \( F = P/P_0 \).

(a) Show that \( F \) depends only on the ratio \( r = v_2/v_1 \) and is equal to \( F(r) = \frac{1}{2}(1 - r^2)(1 + r) \), where \( 0 \leq r \leq 1 \).

(b) Show that the maximum value of \( F(r) \), called the Betz Limit, is \( 16/27 \approx 0.59 \).

(c) Explain why Betz’s formula for \( F(r) \) is not meaningful for \( r \) close to zero. Hint: How much wind would pass through the turbine if \( v_2 \) were zero? Is this realistic?

\[
F = \frac{P}{P_0} = \frac{1}{2} \frac{\rho Av_1 + v_2}{2} \left( v_1^2 - v_2^2 \right) = \frac{1}{2} \frac{\rho Av_1^3}{v_1^2} \left( 1 + \frac{v_2}{v_1} \right) = \frac{1}{2} (1 - r^2)(1 + r).
\]

(b) Based on part (a),

\[
F'(r) = \frac{1}{2} (1 - r^2) - r(1 + r) = -\frac{3}{2} r^2 - r + \frac{1}{2}.
\]

The roots of this quadratic are \( r = -1 \) and \( r = \frac{1}{3} \). Now, \( F(0) = \frac{1}{2} \), \( F(1) = 0 \) and

\[
F \left( \frac{1}{3} \right) = \frac{1}{2} \left( \frac{8}{9} - \frac{4}{3} \right) = \frac{16}{27} \approx 0.59.
\]

Thus, the Betz Limit is \( 16/27 \approx 0.59 \).

(c) If \( v_2 \) were zero, then no air would be passing through the turbine, which is not realistic.

79. The response of a circuit or other oscillatory system to an input of frequency \( \omega \) (“omega”) is described by the function

\[
\phi(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4D^2\omega^2}}
\]

Both \( \omega_0 \) (the natural frequency of the system) and \( D \) (the damping factor) are positive constants. The graph of \( \phi \) is called a resonance curve, and the positive frequency \( \omega_r > 0 \), where \( \phi \) takes its maximum value, if it exists, is called the resonant frequency. Show that \( \omega_r = \sqrt{\omega_0^2 - 2D^2} \) if \( 0 < D < \omega_0/\sqrt{2} \) and that no resonant frequency exists otherwise (Figure 22).
SOLUTION Let \( \phi(\omega) = ((\omega^2_0 - \omega^2)^2 + 4D^2\omega^2)^{-1/2} \). Then
\[
\phi'(\omega) = \frac{2\omega((\omega^2_0 - \omega^2) - 2D^2)}{((\omega^2_0 - \omega^2)^2 + 4D^2\omega^2)^{3/2}}
\]
and the non-negative critical points are \( \omega = 0 \) and \( \omega = \sqrt{\omega^2_0 - 2D^2} \). The latter critical point is positive if and only if \( \omega^2_0 - 2D^2 > 0 \), and since we are given \( D > 0 \), this is equivalent to \( 0 < D < \omega^2_0/\sqrt{2} \).

Define \( \omega_r = \sqrt{\omega^2_0 - 2D^2} \). Now, \( \phi(0) = 1/\omega^2_0 \) and \( \phi(\omega) \to 0 \) as \( \omega \to \infty \). Finally,
\[
\phi(\omega_r) = \frac{1}{2D\sqrt{\omega^2_0 - D^2}}.
\]
which, for \( 0 < D < \omega^2_0/\sqrt{2} \), is larger than \( 1/\omega^2_0 \). Hence, the point \( \omega = \sqrt{\omega^2_0 - 2D^2} \), if defined, is a local maximum.

81. Find the maximum of \( y = x^a - x^b \) on \([0, 1]\) where \( 0 < a < b \). In particular, find the maximum of \( y = x^5 - x^{10} \) on \([0, 1]\).

SOLUTION

- Let \( f(x) = x^a - x^b \). Then \( f'(x) = ax^{a-1} - bx^{b-1} \). Since \( a < b \), \( f'(x) = x^{a-1}(a-bx^{b-a}) = 0 \) implies critical points \( x = 0 \) and \( x = b/(a+b-b-a) \), which is in the interval \([0, 1]\) as \( a < b \) implies \( a/b < 1 \) and consequently \( x = (b/a)^{1/(b-a)} < 1 \). Also, \( f(0) = f(1) = 0 \) and \( a < b \) implies \( x^a > x^b \) on the interval \([0, 1]\), which gives \( f(x) > 0 \) and thus the maximum value of \( f \) on \([0, 1]\) is
\[
f\left(\left(\frac{a}{b}\right)^{1/(b-a)}\right) = \left(\frac{a}{b}\right)^{a/(b-a)} - \left(\frac{a}{b}\right)^{b/(b-a)}.
\]

- Let \( f(x) = x^5 - x^{10} \). Then by part (a), the maximum value of \( f \) on \([0, 1]\) is
\[
f\left(\left(\frac{1}{2}\right)^{1/5}\right) = \left(\frac{1}{2}\right)^{5} - \left(\frac{1}{2}\right)^{10} = \frac{1}{2^5} - \frac{1}{2^{10}} = \frac{1}{4}.
\]

In Exercises 82–84, plot the function using a graphing utility and find its critical points and extreme values on \([-5, 5]\).

83. \( y = \frac{1}{1 + |x - 1|} + \frac{1}{1 + |x - 4|} \)

SOLUTION Let
\[
f(x) = \frac{1}{1 + |x - 1|} + \frac{1}{1 + |x - 4|}.
\]
The plot follows:

We can see on the plot that the critical points of \( f(x) \) lie at the cusps at \( x = 1 \) and \( x = 4 \) and at the location of the horizontal tangent line at \( x = \frac{3}{2} \). With \( f(-5) = \frac{1}{70} \), \( f(1) = f(4) = \frac{5}{4} \), \( f\left(\frac{3}{2}\right) = \frac{7}{4} \) and \( f(5) = \frac{17}{40} \), it follows that the maximum value of \( f(x) \) on \([-5, 5]\) is \( f(1) = f(4) = \frac{5}{4} \) and the minimum value is \( f(-5) = \frac{1}{70} \).

85. (a) Use implicit differentiation to find the critical points on the curve \( 27x^2 = (x^2 + y^2)^3 \).
(b) \( \text{GU} \) Plot the curve and the horizontal tangent lines on the same set of axes.

SOLUTION

(a) Differentiating both sides of the equation \( 27x^2 = (x^2 + y^2)^3 \) with respect to \( x \) yields
\[
54x = 3(x^2 + y^2)^2 \left(2x + 2y \frac{dy}{dx}\right).
\]
Solving for \( \frac{dy}{dx} \) we obtain
\[
\frac{dy}{dx} = \frac{27x - 3x(x^2 + y^2)^2}{3y(x^2 + y)^2} = \frac{x(9 - (x^2 + y^2)^2)}{y(x^2 + y)^2}.
\]

Thus, the derivative is zero when \( x^2 + y^2 = 3 \). Substituting into the equation for the curve, this yields \( x^2 = 1 \), or \( x = \pm 1 \).

There are therefore four points at which the derivative is zero:
\[
(-1, -\sqrt{2}), (-1, \sqrt{2}), (1, -\sqrt{2}), (1, \sqrt{2}).
\]

There are also critical points where the derivative does not exist. This occurs when \( y = 0 \) and gives the following points with vertical tangents:
\[
(0, 0), (\pm \sqrt{27}, 0).
\]

(b) The curve \( 27x^2 = (x^2 + y^2)^3 \) and its horizontal tangents are plotted below.

87. Sketch the graph of a continuous function on \((0, 4)\) having a local minimum but no absolute minimum.

**SOLUTION** Here is the graph of a function \( f \) on \((0, 4)\) with a local minimum value [between \( x = 2 \) and \( x = 4 \)] but no absolute minimum [since \( f(x) \to -\infty \) as \( x \to 0^+ \)].

89. Sketch the graph of a function \( f(x) \) on \([0, 4]\) with a discontinuity such that \( f(x) \) has an absolute minimum but no absolute maximum.

**SOLUTION** Here is the graph of a function \( f \) on \([0, 4]\) that (a) has a discontinuity [at \( x = 4 \)] and (b) has an absolute minimum [at \( x = 0 \)] but no absolute maximum [since \( f(x) \to \infty \) as \( x \to 4^- \)].

**Further Insights and Challenges**

91. Show that the extreme values of \( f(x) = a \sin x + b \cos x \) are \( \pm\sqrt{a^2 + b^2} \).

**SOLUTION** If \( f(x) = a \sin x + b \cos x \), then \( f'(x) = a \cos x - b \sin x \), so that \( f'(x) = 0 \) implies \( a \cos x - b \sin x = 0 \). This implies \( \tan x = \frac{-a}{b} \). Then,
\[
\sin x = \frac{\pm a}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \cos x = \frac{\pm b}{\sqrt{a^2 + b^2}}.
\]

Therefore
\[
f(x) = a \sin x + b \cos x = \frac{\pm a}{\sqrt{a^2 + b^2}} + b \frac{\pm b}{\sqrt{a^2 + b^2}} = \pm \frac{a^2 + b^2}{\sqrt{a^2 + b^2}} = \pm \sqrt{a^2 + b^2}.
\]
93. Show that if the quadratic polynomial \( f(x) = x^2 + rx + s \) takes on both positive and negative values, then its minimum value occurs at the midpoint between the two roots.

**Solution** Let \( f(x) = x^2 + rx + s \) and suppose that \( f(x) \) takes on both positive and negative values. This will guarantee that \( f \) has two real roots. By the quadratic formula, the roots of \( f \) are

\[
x = \frac{-r \pm \sqrt{r^2 - 4s}}{2}.
\]

Observe that the midpoint between these roots is

\[
\frac{1}{2} \left( \frac{-r + \sqrt{r^2 - 4s}}{2} + \frac{-r - \sqrt{r^2 - 4s}}{2} \right) = -\frac{r}{2}.
\]

Next, \( f'(x) = 2x + r = 0 \) when \( x = -\frac{r}{2} \) and, because the graph of \( f(x) \) is an upward opening parabola, it follows that \( f(-\frac{r}{2}) \) is a minimum. Thus, \( f \) takes on its minimum value at the midpoint between the two roots.

95. A cubic polynomial may have a local min and max, or it may have neither (Figure 26). Find conditions on the coefficients \( a \) and \( b \) of

\[
f(x) = \frac{1}{3}x^3 + \frac{1}{2}ax^2 + bx + c
\]

that ensure that \( f \) has neither a local min nor a local max. *Hint:* Apply Exercise 92 to \( f'(x) \).

![Cubic polynomials](image)

**Solution** Let \( f(x) = \frac{1}{3}x^3 + \frac{1}{2}ax^2 + bx + c \). Using Exercise 92, we have \( g(x) = f'(x) = x^2 + ax + b > 0 \) for all \( x \) provided \( b > \frac{1}{2}a^2 \), in which case \( f \) has no critical points and hence no local extrema. (Actually \( b \geq \frac{1}{4}a^2 \) will suffice, since in this case [as we’ll see in a later section] \( f \) has an inflection point but no local extrema.)

97. Prove that if \( f \) is continuous and \( f(a) \) and \( f(b) \) are local minima where \( a < b \), then there exists a value \( c \) between \( a \) and \( b \) such that \( f(c) \) is a local maximum. *(Hint: Apply Theorem 1 to the interval \([a, b]\).* Show that continuity is a necessary hypothesis by sketching the graph of a function (necessarily discontinuous) with two local minima but no local maximum.

**Solution**

- Let \( f(x) \) be a continuous function with \( f(a) \) and \( f(b) \) local minima on the interval \([a, b]\). By Theorem 1, \( f(x) \) must take on both a minimum and a maximum on \([a, b]\). Since local minima occur at \( f(a) \) and \( f(b) \), the maximum must occur at some other point in the interval, call it \( c \), where \( f(c) \) is a local maximum.
- The function graphed here is discontinuous at \( x = 0 \).

4.3 The Mean Value Theorem and Monotonicity

**Preliminary Questions**

1. For which value of \( m \) is the following statement correct? If \( f(2) = 3 \) and \( f(4) = 9 \), and \( f(x) \) is differentiable, then \( f \) has a tangent line of slope \( m \).

**Solution** The Mean Value Theorem guarantees that the function has a tangent line with slope equal to

\[
\frac{f(4) - f(2)}{4 - 2} = \frac{9 - 3}{4 - 2} = 3.
\]

Hence, \( m = 3 \) makes the statement correct.
2. Assume $f$ is differentiable. Which of the following statements does not follow from the MVT?
   
   (a) If $f$ has a secant line of slope 0, then $f$ has a tangent line of slope 0.
   
   (b) If $f(5) < f(9)$, then $f'(c) > 0$ for some $c \in (5, 9)$.
   
   (c) If $f$ has a tangent line of slope 0, then $f$ has a secant line of slope 0.
   
   (d) If $f'(x) > 0$ for all $x$, then every secant line has positive slope.
   
   **Solution** Conclusion (e) does not follow from the Mean Value Theorem. As a counterexample, consider the function $f(x) = x^3$. Note that $f'(0) = 0$, but no secant line has zero slope.
   
3. Can a function that takes on only negative values have a positive derivative? If so, sketch an example.
   
   **Solution** Yes. The figure below displays a function that takes on only negative values but has a positive derivative.

![Graph of derivative](image)

4. For $f(x)$ with derivative as in Figure 12:
   
   (a) Is $f(c)$ a local minimum or maximum?
   
   (b) Is $f(x)$ a decreasing function?

   **Solution**
   
   (a) To the left of $x = c$, the derivative is positive, so $f$ is increasing; to the right of $x = c$, the derivative is negative, so $f$ is decreasing. Consequently, $f(c)$ must be a local maximum.

   (b) No. The derivative is a decreasing function, but as noted in part (a), $f(x)$ is increasing for $x < c$ and decreasing for $x > c$.

**Exercises**

In Exercises 1–8, find a point $c$ satisfying the conclusion of the MVT for the given function and interval.

1. $y = x^{-1}$, $\quad [2, 8]$  
   
   **Solution** Let $f(x) = x^{-1}$, $a = 2$, $b = 8$. Then $f'(x) = -x^{-2}$, and by the MVT, there exists a $c \in (2, 8)$ such that

   $$-\frac{1}{c^2} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{1}{8} - \frac{1}{2} = -\frac{1}{16}.$$  

   Thus $c^2 = 16$ and $c = \pm 4$. Choose $c = 4 \in (2, 8)$.

3. $y = \cos x - \sin x$, $\quad [0, 2\pi]$  
   
   **Solution** Let $f(x) = \cos x - \sin x$, $a = 0$, $b = 2\pi$. Then $f'(x) = -\sin x - \cos x$, and by the MVT, there exists a $c \in (0, 2\pi)$ such that

   $$-\sin c - \cos c = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{1 - 1}{2\pi - 0} = 0.$$  

   Thus $-\sin c = \cos c$. Choose either $c = \frac{3\pi}{4}$ or $c = \frac{7\pi}{4} \in (0, 2\pi)$.

5. $y = x^3$, $\quad [-4, 5]$  
   
   **Solution** Let $f(x) = x^3$, $a = -4$, $b = 5$. Then $f'(x) = 3x^2$, and by the MVT, there exists a $c \in (-4, 5)$ such that

   $$3c^2 = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{125 - (-64)}{5 - (-4)} = \frac{189}{9} = 21.$$  

   Solving for $c$ yields $c^2 = 7$, so $c = \pm \sqrt{7}$. Both of these values are in the interval $[-4, 5]$, so either value can be chosen.
7. \( y = e^{-2x} \), \([0, 3]\)

**SOLUTION** Let \( f(x) = e^{-2x} \), \( a = 0, b = 3 \). Then \( f'(x) = -2e^{-2x} \), and by the MVT, there exists a \( c \in (0, 3) \) such that

\[
-2e^{-2c} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{e^{-6} - 1}{3 - 0} = -e^{-2}.
\]

Solving for \( c \) yields

\[
c = -\frac{1}{2} \ln \left( \frac{1 - e^{-6}}{6} \right) \approx 0.8971 \in (0, 3).
\]

9. **GU** Let \( f(x) = x^5 + x^2 \). The secant line between \( x = 0 \) and \( x = 1 \) has slope 2 (check this), so by the MVT, \( f'(c) = 2 \) for some \( c \in (0, 1) \). Plot \( f(x) \) and the secant line on the same axes. Then plot \( y = 2x + b \) for different values of \( b \) until the line becomes tangent to the graph of \( f \). Zoom in on the point of tangency to estimate \( x \)-coordinate \( c \) of the point of tangency.

**SOLUTION** Let \( f(x) = x^5 + x^2 \). The slope of the secant line between \( x = 0 \) and \( x = 1 \) is

\[
\frac{f(1) - f(0)}{1 - 0} = \frac{2 - 0}{1} = 2.
\]

A plot of \( f(x) \), the secant line between \( x = 0 \) and \( x = 1 \), and the line \( y = 2x - 0.764 \) is shown below at the left. The line \( y = 2x - 0.764 \) appears to be tangent to the graph of \( y = f(x) \). Zooming in on the point of tangency (see below at the right), it appears that the \( x \)-coordinate of the point of tangency is approximately 0.62.

11. Determine the intervals on which \( f'(x) \) is positive and negative, assuming that Figure 13 is the graph of \( f(x) \).

**SOLUTION** The derivative of \( f \) is positive on the intervals \((0, 1)\) and \((3, 5)\) where \( f \) is increasing; it is negative on the intervals \((1, 3)\) and \((5, 6)\) where \( f \) is decreasing.

13. State whether \( f(2) \) and \( f(4) \) are local minima or local maxima, assuming that Figure 13 is the graph of \( f'(x) \).

**SOLUTION**

- \( f'(x) \) makes a transition from positive to negative at \( x = 2 \), so \( f(2) \) is a local maximum.
- \( f'(x) \) makes a transition from negative to positive at \( x = 4 \), so \( f(4) \) is a local minimum.

In Exercises 15–18, sketch the graph of a function \( f(x) \) whose derivative \( f'(x) \) has the given description.

15. \( f'(x) > 0 \) for \( x > 3 \) and \( f'(x) < 0 \) for \( x < 3 \)

**SOLUTION** Here is the graph of a function \( f \) for which \( f'(x) > 0 \) for \( x > 3 \) and \( f'(x) < 0 \) for \( x < 3 \).
17. \( f'(x) \) is negative on \((1, 3)\) and positive everywhere else.

**SOLUTION**  Here is the graph of a function \( f \) for which \( f'(x) \) is negative on \((1, 3)\) and positive elsewhere.

\[
\begin{array}{c|c|c|c}
\text{Symbol} & \text{Meaning} \\
\hline
- & \text{The entity is negative on the given interval.} \\
0 & \text{The entity is zero at the specified point.} \\
+ & \text{The entity is positive on the given interval.} \\
U & \text{The entity is undefined at the specified point.} \\
\nearrow & \text{\( f \) is increasing on the given interval.} \\
\searrow & \text{\( f \) is decreasing on the given interval.} \\
M & \text{\( f \) has a local maximum at the specified point.} \\
m & \text{\( f \) has a local minimum at the specified point.} \\
\neg & \text{There is no local extremum here.} \\
\end{array}
\]

23. \( y = -x^2 + 7x - 17 \)

**SOLUTION**  Let \( f(x) = -x^2 + 7x - 17 \). Then \( f'(x) = 7 - 2x = 0 \) yields the critical point \( c = \frac{7}{2} \).

\[
\begin{array}{c|c|c|c}
x & (-\infty, \frac{7}{2}) & \frac{7}{2} & (\frac{7}{2}, \infty) \\
f' & + & 0 & - \\
f & \nearrow & M & \searrow \\
\end{array}
\]
25. \( y = x^3 - 12x^2 \)

**SOLUTION**  Let \( f(x) = x^3 - 12x^2 \). Then \( f'(x) = 3x^2 - 24x = 3x(x - 8) = 0 \) yields critical points \( c = 0, 8 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\infty, 0)</th>
<th>0</th>
<th>(0, 8)</th>
<th>8</th>
<th>(8, (\infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' )</td>
<td>+</td>
<td>0</td>
<td>–</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>

27. \( y = 3x^4 + 8x^3 - 6x^2 - 24x \)

**SOLUTION**  Let \( f(x) = 3x^4 + 8x^3 - 6x^2 - 24x \). Then

\[
\begin{align*}
  f'(x) &= 12x^3 + 24x^2 - 12x - 24 \\
        &= 12(x^2(x + 2) - 12(x + 2) = 12(x + 2)(x^2 - 1) \\
        &= 12(x - 1)(x + 1)(x + 2) = 0
\end{align*}
\]

yields critical points \( c = -2, -1, 1 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\infty, -2)</th>
<th>-2</th>
<th>(-2, -1)</th>
<th>-1</th>
<th>(-1, 1)</th>
<th>1</th>
<th>(1, (\infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' )</td>
<td>–</td>
<td>0</td>
<td>+</td>
<td>0</td>
<td>–</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>

29. \( y = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + 4 \)

**SOLUTION**  Let \( f(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + 4 \). Then \( f'(x) = x^2 + 3x + 2 = (x + 1)(x + 2) = 0 \) yields critical points \( c = -2, -1 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\infty, -2)</th>
<th>-2</th>
<th>(-2, -1)</th>
<th>-1</th>
<th>(-1, (\infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' )</td>
<td>+</td>
<td>0</td>
<td>–</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>

31. \( y = x^5 + x^3 + 1 \)

**SOLUTION**  Let \( f(x) = x^5 + x^3 + 1 \). Then \( f'(x) = 5x^4 + 3x^2 = x^2(5x^2 + 3) \) yields a single critical point: \( c = 0 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\infty, 0)</th>
<th>0</th>
<th>(0, (\infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' )</td>
<td>+</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>

33. \( y = x^4 - 4x^{3/2} \) (\( x > 0 \))

**SOLUTION**  Let \( f(x) = x^4 - 4x^{3/2} \) for \( x > 0 \). Then \( f'(x) = 4x^3 - 6x^{1/2} = 2x^{1/2}(2x^{5/2} - 3) = 0 \), which gives us the critical point \( c = \left(\frac{3}{2}\right)^{2/5} \). (Note: \( c = 0 \) is not in the interval under consideration.)

<table>
<thead>
<tr>
<th>( x )</th>
<th>(0, \left(\frac{3}{2}\right)^{2/5})</th>
<th>(\left(\frac{3}{2}\right)^{2/5})</th>
<th>(\left(\frac{3}{2}\right)^{2/5}, (\infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' )</td>
<td>–</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>

35. \( y = x + x^{-1} \) (\( x > 0 \))

**SOLUTION**  Let \( f(x) = x + x^{-1} \) for \( x > 0 \). Then \( f'(x) = 1 - x^{-2} = 0 \) yields the critical point \( c = 1 \). (Note: \( c = -1 \) is not in the interval under consideration.)

<table>
<thead>
<tr>
<th>( x )</th>
<th>(0, 1)</th>
<th>1</th>
<th>(1, (\infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' )</td>
<td>–</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>

May 23, 2011
37. \( y = \frac{1}{x^2 + 1} \)

**SOLUTION** Let \( f(x) = \left(x^2 + 1\right)^{-1} \). Then \( f'(x) = -2x \left(x^2 + 1\right)^{-2} = 0 \) yields critical point \( c = 0 \).

\[
\begin{array}{c|ccc}
 x & (-\infty, 0) & 0 & (0, \infty) \\
\hline
f' & + & 0 & - \\
f & \nearrow & M & \searrow
\end{array}
\]

39. \( y = \frac{x^3}{x^2 + 1} \)

**SOLUTION** Let \( f(x) = \frac{x^3}{x^2 + 1}. \) Then

\[
f'(x) = \frac{(x^2 + 1)(3x^2) - x^3(2x)}{(x^2 + 1)^2} = \frac{x^2(x^2 + 3)}{(x^2 + 1)^2} = 0
\]
yields the single critical point \( c = 0 \).

\[
\begin{array}{c|ccc}
 x & (-\infty, 0) & 0 & (0, \infty) \\
\hline
f' & + & 0 & - \\
f & \nearrow & M & \searrow
\end{array}
\]

41. \( y = \theta + \sin \theta + \cos \theta \)

**SOLUTION** Let \( f(\theta) = \theta + \sin \theta + \cos \theta \). Then \( f'(\theta) = 1 + \cos \theta - \sin \theta = 0 \) yields the critical points \( c = \frac{\pi}{2} \) and \( c = \pi \).

\[
\begin{array}{c|cccc}
 \theta & (0, \frac{\pi}{2}) & \frac{\pi}{2} & (\frac{\pi}{2}, \pi) & (\pi, 2\pi) \\
\hline
f' & + & 0 & - & 0 \\
f & \nearrow & M & \searrow & m
\end{array}
\]

43. \( y = \sin^2 \theta + \sin \theta \)

**SOLUTION** Let \( f(\theta) = \sin^2 \theta + \sin \theta \). Then \( f'(\theta) = 2 \sin \theta \cos \theta + \cos \theta = \cos \theta(2 \sin \theta + 1) = 0 \) yields the critical points \( c = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \) and \( \frac{13\pi}{6} \).

\[
\begin{array}{c|cccccccc}
 \theta & (0, \frac{\pi}{6}) & \frac{\pi}{6} & (\frac{\pi}{6}, \frac{7\pi}{6}) & \frac{7\pi}{6} & (\frac{7\pi}{6}, \frac{3\pi}{2}) & \frac{3\pi}{2} & (\frac{3\pi}{2}, \frac{11\pi}{6}) & \frac{11\pi}{6} & (\frac{11\pi}{6}, 2\pi) \\
\hline
f' & + & 0 & - & 0 & + & - & 0 & + \\
f & \nearrow & M & \searrow & m & \nearrow & M & \searrow & m & \nearrow
\end{array}
\]

45. \( y = x + e^{-x} \)

**SOLUTION** Let \( f(x) = x + e^{-x} \). Then \( f'(x) = 1 - e^{-x} \), which yields \( c = 0 \) as the only critical point.

\[
\begin{array}{c|ccc}
 x & (-\infty, 0) & 0 & (0, \infty) \\
\hline
f' & - & 0 & + \\
f & \searrow & m & \nearrow
\end{array}
\]

47. \( y = e^{-x} \cos x, \quad \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \)

**SOLUTION** Let \( f(x) = e^{-x} \cos x \). Then

\[
f'(x) = -e^{-x} \sin x - e^{-x} \cos x = -e^{-x}(\sin x + \cos x),
\]

which yields \( c = -\frac{\pi}{4} \) as the only critical point on the interval \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \).

\[
\begin{array}{c|ccc}
 x & \left[ -\frac{\pi}{2}, -\frac{\pi}{4} \right] & \left(-\frac{\pi}{4}, \frac{\pi}{2} \right) \\
\hline
f' & + & 0 & - \\
f & \nearrow & M & \searrow
\end{array}
\]
49. \( y = \tan^{-1} x - \frac{1}{2} x \)

**SOLUTION** Let \( f(x) = \tan^{-1} x - \frac{1}{2} x \). Then

\[
 f'(x) = \frac{1}{1 + x^2} - \frac{1}{2},
\]

which yields \( c = \pm 1 \) as critical points.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( (-\infty, -1) )</th>
<th>(-1)</th>
<th>((-1, 1))</th>
<th>( 1 )</th>
<th>((1, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' )</td>
<td>-</td>
<td>0</td>
<td>+</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>( f )</td>
<td>( \searrow )</td>
<td>( m )</td>
<td>( \nearrow )</td>
<td>( M )</td>
<td>( \searrow )</td>
</tr>
</tbody>
</table>

51. \( y = x - \ln x \) \((x > 0)\)

**SOLUTION** Let \( f(x) = x - \ln x \). Then \( f'(x) = 1 - x^{-1} \), which yields \( c = 1 \) as the only critical point.

<table>
<thead>
<tr>
<th>( x )</th>
<th>((0, 1))</th>
<th>(1)</th>
<th>((1, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' )</td>
<td>-</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>( f )</td>
<td>( \searrow )</td>
<td>( m )</td>
<td>( \nearrow )</td>
</tr>
</tbody>
</table>

53. Find the minimum value of \( f(x) = x^4 \) for \( x > 0 \).

**SOLUTION** Let \( f(x) = x^4 \). By logarithmic differentiation, we know that \( f'(x) = x^3(1 + \ln x) \). Thus, \( x = \frac{1}{e} \) is the only critical point. Because \( f'(x) < 0 \) for \( 0 < x < \frac{1}{e} \) and \( f'(x) > 0 \) for \( x > \frac{1}{e} \),

\[
 f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{1/e} \approx 0.692201
\]

is the minimum value.

55. Show that \( f(x) = x^3 - 2x^2 + 2x \) is an increasing function. **Hint:** Find the minimum value of \( f'(x) \).

**SOLUTION** Let \( f(x) = x^3 - 2x^2 + 2x \). For all \( x \), we have

\[
 f'(x) = 3x^2 - 4x + 2 = 3 \left(x - \frac{2}{3}\right)^2 + 2 > 0.
\]

Since \( f'(x) > 0 \) for all \( x \), the function \( f \) is everywhere increasing.

57. \( \text{GU} \) Let \( h(x) = \frac{x(x^2 - 1)}{x^2 + 1} \) and suppose that \( f'(x) = h(x) \). Plot \( h(x) \) and use the plot to describe the local extrema and the increasing/decreasing behavior of \( f(x) \). Sketch a plausible graph for \( f(x) \) itself.

**SOLUTION** The graph of \( h(x) \) is shown below at the left. Because \( h(x) \) is negative for \( x < -1 \) and for \( 0 < x < 1 \), it follows that \( f(x) \) is decreasing for \( x < -1 \) and for \( 0 < x < 1 \). Similarly, \( f(x) \) is increasing for \( -1 < x < 0 \) and for \( x > 1 \) because \( h(x) \) is positive on these intervals. Moreover, \( f(x) \) has local minima at \( x = -1 \) and \( x = 1 \) and a local maximum at \( x = 0 \). A plausible graph for \( f(x) \) is shown below at the right.

![Graph of h(x)](image)

![Graph of f(x)](image)

59. Determine where \( f(x) = (1000 - x)^2 + x^2 \) is decreasing. Use this to decide which is larger: 800\(^2\) + 200\(^2\) or 600\(^2\) + 400\(^2\).

**SOLUTION** If \( f(x) = (1000 - x)^2 + x^2 \), then \( f'(x) = -2(1000 - x) + 2x = 4x - 2000 \). \( f'(x) < 0 \) as long as \( x < 500 \). Therefore, \( 800^2 + 200^2 = f(200) > f(400) = 600^2 + 400^2 \).
61. Which values of \( c \) satisfy the conclusion of the MVT on the interval \([a, b]\) if \( f(x) \) is a linear function?

**Solution** Let \( f(x) = px + q \), where \( p \) and \( q \) are constants. Then the slope of every secant line and tangent line of \( f \) is \( p \). Accordingly, considering the interval \([a, b]\), every point \( c \in (a, b) \) satisfies \( f'(c) = p = \frac{f(b) - f(a)}{b - a} \), the conclusion of the MVT.

63. Suppose that \( f(0) = 2 \) and \( f'(x) \leq 3 \) for \( x > 0 \). Apply the MVT to the interval \([0, 4]\) to prove that \( f(4) \leq 14 \). Prove more generally that \( f(x) \leq 2 + 3x \) for all \( x > 0 \).

**Solution** The MVT, applied to the interval \([0, 4]\), guarantees that there exists a \( c \in (0, 4) \) such that

\[
f'(c) = \frac{f(4) - f(0)}{4 - 0} \quad \text{or} \quad f(4) - f(0) = 4f'(c).
\]

Because \( c > 0 \), \( f'(c) \leq 3 \), so \( f(4) - f(0) \leq 12 \). Finally, \( f(4) \leq f(0) + 12 = 14 \).

More generally, let \( x > 0 \). The MVT, applied to the interval \([0, x]\), guarantees there exists a \( c \in (0, x) \) such that

\[
f'(c) = \frac{f(x) - f(0)}{x - 0} \quad \text{or} \quad f(x) - f(0) = f'(c)x.
\]

Because \( c > 0 \), \( f'(c) \leq 3 \), so \( f(x) - f(0) \leq 3x \). Finally, \( f(x) \leq f(0) + 3x = 3x + 2 \).

65. Show that if \( f(2) = 5 \) and \( f'(x) \geq 10 \) for \( x > 2 \), then \( f(x) \geq 10x - 15 \) for all \( x > 2 \).

**Solution** Let \( x > 2 \). The MVT, applied to the interval \([2, x]\), guarantees there exists a \( c \in (2, x) \) such that

\[
f'(c) = \frac{f(x) - f(2)}{x - 2} \quad \text{or} \quad f(x) - f(2) = (x - 2)f'(c).
\]

Because \( f'(x) \geq 10 \), it follows that \( f(x) - f(2) \geq 10(x - 2) \), or \( f(x) \geq f(2) + 10(x - 2) = 10x - 15 \).

### Further Insights and Challenges

67. Prove that if \( f(0) = g(0) \) and \( f'(x) \leq g'(x) \) for \( x \geq 0 \), then \( f(x) \leq g(x) \) for all \( x \geq 0 \). *Hint:* Show that \( f(x) - g(x) \) is nonincreasing.

**Solution** Let \( h(x) = f(x) - g(x) \). By the sum rule, \( h'(x) = f'(x) - g'(x) \). Since \( f'(x) \leq g'(x) \) for all \( x \geq 0 \), \( h'(x) \leq 0 \) for all \( x \geq 0 \). This implies that \( h \) is nonincreasing. Since \( h(0) = f(0) - g(0) = 0 \), \( h(x) \leq 0 \) for all \( x \geq 0 \) (as \( h \) is nonincreasing, it cannot climb above zero). Hence \( f(x) - g(x) \leq 0 \) for all \( x \geq 0 \), and so \( f(x) \leq g(x) \) for \( x \geq 0 \).

69. Use Exercise 67 and the inequality \( \sin x \leq x \) for \( x \geq 0 \) (established in Theorem 3 of Section 2.6) to prove the following assertions for all \( x \geq 0 \) (each assertion follows from the previous one).

(a) \( \cos x \geq 1 - \frac{1}{2}x^2 \)

(b) \( \sin x \geq x - \frac{1}{6}x^3 \)

(c) \( \cos x \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \)

(d) Can you guess the next inequality in the series?

**Solution**

(a) We prove this using Exercise 67: Let \( g(x) = \cos x \) and \( f(x) = 1 - \frac{1}{2}x^2 \). Then \( f(0) = g(0) = 1 \) and \( g'(x) = -\sin x \geq -x = f'(x) \) for \( x \geq 0 \) by Exercise 68. Now apply Exercise 67 to conclude that \( \cos x \geq 1 - \frac{1}{2}x^2 \) for \( x \geq 0 \).

(b) Let \( g(x) = \sin x \) and \( f(x) = x - \frac{1}{6}x^3 \). Then \( f(0) = g(0) = 0 \) and \( g'(x) = \cos x \geq 1 - \frac{1}{2}x^2 = f'(x) \) for \( x \geq 0 \) by part (a). Now apply Exercise 67 to conclude that \( \sin x \geq x - \frac{1}{6}x^3 \) for \( x \geq 0 \).

(c) Let \( g(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \) and \( f(x) = \cos x \). Then \( f(0) = g(0) = 1 \) and \( g'(x) = -\sin x = f'(x) \) for \( x \geq 0 \) by part (b). Now apply Exercise 67 to conclude that \( \cos x \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \) for \( x \geq 0 \).

(d) The next inequality in the series is \( \sin x \leq x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \), valid for \( x \geq 0 \). To construct (d) from (c), we note that the derivative of \( \sin x \) is \( \cos x \), and look for a polynomial (which we currently must do by educated guess) whose derivative is \( 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \). We know the derivative of \( x \) is 1, and that a term whose derivative is \( -\frac{1}{2}x^2 \) should be of the form \( Cx^3 \). \( \frac{1}{2}Cx^3 = \frac{3}{2}C = -\frac{1}{2}x^2 \), so \( C = -\frac{1}{3} \). A term whose derivative is \( \frac{1}{24}x^4 \) should be of the form \( Dx^5 \). From this, \( \frac{1}{24}Dx^5 = 5Dx^4 = \frac{1}{24}x^4 \), so that \( 5D = \frac{1}{24} \) or \( D = \frac{1}{120} \).

71. Assume that \( f'''(x) \) exists and \( f'''(x) = 0 \) for all \( x \). Prove that \( f(x) = mx + b \), where \( m = f'(0) \) and \( b = f(0) \).

**Solution**

- Let \( f'''(x) = 0 \) for all \( x \). Then \( f'(x) = \) constant for all \( x \). Since \( f'(0) = m \), we conclude that \( f'(x) = m \) for all \( x \).
- Let \( g(x) = f(x) - mx. \) Then \( g'(x) = f'(x) - m = m - m = 0 \) which implies that \( g(x) = \) constant for all \( x \) and consequently \( f(x) - mx = \) constant for all \( x \). Rearranging the statement, \( f(x) = mx + \) constant. Since \( f(0) = b \), we conclude that \( f(x) = mx + b \) for all \( x \).
73. Suppose that \( f(x) \) satisfies the following equation (an example of a differential equation):

\[
f''(x) = -f(x)
\]

(a) Show that \( f(x)^2 + f'(x)^2 = f(0)^2 + f'(0)^2 \) for all \( x \). \textit{Hint:} Show that the function on the left has zero derivative.

(b) Verify that \( \sin x \) and \( \cos x \) satisfy Eq. (1), and deduce that \( \sin^2 x + \cos^2 x = 1 \).

\textbf{SOLUTION}

(a) Let \( g(x) = f(x)^2 + f'(x)^2 \). Then

\[
g'(x) = 2f(x)f'(x) + 2f(x)f''(x) = 2f(x)f'(x) + 2f'(x)(-f(x)) = 0,
\]

where we have used the fact that \( f''(x) = -f(x) \). Because \( g'(0) = 0 \) for all \( x \), \( g(x) = f(x)^2 + f'(x)^2 \) must be a constant function. In other words, \( f(x)^2 + f'(x)^2 = C \) for some constant \( C \). To determine the value of \( C \), we can substitute any number for \( x \). In particular, for this problem, we want to substitute \( x = 0 \) and find \( C = f(0)^2 + f'(0)^2 \). Hence,

\[
f(x)^2 + f'(x)^2 = f(0)^2 + f'(0)^2.
\]

(b) Let \( f(x) = \sin x \). Then \( f'(x) = \cos x \) and \( f''(x) = -\sin x \), so \( f''(x) = -f(x) \). Next, let \( f(x) = \cos x \). Then \( f'(x) = -\sin x \), \( f''(x) = -\cos x \), and we again have \( f''(x) = -f(x) \). Finally, if we take \( f(x) = \sin x \), the result from part (a) guarantees that

\[
\sin^2 x + \cos^2 x = \sin^2 0 + \cos^2 0 = 0 + 1 = 1.
\]

75. Use Exercise 74 to prove: \( f(x) = \sin x \) is the unique solution of Eq. (1) such that \( f(0) = 0 \) and \( f'(0) = 1 \); and \( g(x) = \cos x \) is the unique solution such that \( g(0) = 1 \) and \( g'(0) = 0 \). This result can be used to develop all the properties of the trigonometric functions “analytically”—that is, without reference to triangles.

\textbf{SOLUTION} \hspace{1em} In part (b) of Exercise 73, it was shown that \( f(x) = \sin x \) satisfies Eq. (1), and we can directly calculate that \( f(0) = \sin 0 = 0 \) and \( f'(0) = \cos 0 = 1 \). Suppose there is another function, call it \( F(x) \), that satisfies Eq. (1) with the same initial conditions: \( F(0) = 0 \) and \( F'(0) = 1 \). By Exercise 74, it follows that \( F(x) = \sin x \) for all \( x \). Hence, \( f(x) = \sin x \) is the unique solution of Eq. (1) satisfying \( f(0) = 0 \) and \( f'(0) = 1 \). The proof that \( g(x) = \cos x \) is the unique solution of Eq. (1) satisfying \( g(0) = 1 \) and \( g'(0) = 0 \) is carried out in a similar manner.

4.4 The Shape of a Graph

\textbf{Preliminary Questions}

1. If \( f \) is concave up, then \( f' \) is (choose one):
   (a) increasing \hspace{1em} (b) decreasing

\textbf{SOLUTION} \hspace{1em} The correct response is (a): increasing. If the function is concave up, then \( f'' \) is positive. Since \( f'' \) is the derivative of \( f' \), it follows that the derivative of \( f' \) is positive and \( f'' \) must therefore be increasing.

2. What conclusion can you draw if \( f'(c) = 0 \) and \( f''(c) < 0 \)?

\textbf{SOLUTION} \hspace{1em} If \( f'(c) = 0 \) and \( f''(c) < 0 \), then \( f(c) \) is a local maximum.

3. True or False? If \( f(c) \) is a local min, then \( f''(c) \) must be positive.

\textbf{SOLUTION} \hspace{1em} False. \( f''(c) \) could be zero.

4. True or False? If \( f''(x) \) changes from + to − at \( x = c \), then \( f \) has a point of inflection at \( x = c \).

\textbf{SOLUTION} \hspace{1em} False. \( f \) will have a point of inflection at \( x = c \) only if \( x = c \) is in the domain of \( f \).

\textbf{Exercises}

1. Match the graphs in Figure 13 with the description:

\begin{align*}
\text{(a)} & \hspace{1em} f''(x) < 0 \text{ for all } x. & \text{(b)} & \hspace{1em} f''(x) \text{ goes from } + \text{ to } -. \\
\text{(c)} & \hspace{1em} f''(x) > 0 \text{ for all } x. & \text{(d)} & \hspace{1em} f''(x) \text{ goes from } - \text{ to } +. \\
\end{align*}

\textbf{FIGURE 13}

\textbf{SOLUTION}

(a) In C, we have \( f''(x) < 0 \) for all \( x \).

(b) In A, \( f''(x) \) goes from + to −.

(c) In B, we have \( f''(x) > 0 \) for all \( x \).

(d) In D, \( f''(x) \) goes from − to +.
In Exercises 3–18, determine the intervals on which the function is concave up or down and find the points of inflection.

3. \( y = x^2 - 4x + 3 \)

**SOLUTION** Let \( f(x) = x^2 - 4x + 3 \). Then \( f'(x) = 2x - 4 \) and \( f''(x) = 2 > 0 \) for all \( x \). Therefore, \( f \) is concave up everywhere, and there are no points of inflection.

5. \( y = 10x^3 - x^5 \)

**SOLUTION** Let \( f(x) = 10x^3 - x^5 \). Then \( f'(x) = 30x^2 - 5x^4 \) and \( f''(x) = 60x - 20x^3 = 20x(3 - x^2) \). Now, \( f \) is concave up for \( x < -\sqrt{3} \) and for \( 0 < x < \sqrt{3} \) since \( f''(x) > 0 \) there. Moreover, \( f \) is concave down for \( -\sqrt{3} < x < 0 \) and for \( x > \sqrt{3} \) since \( f''(x) < 0 \) there. Finally, because \( f''(x) \) changes sign at \( x = 0 \) and at \( x = \pm \sqrt{3} \), \( f \) has a point of inflection at \( x = 0 \) and at \( x = \pm \sqrt{3} \).

7. \( y = \theta - 2 \sin \theta \quad [0, 2\pi] \)

**SOLUTION** Let \( f(\theta) = \theta - 2 \sin \theta \). Then \( f'(\theta) = 1 - 2 \cos \theta \) and \( f''(\theta) = 2 \sin \theta \). Now, \( f \) is concave up for \( 0 < \theta < \pi \) since \( f''(\theta) > 0 \) there. Moreover, \( f \) is concave down for \( \pi < \theta < 2\pi \) since \( f''(\theta) < 0 \) there. Finally, because \( f''(\theta) \) changes sign at \( \theta = \pi \), \( f(\theta) \) has a point of inflection at \( \theta = \pi \).

9. \( y = x(x - 8\sqrt{x}) \quad (x \geq 0) \)

**SOLUTION** Let \( f(x) = x(x - 8\sqrt{x}) = x^2 - 8x^{3/2} \). Then \( f'(x) = 2x - 12x^{1/2} \) and \( f''(x) = 2 - 6x^{-1/2} \). Now, \( f \) is concave down for \( 0 < x < 9 \) since \( f''(x) < 0 \) there. Moreover, \( f \) is concave up for \( x > 9 \) since \( f''(x) > 0 \) there. Finally, because \( f''(x) \) changes sign at \( x = 9 \), \( f(x) \) has a point of inflection at \( x = 9 \).

11. \( y = (x - 2)(1 - x^3) \)

**SOLUTION** Let \( f(x) = (x - 2)(1 - x^3) = x - x^4 - 2 + 2x^3 \). Then \( f'(x) = 1 - 4x^3 + 6x^2 \) and \( f''(x) = 12x - 12x^2 = 12x(1 - x) = 0 \) at \( x = 0 \) and \( x = 1 \). Now, \( f \) is concave up on \((0, 1)\) since \( f''(x) > 0 \) there. Moreover, \( f \) is concave down on \((-\infty, 0) \cup (1, \infty)\) since \( f''(x) < 0 \) there. Finally, because \( f''(x) \) changes sign at \( x = 0 \) and \( x = 1 \), \( f(x) \) has a point of inflection at both \( x = 0 \) and \( x = 1 \).

13. \( y = \frac{1}{x^2 + 3} \)

**SOLUTION** Let \( f(x) = \frac{1}{x^2 + 3} \). Then \( f'(x) = \frac{-2x}{(x^2 + 3)^2} \) and

\[
f''(x) = \frac{-2(2x^4 + 3) - 8x^2(2x^2 + 3)}{(x^2 + 3)^4} = \frac{6x^2 - 6}{(x^2 + 3)^3}.
\]

Now, \( f \) is concave up for \( |x| > 1 \) since \( f''(x) > 0 \) there. Moreover, \( f \) is concave down for \( |x| < 1 \) since \( f''(x) < 0 \) there. Finally, because \( f''(x) \) changes sign at both \( x = -1 \) and \( x = 1 \), \( f(x) \) has a point of inflection at both \( x = -1 \) and \( x = 1 \).

15. \( y = xe^{-3x} \)

**SOLUTION** Let \( f(x) = xe^{-3x} \). Then \( f'(x) = e^{-3x} - 3xe^{-3x} = (1 - 3x)e^{-3x} \) and \( f''(x) = -3(1 - 3x)e^{-3x} - 3e^{-3x} = (9x - 6)e^{-3x} \). Now, \( f \) is concave down for \( x < \frac{2}{3} \) since \( f''(x) < 0 \) there. Moreover, \( f \) is concave up for \( x > \frac{2}{3} \) since \( f''(x) > 0 \) there. Finally, because \( f''(x) \) changes sign at \( x = \frac{2}{3} \), \( x = \frac{2}{3} \) is a point of inflection.

17. \( y = 2x^2 + \ln x \quad (x > 0) \)

**SOLUTION** Let \( f(x) = 2x^2 + \ln x \). Then \( f'(x) = 4x + x^{-1} \) and \( f''(x) = 4 - x^{-2} \). Now, \( f \) is concave down for \( x < \frac{1}{2} \) since \( f''(x) < 0 \) there. Moreover, \( f \) is concave up for \( x > \frac{1}{2} \) since \( f''(x) > 0 \) there. Finally, because \( f''(x) \) changes sign at \( x = \frac{1}{2} \), \( f(x) \) has a point of inflection at \( x = \frac{1}{2} \).

19. The growth of a sunflower during the first 100 days after sprouting is modeled well by the logistic curve \( y = h(t) \) shown in Figure 15. Estimate the growth rate at the point of inflection and explain its significance. Then make a rough sketch of the first and second derivatives of \( h(t) \).
SOLUTION The point of inflection in Figure 15 appears to occur at $t = 40$ days. The graph below shows the logistic curve with an approximate tangent line drawn at $t = 40$. The approximate tangent line passes roughly through the points $(20, 20)$ and $(60, 240)$. The growth rate at the point of inflection is thus

$$
\frac{240 - 20}{60 - 20} = \frac{220}{40} = 5.5 \text{ cm/day}.
$$

Because the logistic curve changes from concave up to concave down at $t = 40$, the growth rate at this point is the maximum growth rate for the sunflower plant.

![Figure 17](image)

Sketches of the first and second derivative of $h(t)$ are shown below at the left and at the right, respectively.

21. Repeat Exercise 20 but assume that Figure 16 is the graph of the derivative $f'(x)$.

SOLUTION Points of inflection occur when $f''(x)$ changes sign. Consequently, points of inflection occur when $f'(x)$ changes from increasing to decreasing or from decreasing to increasing. In Figure 16, this occurs at $x = b$ and at $x = e$; therefore, $f(x)$ has an inflection point at $x = b$ and another at $x = e$. The function $f(x)$ will be concave down when $f''(x) < 0$ or when $f'(x)$ is decreasing. Thus, $f(x)$ is concave down for $b < x < e$.

23. Figure 17 shows the derivative $f'(x)$ on $[0, 1.2]$. Locate the points of inflection of $f(x)$ and the points where the local minima and maxima occur. Determine the intervals on which $f(x)$ has the following properties:

(a) Increasing  
(b) Decreasing  
(c) Concave up  
(d) Concave down

![Figure 17](image)

SOLUTION Recall that the graph is that of $f'$, not $f$. The inflection points of $f$ occur where $f'$ changes from increasing to decreasing or vice versa because it is at these points that the sign of $f''$ changes. From the graph we conclude that $f$ has points of inflection at $x = 0.17$, $x = 0.64$, and $x = 1$. The local extrema of $f$ occur where $f'$ changes sign. This occurs at $x = 0.4$. Because the sign of $f'$ changes from $+$ to $-$, $f(0.4)$ is a local maximum. There are no local minima.

(a) $f$ is increasing when $f'$ is positive. Hence, $f$ is increasing on $(0, 0.4)$.
(b) $f$ is decreasing when $f'$ is negative. Hence, $f$ is decreasing on $(0.4, 1) \cup (1, 1.2)$.
(c) Now $f$ is concave up where $f'$ is increasing. This occurs on $(0, 0.17) \cup (0.64, 1)$.
(d) Moreover, $f$ is concave down where $f'$ is decreasing. This occurs on $(0.17, 0.64) \cup (1, 1.2)$.

In Exercises 25–38, find the critical points and apply the Second Derivative Test (or state that it fails).

25. $f(x) = x^3 - 12x^2 + 45x$

SOLUTION Let $f(x) = x^3 - 12x^2 + 45x$. Then $f'(x) = 3x^2 - 24x + 45 = 3(x - 3)(x - 5)$, and the critical points are $x = 3$ and $x = 5$. Moreover, $f''(x) = 6x - 24$, so $f''(3) = -6 < 0$ and $f''(5) = 6 > 0$. Therefore, by the Second Derivative Test, $f(3) = 54$ is a local maximum, and $f(5) = 50$ is a local minimum.
27. \( f(x) = 3x^4 - 8x^3 + 6x^2 \)

**Solution** Let \( f(x) = 3x^4 - 8x^3 + 6x^2 \). Then \( f'(x) = 12x^3 - 24x^2 + 12x = 12x(x - 1)^2 = 0 \) at \( x = 0, 1 \) and \( f''(x) = 36x^2 - 48x + 12 \). Thus, \( f''(0) > 0 \), which implies \( f(0) \) is a local minimum; however, \( f''(1) = 0 \), which is inconclusive.

29. \( f(x) = \frac{x^2 - 8x}{x + 1} \)

**Solution** Let \( f(x) = \frac{x^2 - 8x}{x + 1} \). Then

\[
f'(x) = \frac{x^2 + 2x - 8}{(x + 1)^2} \quad \text{and} \quad f''(x) = \frac{2(x + 1)^2 - 2(x^2 + 2x - 8)}{(x + 1)^3}.
\]

Thus, the critical points are \( x = -4 \) and \( x = 2 \). Moreover, \( f''(-4) < 0 \) and \( f''(2) > 0 \). Therefore, by the second derivative test, \( f(-4) = -16 \) is a local maximum and \( f(2) = -4 \) is a local minimum.

31. \( y = 6x^{3/2} - 4x^{1/2} \)

**Solution** Let \( f(x) = 6x^{3/2} - 4x^{1/2} \). Then \( f'(x) = 9x^{1/2} - 2x^{-1/2} = x^{-1/2}(9x - 2) \), so there are two critical points: \( x = 0 \) and \( x = \frac{2}{9} \). Now,

\[
f''(x) = \frac{9}{2}x^{-1/2} + x^{-3/2} = \frac{1}{2}x^{-3/2}(9x + 2).
\]

Thus, \( f'' \left( \frac{2}{9} \right) > 0 \), which implies \( f \left( \frac{2}{9} \right) \) is a local minimum. \( f''(x) \) is undefined at \( x = 0 \), so the Second Derivative Test cannot be applied there.

33. \( f(x) = \sin^2 x + \cos x \), \([0, \pi]\)

**Solution** Let \( f(x) = \sin^2 x + \cos x \). Then \( f'(x) = 2 \sin x \cos x - \sin x = \sin x(2 \cos x - 1) \). On the interval \([0, \pi]\), \( f'(x) = 0 \) at \( x = 0 \), \( x = \frac{\pi}{2} \) and \( x = \pi \). Now,

\[
f''(x) = 2 \cos^2 x - 2 \sin^2 x - \cos x.
\]

Thus, \( f''(0) > 0 \), so \( f(0) \) is a local minimum. On the other hand, \( f'' \left( \frac{\pi}{2} \right) < 0 \), so \( f \left( \frac{\pi}{2} \right) \) is a local maximum. Finally, \( f''(\pi) > 0 \), so \( f(\pi) \) is a local minimum.

35. \( f(x) = xe^{-x^2} \)

**Solution** Let \( f(x) = xe^{-x^2} \). Then \( f'(x) = -2x^2e^{-x^2} + e^{-x^2} = (1 - 2x^2)e^{-x^2} \), so there are two critical points: \( x = \pm \frac{\sqrt{2}}{2} \). Now,

\[
f''(x) = (4x^3 - 2xe^{-x^2} - 4xe^{-x^2} = (4x^3 - 6x)e^{-x^2}.
\]

Thus, \( f'' \left( \frac{\sqrt{2}}{2} \right) < 0 \), so \( f \left( \frac{\sqrt{2}}{2} \right) \) is a local maximum. On the other hand, \( f'' \left( -\frac{\sqrt{2}}{2} \right) > 0 \), so \( f \left( -\frac{\sqrt{2}}{2} \right) \) is a local minimum.

37. \( f(x) = x^3 \ln x \) \((x > 0)\)

**Solution** Let \( f(x) = x^3 \ln x \). Then \( f'(x) = x^2 + 3x^2 \ln x = x^2(1 + 3 \ln x) \), so there is only one critical point: \( x = e^{-1/3} \). Now,

\[
f''(x) = 3x + 2x(1 + 3 \ln x) = x(5 + 6 \ln x).
\]

Thus, \( f'' \left( e^{-1/3} \right) > 0 \), so \( f \left( e^{-1/3} \right) \) is a local minimum.
In Exercises 39–52, find the intervals on which \( f \) is concave up or down, the points of inflection, the critical points, and the local minima and maxima.

**SOLUTION** Here is a table legend for Exercises 39–49.

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>MEANING</th>
</tr>
</thead>
<tbody>
<tr>
<td>−</td>
<td>The entity is negative on the given interval.</td>
</tr>
<tr>
<td>0</td>
<td>The entity is zero at the specified point.</td>
</tr>
<tr>
<td>+</td>
<td>The entity is positive on the given interval.</td>
</tr>
<tr>
<td>U</td>
<td>The entity is undefined at the specified point.</td>
</tr>
<tr>
<td>/</td>
<td>The function ( f, g, \text{ etc.} ) is increasing on the given interval.</td>
</tr>
<tr>
<td>\</td>
<td>The function ( f, g, \text{ etc.} ) is decreasing on the given interval.</td>
</tr>
<tr>
<td>~</td>
<td>The function ( f, g, \text{ etc.} ) is concave up on the given interval.</td>
</tr>
<tr>
<td>~</td>
<td>The function ( f, g, \text{ etc.} ) is concave down on the given interval.</td>
</tr>
<tr>
<td>M</td>
<td>The function ( f, g, \text{ etc.} ) has a local maximum at the specified point.</td>
</tr>
<tr>
<td>m</td>
<td>The function ( f, g, \text{ etc.} ) has a local minimum at the specified point.</td>
</tr>
<tr>
<td>I</td>
<td>The function ( f, g, \text{ etc.} ) has an inflection point here.</td>
</tr>
<tr>
<td>~</td>
<td>There is no local extremum or inflection point here.</td>
</tr>
</tbody>
</table>

39. \( f(x) = x^3 - 2x^2 + x \)

**SOLUTION** Let \( f(x) = x^3 - 2x^2 + x \).
- Then \( f'(x) = 3x^2 - 4x + 1 = (x - 1)(3x - 1) = 0 \) yields \( x = 1 \) and \( x = \frac{1}{3} \) as candidates for extrema.
- Moreover, \( f''(x) = 6x - 4 = 0 \) gives a candidate for a point of inflection at \( x = \frac{2}{3} \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>((-\infty, \frac{1}{3}))</th>
<th>( \frac{1}{3} )</th>
<th>((\frac{1}{3}, 1))</th>
<th>1</th>
<th>((1, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' )</td>
<td>+</td>
<td>0</td>
<td>0</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>( f'' )</td>
<td>/</td>
<td>M</td>
<td>~</td>
<td>m</td>
<td>~</td>
</tr>
</tbody>
</table>

41. \( f(t) = t^2 - t^3 \)

**SOLUTION** Let \( f(t) = t^2 - t^3 \).
- Then \( f'(t) = 2t - 3t^2 = t(2 - 3t) = 0 \) yields \( t = 0 \) and \( t = \frac{2}{3} \) as candidates for extrema.
- Moreover, \( f''(t) = 2 - 6t = 0 \) gives a candidate for a point of inflection at \( t = \frac{1}{3} \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>((-\infty, 0))</th>
<th>0</th>
<th>((0, \frac{2}{3}))</th>
<th>( \frac{2}{3} )</th>
<th>((\frac{2}{3}, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' )</td>
<td>-</td>
<td>0</td>
<td>+</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>( f'' )</td>
<td>~</td>
<td>m</td>
<td>~</td>
<td>M</td>
<td>~</td>
</tr>
</tbody>
</table>

43. \( f(x) = x^2 - 8x^{1/2} \quad (x \geq 0) \)

**SOLUTION** Let \( f(x) = x^2 - 8x^{1/2} \). Note that the domain of \( f \) is \( x \geq 0 \).
- Then \( f'(x) = 2x - 4x^{-1/2} = x^{-1/2}(2x^{3/2} - 4) = 0 \) yields \( x = 0 \) and \( x = \sqrt[3]{2} = (2)^{2/3} \) as candidates for extrema.
- Moreover, \( f''(x) = 2 + 2x^{-3/2} > 0 \) for all \( x \geq 0 \), which means there are no inflection points.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>((0, (2)^{2/3}))</th>
<th>((2)^{2/3})</th>
<th>((2)^{2/3}, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' )</td>
<td>U</td>
<td>-</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>( f'' )</td>
<td>M</td>
<td>~</td>
<td>m</td>
<td>~</td>
</tr>
</tbody>
</table>
45. \( f(x) = \frac{x}{x^2 + 27} \)

**SOLUTION** Let \( f(x) = \frac{x}{x^2 + 27} \)
- Then \( f'(x) = \frac{27 - x^2}{(x^2 + 27)^2} \) yields \( x = \pm 3\sqrt{3} \) as candidates for extrema.
- Moreover, \( f''(x) = \frac{-2x(x^2 + 27)^2 - (27 - x^2)(2)(2x)(x^2 + 27)}{(x^2 + 27)^4} = \frac{2x(x^2 - 81)}{(x^2 + 27)^3} \) yields candidates for a point of inflection at \( x = 0 \) and at \( x = \pm 9 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( (-\infty, -3\sqrt{3}) )</th>
<th>(-3\sqrt{3})</th>
<th>(-3\sqrt{3}, 3\sqrt{3})</th>
<th>( 3\sqrt{3})</th>
<th>( (3\sqrt{3}, \infty) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' )</td>
<td>( - )</td>
<td>( 0 )</td>
<td>( + )</td>
<td>( 0 )</td>
<td>( - )</td>
</tr>
<tr>
<td>( f'' )</td>
<td>( \searrow )</td>
<td>( m )</td>
<td>( \nearrow )</td>
<td>( M )</td>
<td>( \searrow )</td>
</tr>
</tbody>
</table>

47. \( f(\theta) = \theta + \sin \theta, \quad [0, 2\pi] \)

**SOLUTION** Let \( f(\theta) = \theta + \sin \theta \) on \([0, 2\pi]\).
- Then \( f'(\theta) = 1 + \cos \theta = \theta = \pi \) as a candidate for an extremum.
- Moreover, \( f''(\theta) = -\sin \theta = 0 \) gives candidates for a point of inflection at \( \theta = 0, \theta = \pi, \) and at \( \theta = 2\pi \).

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( (0, \pi) )</th>
<th>( \pi )</th>
<th>( (\pi, 2\pi) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' )</td>
<td>( + )</td>
<td>( 0 )</td>
<td>( + )</td>
</tr>
<tr>
<td>( f'' )</td>
<td>( \nearrow )</td>
<td>( \searrow )</td>
<td>( \searrow )</td>
</tr>
</tbody>
</table>

49. \( f(x) = \tan x, \quad \left[ -\frac{\pi}{4}, \frac{\pi}{4} \right] \)

**SOLUTION** Let \( f(x) = \tan x \) on \( \left[ -\frac{\pi}{4}, \frac{\pi}{4} \right] \).
- Then \( f'(x) = \sec^2 x \geq 1 > 0 \) on \( \left[ -\frac{\pi}{4}, \frac{\pi}{4} \right] \).
- Moreover, \( f''(x) = 2 \sec x \cdot \sec x \tan x = 2 \sec^2 x \tan x = 0 \) gives a candidate for a point of inflection at \( x = 0 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \left[ -\frac{\pi}{4}, \frac{\pi}{4} \right] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' )</td>
<td>( + )</td>
</tr>
<tr>
<td>( f'' )</td>
<td>( \nearrow )</td>
</tr>
</tbody>
</table>

51. \( y = (x^2 - 2)e^{-x} \) \( (x > 0) \)

**SOLUTION** Let \( f(x) = (x^2 - 2)e^{-x} \).
- Then \( f'(x) = -(x^2 - 2x - 2)e^{-x} = 0 \) gives \( x = 1 + \sqrt{3} \) as a candidate for an extremum.
- Moreover, \( f''(x) = (x^2 - 4x)e^{-x} = 0 \) gives \( x = 4 \) as a candidate for a point of inflection.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( (0, 1 + \sqrt{3}) )</th>
<th>( 1 + \sqrt{3} )</th>
<th>( (1 + \sqrt{3}, \infty) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' )</td>
<td>( + )</td>
<td>( 0 )</td>
<td>( - )</td>
</tr>
<tr>
<td>( f'' )</td>
<td>( \searrow )</td>
<td>( \nearrow )</td>
<td>( \searrow )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( (0, 4) )</th>
<th>( (4, \infty) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'' )</td>
<td>( - )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( f )</td>
<td>( \nearrow )</td>
<td>( 1 )</td>
</tr>
</tbody>
</table>
53. Sketch the graph of an increasing function such that \( f''(x) \) changes from \(+\) to \( -\) at \( x = 2 \) and from \( -\) to \(+\) at \( x = 4 \). Do the same for a decreasing function.

**SOLUTION** The graph shown below is an increasing function which changes from concave up to concave down at \( x = 2 \) and from concave down to concave up at \( x = 4 \). The graph shown below at the right is a decreasing function which changes from concave up to concave down at \( x = 2 \) and from concave down to concave up at \( x = 4 \).

\begin{align*}
\text{Graph 1: Increasing} &\quad \text{Graph 2: Decreasing}
\end{align*}

In Exercises 54–56, sketch the graph of a function \( f(x) \) satisfying all of the given conditions.

55. (i) \( f'(x) > 0 \) for all \( x \), and
(ii) \( f''(x) < 0 \) for \( x < 0 \) and \( f''(x) > 0 \) for \( x > 0 \).

**SOLUTION** Here is the graph of a function \( f(x) \) satisfying (i) \( f'(x) > 0 \) for all \( x \) and (ii) \( f''(x) < 0 \) for \( x < 0 \) and \( f''(x) > 0 \) for \( x > 0 \).

\begin{align*}
\text{Graph: Increasing and Concave Down}
\end{align*}

57. An infectious flu spreads slowly at the beginning of an epidemic. The infection process accelerates until a majority of the susceptible individuals are infected, at which point the process slows down.

(a) If \( R(t) \) is the number of individuals infected at time \( t \), describe the concavity of the graph of \( R \) near the beginning and end of the epidemic.

(b) Describe the status of the epidemic on the day that \( R(t) \) has a point of inflection.

**SOLUTION**

(a) Near the beginning of the epidemic, the graph of \( R \) is concave up. Near the epidemic’s end, \( R \) is concave down.

(b) “Epidemic subsiding: number of new cases declining.”

59. Water is pumped into a sphere of radius \( R \) at a variable rate in such a way that the water level rises at a constant rate (Figure 18). Let \( V(t) \) be the volume of water in the tank at time \( t \). Sketch the graph \( V(t) \) (approximately, but with the correct concavity). Where does the point of inflection occur?

**SOLUTION** Because water is entering the sphere in such a way that the water level rises at a constant rate, we expect the volume to increase more slowly near the bottom and top of the sphere where the sphere is not as “wide” and to increase more rapidly near the middle of the sphere. The graph of \( V(t) \) should therefore start concave up and change to concave down when the sphere is half full; that is, the point of inflection should occur when the water level is equal to the radius of the sphere. A possible graph of \( V(t) \) is shown below.

\begin{align*}
\text{Graph: Volume vs Time}
\end{align*}

61. **Image Processing** The intensity of a pixel in a digital image is measured by a number \( u \) between 0 and 1. Often, images can be enhanced by rescaling intensities (Figure 19), where pixels of intensity \( u \) are displayed with intensity \( g(u) \) for a suitable function \( g(u) \). One common choice is the sigmoidal correction, defined for constants \( a, b \) by

\[
g(u) = \frac{f(u) - f(0)}{f(1) - f(0)} \quad \text{where} \quad f(u) = \left(1 + e^{b(a-u)}\right)^{-1}
\]
Figure 20 shows that \( g(u) \) reduces the intensity of low-intensity pixels (where \( g(u) < u \)) and increases the intensity of high-intensity pixels.

(a) Verify that \( f'(u) > 0 \) and use this to show that \( g(u) \) increases from 0 to 1 for \( 0 \leq u \leq 1 \).

(b) Where does \( g(u) \) have a point of inflection?

**SOLUTION**

(a) With \( f(u) = (1 + e^{b(a-u)})^{-1} \), it follows that

\[
f'(u) = -(1 + e^{b(a-u)})^{-2} \cdot b e^{b(a-u)} = \frac{be^{b(a-u)}}{(1 + e^{b(a-u)})^2} > 0
\]

for all \( u \). Next, observe that

\[
g(0) = \frac{f(0) - f(0)}{f(1) - f(0)} = 0,
\]

\[
g(1) = \frac{f(1) - f(0)}{f(1) - f(0)} = 1,
\]

and

\[
g'(u) = \frac{1}{f(1) - f(0)} f'(u) > 0
\]

for all \( u \). Thus, \( g(u) \) increases from 0 to 1 for \( 0 \leq u \leq 1 \).

(b) Working from part (a), we find

\[
f''(u) = \frac{b^2 e^{b(a-u)} (2e^{b(a-u)} - 1)}{(1 + e^{b(a-u)})^3}.
\]

Because

\[
g''(u) = \frac{1}{f(1) - f(0)} f''(u),
\]

it follows that \( g(u) \) has a point of inflection when

\[2e^{b(a-u)} - 1 = 0 \quad \text{or} \quad u = a + \frac{1}{b} \ln 2.\]

**Further Insights and Challenges**

In Exercises 63–65, assume that \( f(x) \) is differentiable.

63. **Proof of the Second Derivative Test** Let \( c \) be a critical point such that \( f''(c) > 0 \) (the case \( f''(c) < 0 \) is similar).

(a) Show that \( f''(c) = \lim_{h \to 0} \frac{f'(c + h) - f'(c)}{h} \).

(b) Use (a) to show that there exists an open interval \((a, b)\) containing \( c \) such that \( f'(x) < 0 \) if \( a < x < c \) and \( f'(x) > 0 \) if \( c < x < b \). Conclude that \( f(c) \) is a local minimum.

**SOLUTION**

(a) Because \( c \) is a critical point, either \( f'(c) = 0 \) or \( f'(c) \) does not exist; however, \( f''(c) \) exists, so \( f'(c) \) must also exist. Therefore, \( f'(c) = 0 \). Now, from the definition of the derivative, we have

\[
f''(c) = \lim_{h \to 0} \frac{f'(c + h) - f'(c)}{h} = \lim_{h \to 0} \frac{f'(c + h)}{h}.
\]
(b) We are given that \( f''(c) > 0 \). By part (a), it follows that

\[
\lim_{h \to 0} \frac{f'(c + h)}{h} > 0;
\]

in other words, for sufficiently small \( h \),

\[
\frac{f'(c + h)}{h} > 0.
\]

Now, if \( h \) is sufficiently small but negative, then \( f'(c + h) \) must also be negative (so that the ratio \( f'(c + h)/h \) will be positive) and \( c + h < c \). On the other hand, if \( h \) is sufficiently small but positive, then \( f'(c + h) \) must also be positive and \( c + h > c \). Thus, there exists an open interval \((a, b)\) containing \( c \) such that \( f'(x) < 0 \) for \( a < x < c \) and \( f'(x) > 0 \) for \( c < x < b \). Finally, because \( f'(x) \) changes from negative to positive at \( x = c \), \( f(c) \) must be a local minimum.

65. Assume that \( f''(x) \) exists and let \( c \) be a point of inflection of \( f(x) \).

(a) Use the method of Exercise 64 to prove that the tangent line at \( x = c \) crosses the graph (Figure 21). Hint: Show that \( G(x) \) changes sign at \( x = c \).

(b) Verify this conclusion for \( f(x) = \frac{x}{3x^2 + 1} \) by graphing \( f(x) \) and the tangent line at each inflection point on the same set of axes.

![Figure 21](https://example.com/tangent-line-inflection.png)

**SOLUTION**

(a) Let \( G(x) = f(x) - f'(c)(x - c) - f(c) \). Then, as in Exercise 63, \( G(c) = G'(c) = 0 \) and \( G''(x) = f''(x) \). If \( f''(x) \) changes from positive to negative at \( x = c \), then so does \( G''(x) \) and \( G'(x) \) is increasing for \( x < c \) and decreasing for \( x > c \). This means that \( G'(x) < 0 \) for \( x < c \) and \( G'(x) > 0 \) for \( x > c \). This in turn implies that \( G(x) \) is decreasing, so \( G(x) > 0 \) for \( x < c \) but \( G(x) < 0 \) for \( x > c \). On the other hand, if \( f''(x) \) changes from negative to positive at \( x = c \), then so does \( G''(x) \) and \( G'(x) \) is decreasing for \( x < c \) and increasing for \( x > c \). Thus, \( G'(x) < 0 \) for \( x < c \) and \( G'(x) > 0 \) for \( x > c \). This in turn implies that \( G(x) \) is increasing, so \( G(x) < 0 \) for \( x < c \) and \( G(x) > 0 \) for \( x > c \). In either case, \( G(x) \) changes sign at \( x = c \), and the tangent line at \( x = c \) crosses the graph of the function.

(b) Let \( f(x) = \frac{x}{3x^2 + 1} \). Then

\[
f'(x) = \frac{1 - 3x^2}{(3x^2 + 1)^2} \quad \text{and} \quad f''(x) = \frac{-18x(1 - x^2)}{(3x^2 + 1)^3}.
\]

Therefore \( f(x) \) has a point of inflection at \( x = 0 \) and at \( x = \pm 1 \). The figure below shows the graph of \( y = f(x) \) and its tangent lines at each of the points of inflection. It is clear that each tangent line crosses the graph of \( f(x) \) at the inflection point.

![Graph](https://example.com/graph.png)

67. Let \( f(x) \) be a polynomial of degree \( n \geq 2 \). Show that \( f(x) \) has at least one point of inflection if \( n \) is odd. Then give an example to show that \( f(x) \) need not have a point of inflection if \( n \) is even.

**SOLUTION** Let \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) be a polynomial of degree \( n \). Then \( f'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + 2 a_2 x + a_1 \) and \( f''(x) = n(n-1) a_n x^{n-2} + (n-2) a_{n-1} x^{n-3} + \cdots + 6 a_3 x + 2 a_2 \). If \( n \geq 3 \) and is odd, then \( n - 2 \) is also odd and \( f''(x) \) is a polynomial of odd degree. Therefore \( f''(x) \) must take on both positive and negative values. It follows that \( f''(x) \) has at least one root \( c \) such that \( f''(c) \) changes sign at \( c \). The function \( f(x) \) will then have a point of inflection at \( x = c \). On the other hand, the functions \( f(x) = x^2, x^4 \) and \( x^8 \) are polynomials of even degree that do not have any points of inflection.
4.5 L'Hôpital's Rule

**Preliminary Questions**

1. What is wrong with applying L'Hôpital’s Rule to \( \lim_{x \to 0} \frac{x^2 - 2x}{3x - 2} \)?

**Solution**

As \( x \to 0 \),

\[
\frac{x^2 - 2x}{3x - 2}
\]

is not of the form \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \), so L'Hôpital's Rule cannot be used.

2. Does L'Hôpital’s Rule apply to \( \lim_{x \to a} f(x)g(x) \) if \( f(x) \) and \( g(x) \) both approach \( \infty \) as \( x \to a \)?

**Solution**

No. L'Hôpital’s Rule only applies to limits of the form \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \).

**Exercises**

In Exercises 1–10, use L'Hôpital’s Rule to evaluate the limit, or state that L'Hôpital’s Rule does not apply.

1. \( \lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 4} \)

**Solution**

Because the quotient is not indeterminate at \( x = 3 \),

\[
\frac{2x^2 - 5x - 3}{x - 4} \bigg|_{x=3} = \frac{18 - 15 - 3}{3 - 4} = \frac{0}{-1},
\]

L'Hôpital’s Rule does not apply.

3. \( \lim_{x \to 4} \frac{x^3 - 64}{x^2 + 16} \)

**Solution**

Because the quotient is not indeterminate at \( x = 4 \),

\[
\frac{x^3 - 64}{x^2 + 16} \bigg|_{x=4} = \frac{64 - 64}{16 + 16} = \frac{0}{32}.
\]

L'Hôpital’s Rule does not apply.

5. \( \lim_{x \to 9} \frac{x^{1/2} + x - 6}{x^{3/2} - 27} \)

**Solution**

Because the quotient is not indeterminate at \( x = 9 \),

\[
\frac{x^{1/2} + x - 6}{x^{3/2} - 27} \bigg|_{x=9} = \frac{3 + 9 - 6}{27 - 27} = \frac{6}{0},
\]

L'Hôpital’s Rule does not apply.

7. \( \lim_{x \to 0} \frac{\sin 4x}{x^2 + 3x + 1} \)

**Solution**

Because the quotient is not indeterminate at \( x = 0 \),

\[
\frac{\sin 4x}{x^2 + 3x + 1} \bigg|_{x=0} = \frac{0}{0 + 0 + 1} = \frac{0}{1},
\]

L'Hôpital’s Rule does not apply.

9. \( \lim_{x \to 0} \frac{\cos 2x - 1}{\sin 5x} \)

**Solution**

The functions \( \cos 2x - 1 \) and \( \sin 5x \) are differentiable, but the quotient is indeterminate at \( x = 0 \),

\[
\frac{\cos 2x - 1}{\sin 5x} \bigg|_{x=0} = \frac{1 - 1}{0} = \frac{0}{0},
\]

so L'Hôpital’s Rule applies. We find

\[
\lim_{x \to 0} \frac{\cos 2x - 1}{\sin 5x} = \lim_{x \to 0} -2 \sin 2x \cdot \frac{5}{\cos 5x} = \frac{0}{5} = 0.
\]
In Exercises 11–16, show that L'Hôpital's Rule is applicable to the limit as \( x \to \pm \infty \) and evaluate.

11. \( \lim_{x \to \infty} \frac{9x + 4}{3 - 2x} \)

**SOLUTION** As \( x \to \infty \), the quotient \( \frac{9x + 4}{3 - 2x} \) is of the form \( \frac{\infty}{\infty} \), so L'Hôpital's Rule applies. We find

\[
\lim_{x \to \infty} \frac{9x + 4}{3 - 2x} = \lim_{x \to \infty} \frac{9}{-2} = -\frac{9}{2}.
\]

13. \( \lim_{x \to \infty} \frac{\ln x}{e^x} \)

**SOLUTION** As \( x \to \infty \), the quotient \( \frac{\ln x}{e^x} \) is of the form \( \frac{\infty}{\infty} \), so L'Hôpital's Rule applies. We find

\[
\lim_{x \to \infty} \frac{\ln x}{e^x} = \lim_{x \to \infty} \frac{\frac{1}{x}}{e^x} = \lim_{x \to \infty} \frac{1}{xe^x} = 0.
\]

15. \( \lim_{x \to \infty} \frac{\ln(x^4 + 1)}{x} \)

**SOLUTION** As \( x \to \infty \), the quotient \( \frac{\ln(x^4 + 1)}{x} \) is of the form \( \frac{\infty}{\infty} \), so L'Hôpital's Rule applies. Here, we use L'Hôpital's Rule twice to find

\[
\lim_{x \to \infty} \frac{\ln(x^4 + 1)}{x} = \lim_{x \to \infty} \frac{\frac{4x^3}{x^4 + 1}}{1} = \lim_{x \to \infty} \frac{12x^2}{4x^3} = \lim_{x \to \infty} \frac{3}{x} = 0.
\]

In Exercises 17–54, evaluate the limit.

17. \( \lim_{x \to 1} \frac{\sqrt{8 + x} - 3^{1/3}}{x^2 - 3x + 2} \)

**SOLUTION** \( \lim_{x \to 1} \frac{\sqrt{8 + x} - 3^{1/3}}{x^2 - 3x + 2} = \lim_{x \to 1} \frac{\frac{1}{2}(8 + x)^{-1/2} - x^{-2/3}}{2x - 3} = \frac{1}{6} - \frac{1}{6} = \frac{5}{6} \).

19. \( \lim_{x \to \infty} \frac{3x - 2}{1 - 5x} \)

**SOLUTION** \( \lim_{x \to \infty} \frac{3x - 2}{1 - 5x} = \lim_{x \to \infty} \frac{3}{-5} = -\frac{3}{5} \).

21. \( \lim_{x \to \infty} \frac{7x^2 + 4x}{9 - 3x^2} \)

**SOLUTION** \( \lim_{x \to \infty} \frac{7x^2 + 4x}{9 - 3x^2} = \lim_{x \to \infty} \frac{14x + 4}{-6x} = \lim_{x \to \infty} \frac{7}{-3} = -\frac{7}{3} \).

23. \( \lim_{x \to \infty} \frac{(1 + 3x)^{1/2} - 2}{(1 + 7x)^{1/3} - 2} \)

**SOLUTION** Apply L'Hôpital's Rule once:

\[
\lim_{x \to 1} \frac{(1 + 3x)^{1/2} - 2}{(1 + 7x)^{1/3} - 2} = \lim_{x \to 1} \frac{\frac{3}{2}(1 + 3x)^{-1/2}}{\frac{7}{3}(1 + 7x)^{-2/3}} = \frac{(\frac{3}{2})^{1/2}}{(\frac{7}{3})^{2/3}} = \frac{9}{7}.
\]

25. \( \lim_{x \to 0} \frac{\sin 2x}{\sin 7x} \)

**SOLUTION** \( \lim_{x \to 0} \frac{\sin 2x}{\sin 7x} = \lim_{x \to 0} \frac{2 \cos 2x}{7 \cos 7x} = \frac{2}{7} \).

27. \( \lim_{x \to 0} \frac{\tan x}{x} \)

**SOLUTION** \( \lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sec^2 x}{1} = 1 \).
29. \[ \lim_{x \to 0} \frac{\sin x - x \cos x}{x - \sin x} \]

**SOLUTION**

\[ \lim_{x \to 0} \frac{\sin x - x \cos x}{x - \sin x} = \lim_{x \to 0} \frac{x \sin x}{\sin x} = \lim_{x \to 0} \frac{\cos x + x \cos x - x \sin x}{\cos x} = 2. \]

31. \[ \lim_{x \to 0} \frac{\cos(x + \frac{x}{2})}{\sin x} \]

**SOLUTION**

\[ \lim_{x \to 0} \frac{\cos(x + \frac{x}{2})}{\sin x} = \lim_{x \to 0} \frac{-\sin(x + \frac{x}{2})}{\cos x} = -1. \]

33. \[ \lim_{x \to \pi/2} \frac{\cos x}{\sin(2x)} \]

**SOLUTION**

\[ \lim_{x \to \pi/2} \frac{\cos x}{\sin(2x)} = \lim_{x \to \pi/2} \frac{-\sin x}{2 \cos(2x)} = \frac{1}{2}. \]

35. \[ \lim_{x \to \pi/2} (\sec x - \tan x) \]

**SOLUTION**

\[ \lim_{x \to \pi/2} (\sec x - \tan x) = \lim_{x \to \pi/2} \left( \frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) = \lim_{x \to \pi/2} \left( \frac{1 - \sin x}{\cos x} \right) = \lim_{x \to \pi/2} \left( \frac{-\cos x}{\sin x} \right) = 0. \]

37. \[ \lim_{x \to 1} \left( \frac{\pi x}{2} \right) \ln x \]

**SOLUTION**

\[ \lim_{x \to 1} \frac{\pi x}{2} \ln x = \lim_{x \to 1} \ln x = \lim_{x \to 1} \cot \left( \frac{\pi x}{2} \right) = \lim_{x \to 1} \left( 1 - \frac{\pi}{2} \cot \left( \frac{\pi x}{2} \right) \right) = \lim_{x \to 1} \left( \frac{\ln 2}{\cos x} \right) = \frac{\ln 2}{\cos \frac{\pi}{2}} = \frac{2}{\pi}. \]

39. \[ \lim_{x \to 0} \frac{e^x - 1}{\sin x} \]

**SOLUTION**

\[ \lim_{x \to 0} \frac{e^x - 1}{\sin x} = \lim_{x \to 0} \frac{e^x}{1} = 1. \]

41. \[ \lim_{x \to 0} \frac{e^{2x} - 1 - x}{x^2} \]

**SOLUTION**

\[ \lim_{x \to 0} \frac{e^{2x} - 1 - x}{x^2} = \lim_{x \to 0} \frac{2e^{2x} - 1}{2x} \] which does not exist.

43. \[ \lim_{t \to 0^+} (\sin t)(\ln t) \]

**SOLUTION**

\[ \lim_{t \to 0^+} (\sin t)(\ln t) = \lim_{t \to 0^+} \frac{\ln t}{t} = \lim_{t \to 0^+} \frac{1}{t} \cot t = \lim_{t \to 0^+} \frac{-\sin^2 t}{t \cos t} = \lim_{t \to 0^+} \frac{-2 \sin t \cos t}{t \cos t} = 0. \]

45. \[ \lim_{x \to 0} \frac{a^x - 1}{x} \] \(a > 0\)

**SOLUTION**

\[ \lim_{x \to 0} \frac{a^x - 1}{x} = \ln a \cdot \ln a = \ln a. \]

47. \[ \lim_{x \to 1} (1 + \ln x)^{1/(x-1)} \]

**SOLUTION**

\[ \lim_{x \to 1} \ln(1 + \ln x)^{1/(x-1)} = \lim_{x \to 1} \frac{1}{x-1} \ln(1 + \ln x) = \lim_{x \to 1} \frac{1}{x-1} \ln \left( \frac{x+1}{x} \right) = 1. \]

Hence,

\[ \lim_{x \to 1} (1 + \ln x)^{1/(x-1)} = \lim_{x \to 1} e^{(1+\ln x)^{1/(x-1)}} = e. \]
49. \( \lim_{x \to 0} (\cos x)^{3/x^2} \)

**SOLUTION**

\[
\lim_{x \to 0} \ln((\cos x)^{3/x^2}) = \lim_{x \to 0} \frac{3 \ln \cos x}{x^2} = \lim_{x \to 0} \frac{3 \tan x}{2x} = \lim_{x \to 0} \frac{3 \sec^2 x}{2} = -\frac{3}{2}.
\]

Hence, \( \lim_{x \to 0} (\cos x)^{3/x^2} = e^{-3/2} \).

51. \( \lim_{x \to 0} \frac{\sin^{-1} x}{x} \)

**SOLUTION**

\[
\lim_{x \to 0} \frac{\sin^{-1} x}{x} = \lim_{x \to 0} \frac{1}{\sqrt{1-x}} = 1.
\]

53. \( \lim_{x \to 1} \frac{\tan^{-1} x - \frac{\pi}{4}}{\frac{\pi}{4} - x} \)

**SOLUTION**

\[
\lim_{x \to 1} \frac{\tan^{-1} x - \frac{\pi}{4}}{\tan(\pi x/4) - 1} = \lim_{x \to 1} \frac{1}{\frac{\pi}{4} \sec^2(\pi x/4)} = \frac{1}{\frac{\pi}{4}} = \frac{4}{\pi}.
\]

55. Evaluate \( \lim_{x \to \pi/2} \frac{\cos mx}{\cos nx} \), where \( m, n \neq 0 \) are integers.

**SOLUTION** Suppose \( m \) and \( n \) are even. Then there exist integers \( k \) and \( l \) such that \( m = 2k \) and \( n = 2l \) and

\[
\lim_{x \to \pi/2} \frac{\cos mx}{\cos nx} = \frac{\cos k\pi}{\cos l\pi} = (-1)^{k-l}.
\]

Now, suppose \( m \) is even and \( n \) is odd. Then

\[
\lim_{x \to \pi/2} \frac{\cos mx}{\cos nx}
\]

does not exist (from one side the limit tends toward \(-\infty\), while from the other side the limit tends toward \(+\infty\)). Third, suppose \( m \) is odd and \( n \) is even. Then

\[
\lim_{x \to \pi/2} \frac{\cos mx}{\cos nx} = 0.
\]

Finally, suppose \( m \) and \( n \) are odd. This is the only case when the limit is indeterminate. Then there exist integers \( k \) and \( l \) such that \( m = 2k + 1, n = 2l + 1 \) and, by L'Hôpital's Rule,

\[
\lim_{x \to \pi/2} \frac{\cos mx}{\cos nx} = \lim_{x \to \pi/2} \frac{-m \sin mx}{-n \sin nx} = (-1)^{k-l} \frac{m}{n}.
\]

To summarize,

\[
\lim_{x \to \pi/2} \frac{\cos mx}{\cos nx} = \begin{cases} 
(-1)^{\left(\frac{m+n}{2}\right)}, & m, n \text{ even} \\
\text{does not exist}, & m \text{ even, } n \text{ odd} \\
0, & m \text{ odd, } n \text{ even} \\
(-1)^{\left(\frac{m-n}{2}\right)} \frac{m}{\pi}, & m, n \text{ odd}
\end{cases}
\]

57. Prove the following limit formula for \( e \):

\[
e = \lim_{x \to 0} (1 + x)^{1/x}
\]

Then find a value of \( x \) such that \(|(1 + x)^{1/x} - e| \leq 0.001 \).

**SOLUTION** Using L'Hôpital's Rule,

\[
\lim_{x \to 0} \frac{\ln(1 + x)}{x} = \lim_{x \to 0} \frac{1}{1 + x} = 1.
\]
Thus,

$$\lim_{x \to 0} \ln \left(1 + \frac{1}{x}\right) = \lim_{x \to 0} \frac{1}{x} \ln(1 + x) = \lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1,$$

and $$\lim_{x \to 0^+} (1 + x)^{1/x} = e^1 = e.$$ For $$x = 0.0005,$$

$$\left| (1 + x)^{1/x} - e \right| = |(1.0005)^{2000} - e| \approx 6.79 \times 10^{-4} < 0.001.$$

59. Let $$f(x) = x^{1/x}$$ for $$x > 0.$$ 
(a) Calculate $$\lim_{x \to 0^+} f(x)$$ and $$\lim_{x \to \infty} f(x).$$ 
(b) Find the maximum value of $$f(x),$$ and determine the intervals on which $$f(x)$$ is increasing or decreasing.

**SOLUTION**

(a) Let $$f(x) = x^{1/x}.$$ Note that $$\lim_{x \to 0^+} x^{1/x}$$ is not indeterminate. As $$x \to 0^+,$$ the base of the function tends toward 0 and the exponent tends toward $$+\infty.$$ Both of these factors force $$x^{1/x}$$ toward 0. Thus, $$\lim_{x \to 0^+} f(x) = 0.$$ On the other hand, $$\lim_{x \to \infty} f(x)$$ is indeterminate. We calculate this limit as follows:

$$\lim_{x \to \infty} \ln f(x) = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1}{x} = 0,$$

so $$\lim_{x \to \infty} f(x) = e^0 = 1.$$

(b) Again, let $$f(x) = x^{1/x},$$ so that $$\ln f(x) = \frac{1}{x} \ln x.$$ To find the derivative $$f',$$ we apply the derivative to both sides:

$$\frac{d}{dx} \ln f(x) = \frac{d}{dx} \left( \frac{1}{x} \ln x \right)$$

$$\frac{1}{f(x)} f'(x) = -\frac{\ln x}{x^2} + \frac{1}{x^2}$$

$$f'(x) = f(x) \left( -\frac{\ln x}{x^2} + \frac{1}{x^2} \right) = \frac{x^{1/x}}{x^2} (1 - \ln x)$$

Thus, $$f$$ is increasing for $$0 < x < e,$$ is decreasing for $$x > e$$ and has a maximum at $$x = e.$$ The maximum value is $$f(e) = e^{1/e} \approx 1.444668.$$

61. Determine whether $$f << g$$ or $$g << f$$ (or neither) for the functions $$f(x) = \log_{10} x$$ and $$g(x) = \ln x.$$

**SOLUTION** Because

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{\log_{10} x}{\ln x} = \lim_{x \to \infty} \frac{\ln x}{\ln 10} = \frac{1}{\ln 10},$$

neither $$f << g$$ or $$g << f$$ is satisfied.

63. Just as exponential functions are distinguished by their rapid rate of increase, the logarithm functions grow particularly slowly. Show that $$\ln x << x^a$$ for all $$a > 0.$$

**SOLUTION** Using L'Hôpital's Rule:

$$\lim_{x \to \infty} \frac{\ln x}{x^a} = \lim_{x \to \infty} \frac{x^{-1}}{ax^{a-1}} = \lim_{x \to \infty} \frac{1}{ax^{a-1}} = 0;$$

hence, $$\ln x << (x^a).$$

65. Determine whether $$\sqrt{x} << e^{\sqrt{x}}$$ or $$e^{\sqrt{x}} << \sqrt{x}.$$ Hint: Use the substitution $$u = \ln x$$ instead of L'Hôpital's Rule.

**SOLUTION** Let $$u = \ln x,$$ then $$x = e^u,$$ and as $$x \to \infty, u \to \infty.$$ So

$$\lim_{x \to \infty} \frac{e^{\sqrt{x}}}{\sqrt{x}} = \lim_{u \to \infty} \frac{e^{\sqrt{u}}}{\sqrt{u}} = \lim_{u \to \infty} e^{\sqrt{u} - \frac{1}{2}}.$$

We need to examine $$\lim_{u \to \infty} (\sqrt{u} - \frac{u}{2}).$$ Since

$$\lim_{u \to \infty} \frac{u/2}{\sqrt{u}} = \lim_{u \to \infty} \frac{1}{2\sqrt{u}} = 0 = \lim_{u \to \infty} \sqrt{u} = \infty,$$
\[ \sqrt{u} = o(u/2) \text{ and } \lim_{u \to \infty} \left( \sqrt{u} - \frac{u}{2} \right) = -\infty. \text{ Thus} \]

\[ \lim_{u \to \infty} e^{\sqrt{u} - \frac{u}{2}} = e^{-\infty} = 0 \implies \lim_{u \to \infty} \frac{e^{\sqrt{u}}}{\sqrt{u}} = 0 \]

and \( e^{\ln x} \ll \sqrt{x} \).

67. Assumptions Matter Let \( f(x) = x(2 + \sin x) \) and \( g(x) = x^2 + 1 \).
(a) Show directly that \( \lim_{x \to \infty} f(x)/g(x) = 0 \).
(b) Show that \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty \), but \( \lim_{x \to \infty} f'(x)/g'(x) \) does not exist.
Do (a) and (b) contradict L'Hôpital's Rule? Explain.

**SOLUTION**
(a) \( 1 \leq 2 + \sin x \leq 3 \), so
\[ \frac{x}{x^2 + 1} \leq \frac{x(2 + \sin x)}{x^2 + 1} \leq \frac{3x}{x^2 + 1} \]

Since,
\[ \lim_{x \to \infty} \frac{x}{x^2 + 1} = \lim_{x \to \infty} \frac{x}{x^2 + 1} = 0, \]

it follows by the Squeeze Theorem that
\[ \lim_{x \to \infty} \frac{x(2 + \sin x)}{x^2 + 1} = 0. \]

(b) \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} x(2 + \sin x) \geq \lim_{x \to \infty} x = \infty \) and \( \lim_{x \to \infty} g(x) = \lim_{x \to \infty} (x^2 + 1) = \infty \), but
\[ \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{x(2 + \sin x)}{2x} \]
does not exist since \( \cos x \) oscillates. This does not violate L'Hôpital's Rule since the theorem clearly states
\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} \]
"provided the limit on the right exists."

69. Let \( G(b) = \lim_{x \to \infty} (1 + b^x)^{1/x} \).
(a) Use the result of Exercise 68 to evaluate \( G(b) \) for all \( b > 0 \).
(b) Verify your result graphically by plotting \( y = (1 + b^x)^{1/x} \) together with the horizontal line \( y = G(b) \) for the values \( b = 0.25, 0.5, 2, 3 \).

**SOLUTION**
(a) Using Exercise 68, we see that \( G(b) = e^{H(b)} \). Thus, \( G(b) = 1 \) if \( 0 \leq b \leq 1 \) and \( G(b) = b \) if \( b > 1 \).
(b)
In Exercises 71–73, let

\[ f(x) = \begin{cases} 
  e^{-1/x^2} & \text{for } x \neq 0 \\
  0 & \text{for } x = 0 
\end{cases} \]

These exercises show that \( f(x) \) has an unusual property: All of its derivatives at \( x = 0 \) exist and are equal to zero.

71. Show that \( \lim_{x \to 0} \frac{f(x)}{x^k} = 0 \) for all \( k \). \textbf{Hint:} Let \( t = x^{-1} \) and apply the result of Exercise 70.

\textbf{Solution} \quad \lim_{x \to 0} \frac{f(x)}{x^k} = \lim_{x \to 0} \frac{1}{x^k e^{1/x^2}}. \text{ Let } t = 1/x. \text{ As } x \to 0, \ t \to \infty. \text{ Thus,}

\[ \lim_{x \to 0} \frac{1}{x^k e^{1/x^2}} = \lim_{t \to \infty} \frac{t^k}{e^t} = 0 \]

by Exercise 70.

73. Show that for \( k \geq 1 \) and \( x \neq 0 \),

\[ f^{(k)}(x) = \frac{P(x)e^{-1/x^2}}{x^r} \]

for some polynomial \( P(x) \) and some exponent \( r \geq 1 \). Use the result of Exercise 71 to show that \( f^{(k)}(0) \) exists and is equal to zero for all \( k \geq 1 \).

\textbf{Solution} \quad \text{For } x \neq 0, \ f'(x) = e^{-1/x^2} \left( \frac{2}{x^3} \right). \text{ Here } P(x) = 2 \text{ and } r = 3. \text{ Assume } f^{(k)}(x) = \frac{P(x)e^{-1/x^2}}{x^r}. \text{ Then}

\[ f^{(k+1)}(x) = e^{-1/x^2} \left( \frac{x^3 P'(x) + (2 - rx^2)P(x)}{x^{r+3}} \right) \]

which is of the form desired.

Moreover, from Exercise 72, \( f'(0) = 0 \). Suppose \( f^{(k)}(0) = 0 \). Then

\[ f^{(k+1)}(0) = \lim_{x \to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} = \lim_{x \to 0} \frac{P(x)e^{-1/x^2}}{x^{r+1}} = 0 = \lim_{x \to 0} \frac{f(x)}{x^{r+1}} = 0. \]

\textbf{Further Insights and Challenges}

75. The Second Derivative Test for critical points fails if \( f''(c) = 0 \). This exercise develops a \textbf{Higher Derivative Test} based on the sign of the first nonzero derivative. Suppose that

\[ f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0, \ \text{ but } f^{(n)}(c) \neq 0 \]

(a) Show, by applying L'Hôpital's Rule \( n \) times, that

\[ \lim_{x \to c} \frac{f(x) - f(c)}{(x - c)^n} = \frac{1}{n!} f^{(n)}(c) \]

where \( n! = n(n - 1)(n - 2) \cdots (2)(1) \).

(b) Use (a) to show that if \( n \) is even, then \( f(c) \) is a local minimum if \( f^{(n)}(c) > 0 \) and is a local maximum if \( f^{(n)}(c) < 0 \). \textbf{Hint:} If \( n \) is even, then \( (x - c)^n > 0 \) for \( x \neq a \), so \( (x - c)^n f(x) - f(c) \) must be positive for \( x \) near \( c \) if \( f^{(n)}(c) > 0 \).

(c) Use (a) to show that if \( n \) is odd, then \( f(c) \) is neither a local minimum nor a local maximum.

\textbf{Solution} \quad \text{(a) Repeated application of L'Hôpital's rule yields}

\[ \lim_{x \to c} \frac{f(x) - f(c)}{(x - c)^n} = \lim_{x \to c} \frac{f'(x)}{n(x - c)^{n-1}} = \frac{f''(x)}{n(n-1)(x - c)^{n-2}} = \lim_{x \to c} \frac{f'''(x)}{n(n-1)(n-2)(x - c)^{n-3}} = \cdots = \frac{1}{n!} f^{(n)}(c) \]
(b) Suppose \( n \) is even. Then \( (x - c)^n > 0 \) for all \( x \neq c \). If \( f^{(n)}(c) > 0 \), it follows that \( f(x) - f(c) \) must be positive for \( x \) near \( c \). In other words, \( f(x) > f(c) \) for \( x \) near \( c \) and \( f(c) \) is a local minimum. On the other hand, if \( f^{(n)}(c) < 0 \), it follows that \( f(x) - f(c) \) must be negative for \( x \) near \( c \). In other words, \( f(x) < f(c) \) for \( x \) near \( c \) and \( f(c) \) is a local maximum.

(c) If \( n \) is odd, then \( (x - c)^n > 0 \) for \( x > c \) but \( (x - c)^n < 0 \) for \( x < c \). If \( f^{(n)}(c) > 0 \), it follows that \( f(x) - f(c) \) must be positive for \( x \) near \( c \) and \( x > c \) but is negative for \( x \) near \( c \) and \( x < c \). In other words, \( f(x) > f(c) \) for \( x \) near \( c \) and \( x > c \) but \( f(x) < f(c) \) for \( x \) near \( c \) and \( x < c \). Thus, \( f(c) \) is neither a local minimum nor a local maximum. We obtain a similar result if \( f^{(n)}(c) < 0 \).

77. We expended a lot of effort to evaluate \( \lim_{x \to 0} \frac{\sin x}{x} \) in Chapter 2. Show that we could have evaluated it easily using L'Hôpital's Rule. Then explain why this method would involve circular reasoning.

**Solution** \[
\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1. \]
To use L'Hôpital’s Rule to evaluate \( \lim_{x \to 0} \frac{\sin x}{x} \), we must know that the derivative of \( \sin x \) is \( \cos x \), but to determine the derivative of \( \sin x \), we must be able to evaluate \( \lim_{x \to 0} \frac{\sin x}{x} \). 

79. Patience Required Use L'Hôpital’s Rule to evaluate and check your answers numerically:

(a) \( \lim_{x \to 0^+} \left( \frac{\sin x}{x} \right)^{1/2} \)

**Solution** (a) We start by evaluating

\[
\lim_{x \to 0^+} \ln \left( \frac{\sin x}{x} \right)^{1/2} = \lim_{x \to 0^+} \frac{\ln(\sin x) - \ln x}{x^2}.
\]

Repeatedly using L'Hôpital’s Rule, we find

\[
\lim_{x \to 0^+} \ln \left( \frac{\sin x}{x} \right)^{1/2} = \lim_{x \to 0^+} \frac{\cot x - x^{-1}}{2x} = \lim_{x \to 0^+} \frac{x \cos x - \sin x}{2x^2 \sin x} = \lim_{x \to 0^+} \frac{-x \sin x}{2x^2 \cos x + 4x \sin x} = \lim_{x \to 0^+} \frac{-x \cos x - \sin x}{8x \cos x + 4 \sin x - 2x^2 \sin x} = \lim_{x \to 0^+} \frac{-2x \cos x + x \sin x}{12 \cos x - 2x^2 \cos x - 12x \sin x} = \frac{2}{12} = \frac{1}{6}.
\]

Therefore, \( \lim_{x \to 0^+} \left( \frac{\sin x}{x} \right)^{1/2} = e^{-1/6} \). Numerically we find:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.1</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \left( \frac{\sin x}{x} \right)^{1/2} )</td>
<td>0.841471</td>
<td>0.846435</td>
</tr>
</tbody>
</table>

Note that \( e^{-1/6} \approx 0.846481724 \).

(b) Repeatedly using L'Hôpital’s Rule and simplifying, we find

\[
\lim_{x \to 0} \left( \frac{1}{\sin^2 x} - \frac{1}{x^2} \right) = \lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \lim_{x \to 0} \frac{2x - 2 \sin x \cos x}{(2 \sin x \cos x) + 2x \sin^2 x} = \lim_{x \to 0} \frac{2x - 2 \sin 2x}{2 - 2 \cos 2x} = \lim_{x \to 0} \frac{2x - 2 \sin 2x}{2 \cos 2x} = \lim_{x \to 0} \frac{2x - 2 \sin 2x}{2} = \frac{1}{3}.
\]
Numerically we find:

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>0.1</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{\sin^2 x} - \frac{1}{x^2})</td>
<td>0.412283</td>
<td>0.334001</td>
<td>0.333340</td>
</tr>
</tbody>
</table>

4.6 Graph Sketching and Asymptotes

Preliminary Questions

1. Sketch an arc where \(f'\) and \(f''\) have the sign combination ++. Do the same for −−.

**Solution** An arc with the sign combination ++ (increasing, concave up) is shown below at the left. An arc with the sign combination −− (decreasing, concave up) is shown below at the right.

[Graphs of arc with ++ and −− sign combinations]

2. If the sign combination of \(f'\) and \(f''\) changes from ++ to +− at \(x = c\), then (choose the correct answer):
   (a) \(f(c)\) is a local min
   (b) \(f(c)\) is a local max
   (c) \(c\) is a point of inflection

**Solution** Because the sign of the second derivative changes at \(x = c\), the correct response is (c): \(c\) is a point of inflection.

3. The second derivative of the function \(f(x) = (x - 4)^{-1}\) is \(f''(x) = 2(x - 4)^{-3}\). Although \(f''(x)\) changes sign at \(x = 4\), \(f(x)\) does not have a point of inflection at \(x = 4\). Why not?

**Solution** The function \(f\) does not have a point of inflection at \(x = 4\) because \(x = 4\) is not in the domain of \(f\).

Exercises

1. Determine the sign combinations of \(f'\) and \(f''\) for each interval A–G in Figure 16.

**Solution**

- In A, \(f\) is decreasing and concave up, so \(f' < 0\) and \(f'' > 0\).
- In B, \(f\) is increasing and concave up, so \(f' > 0\) and \(f'' > 0\).
- In C, \(f\) is increasing and concave down, so \(f' > 0\) and \(f'' < 0\).
- In D, \(f\) is decreasing and concave down, so \(f' < 0\) and \(f'' < 0\).
- In E, \(f\) is decreasing and concave up, so \(f' < 0\) and \(f'' > 0\).
- In F, \(f\) is increasing and concave up, so \(f' > 0\) and \(f'' > 0\).
- In G, \(f\) is increasing and concave down, so \(f' > 0\) and \(f'' < 0\).

In Exercises 3–6, draw the graph of a function for which \(f'\) and \(f''\) take on the given sign combinations.

3. ++, +−, −−

**Solution** This function changes from concave up to concave down at \(x = -1\) and from increasing to decreasing at \(x = 0\).
5. $-+, --, +$

**Solution**  The function is decreasing everywhere and changes from concave up to concave down at $x = -1$ and from concave down to concave up at $x = -\frac{1}{2}$.

7. Sketch the graph of $y = x^2 - 5x + 4$.

**Solution**  Let $f(x) = x^2 - 5x + 4$. Then $f'(x) = 2x - 5$ and $f''(x) = 2$. Hence $f$ is decreasing for $x < 5/2$, is increasing for $x > 5/2$, has a local minimum at $x = 5/2$ and is concave up everywhere.

9. Sketch the graph of $f(x) = x^3 - 3x^2 + 2$. Include the zeros of $f(x)$, which are $x = 1$ and $1 \pm \sqrt{3}$ (approximately $-0.73, 2.73$).

**Solution**  Let $f(x) = x^3 - 3x^2 + 2$. Then $f'(x) = 3x^2 - 6x = 3x(x - 2) = 0$ yields $x = 0, 2$ and $f''(x) = 6x - 6$. Thus $f$ is concave down for $x < 1$, is concave up for $x > 1$, has an inflection point at $x = 1$, is increasing for $x < 0$ and for $x > 2$, is decreasing for $0 < x < 2$, has a local maximum at $x = 0$, and has a local minimum at $x = 2$.

11. Extend the sketch of the graph of $f(x) = \cos x + \frac{1}{2}x$ in Example 4 to the interval $[0, 5\pi]$.

**Solution**  Let $f(x) = \cos x + \frac{1}{2}x$. Then $f'(x) = -\sin x + \frac{1}{2} = 0$ yields critical points at $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$, $\frac{25\pi}{6}$, and $\frac{29\pi}{6}$. Moreover, $f''(x) = -\cos x$ so there are points of inflection at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$, and $\frac{9\pi}{2}$.
In Exercises 13–34, find the transition points, intervals of increase/decrease, concavity, and asymptotic behavior. Then sketch the graph, with this information indicated.

13. \( y = x^3 + 24x^2 \)

**Solution** Let \( f(x) = x^3 + 24x^2 \). Then \( f'(x) = 3x^2 + 48x = 3x(x + 16) \) and \( f''(x) = 6x + 48 \). This shows that \( f \) has critical points at \( x = 0 \) and \( x = -16 \) and a candidate for an inflection point at \( x = -8 \).

<table>
<thead>
<tr>
<th>Interval</th>
<th>(−∞, −16)</th>
<th>(−16, −8)</th>
<th>(−8, 0)</th>
<th>(0, ∞)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Signs of ( f' )</td>
<td>−−</td>
<td>−</td>
<td>−</td>
<td>+</td>
</tr>
<tr>
<td>Signs of ( f'' )</td>
<td>−−</td>
<td>++</td>
<td>−</td>
<td>−</td>
</tr>
</tbody>
</table>

Thus, there is a local maximum at \( x = -16 \), a local minimum at \( x = 0 \), and an inflection point at \( x = -8 \). Moreover,

\[
\lim_{x \to -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty.
\]

Here is a graph of \( f \) with these transition points highlighted as in the graphs in the textbook.

15. \( y = x^2 - 4x^3 \)

**Solution** Let \( f(x) = x^2 - 4x^3 \). Then \( f'(x) = 2x - 12x^2 = 2x(1 - 6x) \) and \( f''(x) = 2 - 24x \). Critical points are at \( x = 0 \) and \( x = \frac{1}{6} \), and the sole candidate point of inflection is at \( x = \frac{1}{12} \).

<table>
<thead>
<tr>
<th>Interval</th>
<th>(−∞, 0)</th>
<th>( (0, \frac{1}{12}) )</th>
<th>( \left( \frac{1}{12}, \frac{1}{6} \right) )</th>
<th>( \left( \frac{1}{6}, \infty \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Signs of ( f' )</td>
<td>−−</td>
<td>−−</td>
<td>−−</td>
<td>+−</td>
</tr>
<tr>
<td>Signs of ( f'' )</td>
<td>−−</td>
<td>++</td>
<td>−−</td>
<td>−−</td>
</tr>
</tbody>
</table>

Thus, \( f(0) \) is a local minimum, \( f\left(\frac{1}{12}\right) \) is a local maximum, and there is a point of inflection at \( x = \frac{1}{12} \). Moreover,

\[
\lim_{x \to -\infty} f(x) = \infty.
\]

Here is the graph of \( f \) with transition points highlighted as in the textbook:

17. \( y = 4 - 2x^2 + \frac{1}{5}x^4 \)

**Solution** Let \( f(x) = \frac{1}{5}x^4 - 2x^2 + 4 \). Then \( f'(x) = \frac{2}{5}x^3 - 4x = \frac{2}{5}x \left( x^2 - 6 \right) \) and \( f''(x) = 2x^2 - 4 \). This shows that \( f \) has critical points at \( x = 0 \) and \( x = \pm\sqrt{6} \) and has candidates for points of inflection at \( x = \pm\sqrt{2} \).

<table>
<thead>
<tr>
<th>Interval</th>
<th>(−∞, −( \sqrt{6} ))</th>
<th>(−( \sqrt{6} ), −( \sqrt{2} ))</th>
<th>(−( \sqrt{2} ), 0)</th>
<th>(0, ( \sqrt{2} ))</th>
<th>(( \sqrt{2} ), ( \sqrt{6} ))</th>
<th>(( \sqrt{6} ), ∞)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Signs of ( f' )</td>
<td>−−</td>
<td>++</td>
<td>−−</td>
<td>−−</td>
<td>−+</td>
<td>+−</td>
</tr>
<tr>
<td>Signs of ( f'' )</td>
<td>−−</td>
<td>+−</td>
<td>−−</td>
<td>−−</td>
<td>−−</td>
<td>−+</td>
</tr>
</tbody>
</table>

Thus, \( f \) has local minima at \( x = \pm\sqrt{6} \), a local maximum at \( x = 0 \), and inflection points at \( x = \pm\sqrt{2} \). Moreover,

\[
\lim_{x \to -\infty} f(x) = \infty.
\]

Here is a graph of \( f \) with transition points highlighted.
19. \( y = x^5 + 5x \)

**Solution** Let \( f(x) = x^5 + 5x \). Then \( f'(x) = 5x^4 + 5 = 5(x^4 + 1) \) and \( f''(x) = 20x^3 \). \( f'(x) > 0 \) for all \( x \), so the graph has no critical points and is always increasing. \( f''(x) = 0 \) at \( x = 0 \). Sign analyses reveal that \( f''(x) \) changes from negative to positive at \( x = 0 \), so that the graph of \( f(x) \) has an inflection point at \((0, 0)\). Moreover,

\[
\lim_{x \to -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty.
\]

Here is a graph of \( f \) with transition points highlighted.

\[\text{Graph image}\]

21. \( y = x^4 - 3x^3 + 4x \)

**Solution** Let \( f(x) = x^4 - 3x^3 + 4x \). Then 
\[
\begin{align*}
f'(x) &= 4x^3 - 9x^2 + 4 = (4x^2 - x - 2)(x - 2) \\
f''(x) &= 12x^2 - 18x = 6x(2x - 3).
\end{align*}
\]

This shows that \( f \) has critical points at \( x = 2 \) and \( x = \frac{1 \pm \sqrt{33}}{8} \) and candidate points of inflection at \( x = 0 \) and \( x = \frac{3}{2} \). Sign analyses reveal that \( f'(x) \) changes from negative to positive at \( x = \frac{1 - \sqrt{33}}{8} \), from positive to negative at \( x = \frac{1 + \sqrt{33}}{8} \), and again from negative to positive at \( x = 2 \). Therefore, \( f\left(\frac{1 - \sqrt{33}}{8}\right) \) and \( f(2) \) are local minima of \( f(x) \), and \( f\left(\frac{1 + \sqrt{33}}{8}\right) \) is a local maximum. Further sign analyses reveal that \( f''(x) \) changes from positive to negative at \( x = 0 \) and from negative to positive at \( x = \frac{3}{2} \), so that there are points of inflection both at \( x = 0 \) and \( x = \frac{3}{2} \). Moreover,

\[
\lim_{x \to -\infty} f(x) = \infty.
\]

Here is a graph of \( f(x) \) with transition points highlighted.

\[\text{Graph image}\]

23. \( y = x^7 - 14x^6 \)

**Solution** Let \( f(x) = x^7 - 14x^6 \). Then \( f'(x) = 7x^6 - 84x^5 = 7x^5(x - 12) \) and \( f''(x) = 42x^5 - 420x^4 = 42x^4(x - 10) \). Critical points are at \( x = 0 \) and \( x = 12 \), and candidate inflection points are at \( x = 0 \) and \( x = 10 \). Sign analyses reveal that \( f'(x) \) changes from positive to negative at \( x = 0 \) and from negative to positive at \( x = 12 \). Therefore \( f(0) \) is a local maximum and \( f(12) \) is a local minimum. Also, \( f''(x) \) changes from negative to positive at \( x = 10 \). Therefore, there is a point of inflection at \( x = 10 \). Moreover,

\[
\lim_{x \to -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty.
\]

Here is a graph of \( f \) with transition points highlighted.

\[\text{Graph image}\]
25. \( y = x - 4\sqrt{x} \)

**Solution** Let \( f(x) = x - 4\sqrt{x} = x - 4x^{1/2} \). Then \( f'(x) = 1 - 2x^{-1/2} \). This shows that \( f \) has critical points at \( x = 0 \) (where the derivative does not exist) and at \( x = 4 \) (where the derivative is zero). Because \( f'(x) < 0 \) for \( 0 < x < 4 \) and \( f'(x) > 0 \) for \( x > 4 \), \( f \) has a local minimum at \( x = 4 \). Moreover, \( f''(x) = x^{-3/2} > 0 \) for all \( x > 0 \), so the graph is always concave up. Moreover,

\[
\lim_{x \to \infty} f(x) = \infty.
\]

Here is a graph of \( f \) with transition points highlighted.

27. \( y = x(8 - x)^{1/3} \)

**Solution** Let \( f(x) = x(8 - x)^{1/3} \). Then

\[
f'(x) = x \cdot \left( \frac{1}{3} (8 - x)^{-2/3} (-1) + (8 - x)^{1/3} \right) = \frac{24 - 4x}{3(8 - x)^{5/3}}
\]

and similarly

\[
f''(x) = \frac{4x - 48}{9(8 - x)^{5/3}}.
\]

Critical points are at \( x = 8 \) and \( x = 6 \), and candidate inflection points are \( x = 8 \) and \( x = 12 \). Sign analyses reveal that \( f'(x) \) changes from positive to negative at \( x = 6 \) and \( f'(x) \) remains negative on either side of \( x = 8 \). Moreover, \( f''(x) \) changes from negative to positive at \( x = 8 \) and from positive to negative at \( x = 12 \). Therefore, \( f \) has a local maximum at \( x = 6 \) and inflection points at \( x = 8 \) and \( x = 12 \). Moreover,

\[
\lim_{x \to \pm \infty} f(x) = -\infty.
\]

Here is a graph of \( f \) with the transition points highlighted.

29. \( y = xe^{-x^2} \)

**Solution** Let \( f(x) = xe^{-x^2} \). Then

\[
f'(x) = -2xe^{-x^2} + e^{-x^2} = (1 - 2x^2)e^{-x^2},
\]

and

\[
f''(x) = (4x^3 - 2x)e^{-x^2} - 4xe^{-x^2} = 2x(2x^2 - 3)e^{-x^2}.
\]

There are critical points at \( x = \pm \frac{\sqrt{3}}{2} \), and \( x = 0 \) and \( x = \pm \frac{\sqrt{3}}{2} \) are candidates for inflection points. Sign analysis shows that \( f'(x) \) changes from negative to positive at \( x = -\frac{\sqrt{3}}{2} \) and from positive to negative at \( x = \frac{\sqrt{3}}{2} \). Moreover, \( f''(x) \) changes from negative to positive at both \( x = \pm \frac{\sqrt{3}}{2} \) and from positive to negative at \( x = 0 \). Therefore, \( f \) has a local minimum at \( x = -\frac{\sqrt{3}}{2} \), a local maximum at \( x = \frac{\sqrt{3}}{2} \) and inflection points at \( x = 0 \) and \( x = \pm \frac{\sqrt{3}}{2} \). Moreover,

\[
\lim_{x \to \pm \infty} f(x) = 0,
\]

so the graph has a horizontal asymptote at \( y = 0 \). Here is a graph of \( f \) with the transition points highlighted.
31. \( y = x - 2 \ln x \)

**SOLUTION** Let \( f(x) = x - 2 \ln x \). Note that the domain of \( f \) is \( x > 0 \). Now,

\[
f'(x) = 1 - \frac{2}{x} \quad \text{and} \quad f''(x) = \frac{2}{x^2}.
\]

The only critical point is \( x = 2 \). Sign analysis shows that \( f'(x) \) changes from negative to positive at \( x = 2 \), so \( f(2) \) is a local minimum. Further, \( f''(x) > 0 \) for \( x > 0 \), so the graph is always concave up. Moreover,

\[
\lim_{x \to \infty} f(x) = \infty.
\]

Here is a graph of \( f \) with the transition points highlighted.

33. \( y = x - x^2 \ln x \)

**SOLUTION** Let \( f(x) = x - x^2 \ln x \). Then \( f'(x) = 1 - x - 2x \ln x \) and \( f''(x) = -3 - 2 \ln x \). There is a critical point at \( x = 1 \), and \( x = e^{-3/2} \approx 0.223 \) is a candidate inflection point. Sign analysis shows that \( f'(x) \) changes from positive to negative at \( x = 1 \) and that \( f''(x) \) changes from positive to negative at \( x = e^{-3/2} \). Therefore, \( f \) has a local maximum at \( x = 1 \) and a point of inflection at \( x = e^{-3/2} \). Moreover,

\[
\lim_{x \to \infty} f(x) = -\infty.
\]

Here is a graph of \( f \) with the transition points highlighted.

35. Sketch the graph of \( f(x) = 18(x - 3)(x - 1)^{2/3} \) using the formulas

\[
f'(x) = \frac{30(x - \frac{9}{5})}{(x - 1)^{1/3}}, \quad f''(x) = \frac{20(x - \frac{3}{5})}{(x - 1)^{4/3}}
\]

**SOLUTION**

\[
f'(x) = \frac{30(x - \frac{9}{5})}{(x - 1)^{1/3}}
\]

yields critical points at \( x = \frac{9}{5}, x = 1 \).

\[
f''(x) = \frac{20(x - \frac{3}{5})}{(x - 1)^{4/3}}
\]

yields potential inflection points at \( x = \frac{3}{5}, x = 1 \).
There are critical points at \( x = \pm 3 \), a local maximum at \( x = 1 \) (at which the graph has a cusp), and a local minimum at \( x = \pm \frac{2}{3} \). The sketch looks something like this.

\[
\begin{array}{c|c}
\text{Interval} & \text{signs of } f' \text{ and } f'' \\
\hline
(-\infty, \frac{2}{3}) & +-- \\
(\frac{2}{3}, 1) & ++ \\
(1, \frac{9}{2}) & -- \\
(\frac{9}{2}, \infty) & ++ \\
\end{array}
\]

The graph has an inflection point at \( x = \frac{3}{2} \), a local maximum at \( x = 1 \) (at which the graph has a cusp), and a local minimum at \( x = \frac{2}{3} \). The sketch looks something like this.

\[
\begin{array}{c|c}
\text{Interval} & \text{signs of } f' \text{ and } f'' \\
\hline
(-\infty, \frac{2}{3}) & +-- \\
(\frac{2}{3}, 1) & ++ \\
(1, \frac{9}{2}) & -- \\
(\frac{9}{2}, \infty) & ++ \\
\end{array}
\]

\textit{CSE} In Exercises 37–40, sketch the graph of the function, indicating all transition points. If necessary, use a graphing utility or computer algebra system to locate the transition points numerically.

37. \( y = x^2 - 10 \ln(x^2 + 1) \)

**SOLUTION** Let \( f(x) = x^2 - 10 \ln(x^2 + 1) \). Then \( f'(x) = 2x - \frac{20x}{x^2 + 1} \) and

\[
f''(x) = 2 - \frac{(x^2 + 1)(20) - (20x)(2x)}{(x^2 + 1)^2} = x^4 + 12x^2 - 9 \quad \frac{(x^2 + 1)^2}{(x^2 + 1)^2}.
\]

There are critical points at \( x = 0 \) and \( x = \pm 3 \), and \( x = \pm \sqrt{-6 + 3\sqrt{3}} \) are candidates for inflection points. Sign analysis shows that \( f'(x) \) changes from negative to positive at \( x = \pm 3 \) and from positive to negative at \( x = 0 \). Moreover, \( f''(x) \) changes from positive to negative at \( x = -\sqrt{-6 + 3\sqrt{3}} \) and from negative to positive at \( x = \sqrt{-6 + 3\sqrt{3}} \). Therefore, \( f \) has a local maximum at \( x = 0 \), local minima at \( x = \pm 3 \) and points of inflection at \( x = \pm \sqrt{-6 + 3\sqrt{3}} \). Here is a graph of \( f \) with the transition points highlighted.

39. \( y = x^4 - 4x^2 + x + 1 \)

**SOLUTION** Let \( f(x) = x^4 - 4x^2 + x + 1 \). Then \( f'(x) = 4x^3 - 8x + 1 \) and \( f''(x) = 12x^2 - 8 \). The critical points are \( x = -1.473 \), \( x = 0.126 \) and \( x = 1.347 \), while the candidates for points of inflection are \( x = \pm \sqrt{2 \over 3} \). Sign analysis reveals that \( f'(x) \) changes from negative to positive at \( x = -1.473 \), from positive to negative at \( x = 0.126 \) and from negative to positive at \( x = 1.347 \). For the second derivative, \( f''(x) \) changes from positive to negative at \( x = -\sqrt{2 \over 3} \) and from negative to positive at \( x = \sqrt{2 \over 3} \). Therefore, \( f \) has local minima at \( x = -1.473 \) and \( x = 1.347 \), a local maximum at \( x = 0.126 \) and points of inflection at \( x = \pm \sqrt{2 \over 3} \). Moreover,

\[
\lim_{x \to \pm \infty} f(x) = \pm \infty.
\]

Here is a graph of \( f \) with the transition points highlighted.
In Exercises 41–46, sketch the graph over the given interval, with all transition points indicated.

41. \( y = x + \sin x \), \([-0, 2\pi]\)

**SOLUTION** Let \( f(x) = x + \sin x \). Setting \( f'(x) = 1 + \cos x = 0 \) yields \( \cos x = -1 \), so that \( x = \pi \) is the lone critical point on the interval \([0, 2\pi]\). Setting \( f''(x) = -\sin x = 0 \) yields potential points of inflection at \( x = 0, \pi, 2\pi \) on the interval \([0, 2\pi]\).

<table>
<thead>
<tr>
<th>Interval</th>
<th>signs of ( f' ) and ( f'' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, \pi))</td>
<td>++</td>
</tr>
<tr>
<td>((\pi, 2\pi))</td>
<td>++</td>
</tr>
</tbody>
</table>

The graph has an inflection point at \( x = \pi \), and no local maxima or minima. Here is a sketch of the graph of \( f(x) \):

43. \( y = 2\sin x - \cos^2 x \), \([-0, 2\pi]\)

**SOLUTION** Let \( f(x) = 2\sin x - \cos^2 x \). Then \( f'(x) = 2\cos x - 2\cos x (-\sin x) = \sin 2x + 2\cos x \) and \( f''(x) = 2\cos 2x - 2\sin x \). Setting \( f'(x) = 0 \) yields \( 2\sin 2x = -2\cos x \), so that \( 2\sin x \cos x = -2\cos x \). This implies \( \cos x = 0 \) or \( \sin x = -1 \), so that \( x = \frac{\pi}{2} \) or \( \frac{3\pi}{2} \). Setting \( f''(x) = 0 \) yields \( 2\cos 2x = 2\sin x \), so that \( 2\sin(\frac{\pi}{2} - 2x) = 2\sin x \), or \( \frac{\pi}{2} - 2x = x \pm 2\pi \). This yields \( 3x = \frac{\pi}{2} + 2n\pi \), or \( x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6} = \frac{3\pi}{2} \).

<table>
<thead>
<tr>
<th>Interval</th>
<th>signs of ( f' ) and ( f'' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, \frac{\pi}{6}))</td>
<td>++</td>
</tr>
<tr>
<td>((\frac{\pi}{6}, \frac{\pi}{2}))</td>
<td>+−</td>
</tr>
<tr>
<td>((\frac{\pi}{2}, \frac{5\pi}{6}))</td>
<td>−−</td>
</tr>
<tr>
<td>((\frac{5\pi}{6}, \frac{3\pi}{2}))</td>
<td>−+</td>
</tr>
<tr>
<td>((\frac{3\pi}{2}, 2\pi))</td>
<td>++</td>
</tr>
</tbody>
</table>

The graph has a local maximum at \( x = \frac{\pi}{6} \), a local minimum at \( x = \frac{3\pi}{2} \), and inflection points at \( x = \frac{\pi}{6} \) and \( x = \frac{5\pi}{6} \). Here is a graph of \( f(x) \) without transition points highlighted.

45. \( y = \sin x + \sqrt{3}\cos x \), \([0, \pi]\)

**SOLUTION** Let \( f(x) = \sin x + \sqrt{3}\cos x \). Setting \( f'(x) = \cos x - \sqrt{3}\sin x = 0 \) yields \( \tan x = \frac{1}{\sqrt{3}} \). In the interval \([0, \pi]\), the solution is \( x = \frac{\pi}{6} \). Setting \( f''(x) = -\sin x - \sqrt{3}\cos x = 0 \) yields \( \tan x = -\sqrt{3} \). In the interval \([0, \pi]\), the lone solution is \( x = \frac{2\pi}{3} \).

<table>
<thead>
<tr>
<th>Interval</th>
<th>signs of ( f' ) and ( f'' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, \pi/6))</td>
<td>+−</td>
</tr>
<tr>
<td>((\pi/6, 2\pi/3))</td>
<td>−−</td>
</tr>
<tr>
<td>((2\pi/3, \pi))</td>
<td>−+</td>
</tr>
</tbody>
</table>
The graph has a local maximum at \( x = \frac{\pi}{6} \) and a point of inflection at \( x = \frac{5\pi}{3} \). A plot without the transition points highlighted is given below:

47. Are all sign transitions possible? Explain with a sketch why the transitions \( + + \rightarrow - - \) and \( - - \rightarrow + + \) do not occur if the function is differentiable. (See Exercise 76 for a proof.)

**Solution** In both cases, there is a point where \( f \) is not differentiable at the transition from increasing to decreasing or decreasing to increasing.

49. Which of the graphs in Figure 18 cannot be the graph of a polynomial? Explain.

**Solution** Polynomials are everywhere differentiable. Accordingly, graph (B) cannot be the graph of a polynomial, since the function in (B) has a cusp (sharp corner), signifying nondifferentiability at that point.

51. Match the graphs in Figure 20 with the two functions \( y = \frac{3x^2}{x^2 - 1} \) and \( y = \frac{3x^2}{x^2 - 1} \). Explain.

**Solution** Since \( \lim_{x \to \pm\infty} \frac{3x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{3}{1} = 3 \), the graph of \( y = \frac{3x^2}{x^2 - 1} \) has a horizontal asymptote of \( y = 3 \); hence, the right curve is the graph of \( f(x) = \frac{3x^2}{x^2 - 1} \). Since
\[
\lim_{x \to \pm\infty} \frac{3x}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{3}{x} \cdot \lim_{x \to \pm\infty} x^{-1} = 0,
\]
the graph of \( y = \frac{3x}{x^2 - 1} \) has a horizontal asymptote of \( y = 0 \); hence, the left curve is the graph of \( f(x) = \frac{3x}{x^2 - 1} \).
In Exercises 53–70, sketch the graph of the function. Indicate the transition points and asymptotes.

53. \( y = \frac{1}{3x - 1} \)

**Solution** Let \( f(x) = \frac{1}{3x - 1} \). Then \( f'(x) = -\frac{3}{(3x - 1)^2} \), so that \( f \) is decreasing for all \( x \neq \frac{1}{3} \). Moreover, \( f''(x) = \frac{18}{(3x - 1)^3} \), so that \( f \) is concave up for \( x > \frac{1}{3} \) and concave down for \( x < \frac{1}{3} \). Because \( \lim_{x \to \pm\infty} \frac{1}{3x - 1} = 0 \), \( f \) has a horizontal asymptote at \( y = 0 \). Finally, \( f \) has a vertical asymptote at \( x = \frac{1}{3} \) with

\[
\lim_{x \to \frac{1}{3}^-} \frac{1}{3x - 1} = -\infty \quad \text{and} \quad \lim_{x \to \frac{1}{3}^+} \frac{1}{3x - 1} = \infty.
\]

55. \( y = \frac{x + 3}{x - 2} \)

**Solution** Let \( f(x) = \frac{x + 3}{x - 2} \). Then \( f'(x) = -\frac{5}{(x - 2)^2} \), so that \( f \) is decreasing for all \( x \neq 2 \). Moreover, \( f''(x) = \frac{10}{(x - 2)^3} \), so that \( f \) is concave up for \( x > 2 \) and concave down for \( x < 2 \). Because \( \lim_{x \to \pm\infty} \frac{x + 3}{x - 2} = 1 \), \( f \) has a horizontal asymptote at \( y = 1 \). Finally, \( f \) has a vertical asymptote at \( x = 2 \) with

\[
\lim_{x \to 2^-} \frac{x + 3}{x - 2} = -\infty \quad \text{and} \quad \lim_{x \to 2^+} \frac{x + 3}{x - 2} = \infty.
\]

57. \( y = \frac{1}{x} + \frac{1}{x - 1} \)

**Solution** Let \( f(x) = \frac{1}{x} + \frac{1}{x - 1} \). Then \( f'(x) = -\frac{2x^2 - 2x + 1}{x^2(x - 1)^2} \), so that \( f \) is decreasing for all \( x \neq 0, 1 \). Moreover, \( f''(x) = \frac{2(2x^3 - 3x^2 + 3x - 1)}{x^3(x - 1)^3} \), so that \( f \) is concave up for \( 0 < x < \frac{1}{2} \) and \( x > 1 \) and concave down for \( x < 0 \) and \( \frac{1}{2} < x < 1 \). Because \( \lim_{x \to \pm\infty} \left( \frac{1}{x} + \frac{1}{x - 1} \right) = 0 \), \( f \) has a horizontal asymptote at \( y = 0 \). Finally, \( f \) has vertical asymptotes at \( x = 0 \) and \( x = 1 \) with

\[
\lim_{x \to 0^-} \left( \frac{1}{x} + \frac{1}{x - 1} \right) = -\infty \quad \text{and} \quad \lim_{x \to 0^+} \left( \frac{1}{x} + \frac{1}{x - 1} \right) = \infty
\]

and

\[
\lim_{x \to 1^-} \left( \frac{1}{x} + \frac{1}{x - 1} \right) = -\infty \quad \text{and} \quad \lim_{x \to 1^+} \left( \frac{1}{x} + \frac{1}{x - 1} \right) = \infty.
\]
59. \( y = \frac{1}{x(x-2)} \)

**SOLUTION** Let \( f(x) = \frac{1}{x(x-2)} \). Then \( f'(x) = \frac{2(1-x)}{x^2(x-2)^2} \), so that \( f \) is increasing for \( x < 0 \) and \( 0 < x < 1 \) and decreasing for \( 1 < x < 2 \) and \( x > 2 \). Moreover, \( f''(x) = \frac{2(3x^2 - 6x + 4)}{x^4(x-2)^3} \), so that \( f \) is concave up for \( x < 0 \) and \( x > 2 \) and concave down for \( 0 < x < 2 \). Because \( \lim_{x \to \pm\infty} \left( \frac{1}{x(x-2)} \right) = 0 \), \( f \) has a horizontal asymptote at \( y = 0 \). Finally, \( f \) has vertical asymptotes at \( x = 0 \) and \( x = 2 \) with

\[
\lim_{x \to 0^-} \left( \frac{1}{x(x-2)} \right) = +\infty \quad \text{and} \quad \lim_{x \to 0^+} \left( \frac{1}{x(x-2)} \right) = -\infty.
\]

\[
\lim_{x \to 2^-} \left( \frac{1}{x(x-2)} \right) = \infty \quad \text{and} \quad \lim_{x \to 2^+} \left( \frac{1}{x(x-2)} \right) = -\infty.
\]

61. \( y = \frac{1}{x^2 - 6x + 8} \)

**SOLUTION** Let \( f(x) = \frac{1}{x^2 - 6x + 8} = \frac{1}{(x-2)(x-4)} \). Then \( f'(x) = \frac{6-2x}{(x^2-6x+8)^2} \), so that \( f \) is increasing for \( x < 2 \) and for \( 2 < x < 3 \), is decreasing for \( 3 < x < 4 \) and for \( x > 4 \), and has a local maximum at \( x = 3 \). Moreover, \( f''(x) = \frac{2(3x^2 - 18x + 28)}{(x^2 - 6x + 8)^3} \), so that \( f \) is concave up for \( x < 2 \) and for \( x > 4 \) and is concave down for \( 2 < x < 4 \). Because \( \lim_{x \to \pm\infty} \frac{1}{x^2 - 6x + 8} = 0 \), \( f \) has a horizontal asymptote at \( y = 0 \). Finally, \( f \) has vertical asymptotes at \( x = 2 \) and \( x = 4 \), with

\[
\lim_{x \to 2^-} \left( \frac{1}{x^2 - 6x + 8} \right) = \infty \quad \text{and} \quad \lim_{x \to 2^+} \left( \frac{1}{x^2 - 6x + 8} \right) = -\infty
\]

and

\[
\lim_{x \to 4^-} \left( \frac{1}{x^2 - 6x + 8} \right) = -\infty \quad \text{and} \quad \lim_{x \to 4^+} \left( \frac{1}{x^2 - 6x + 8} \right) = \infty.
\]

63. \( y = 1 - \frac{3}{x} + \frac{4}{x^3} \)

**SOLUTION** Let \( f(x) = 1 - \frac{3}{x} + \frac{4}{x^3} \). Then

\[
f'(x) = \frac{3}{x^2} - \frac{12}{x^3} = \frac{3(x-2)(x+2)}{x^4},
\]

so that \( f \) is increasing for \( |x| > 2 \) and decreasing for \( -2 < x < 0 \) and for \( 0 < x < 2 \). Moreover,

\[
f''(x) = \frac{6}{x^3} + \frac{48}{x^5} = \frac{6(8 - x^2)}{x^5}.
\]
so that $f$ is concave down for $-2\sqrt{2} < x < 0$ and for $x > 2\sqrt{2}$, while $f$ is concave up for $x < -2\sqrt{2}$ and for $0 < x < 2\sqrt{2}$. Because

$$\lim_{x \to \pm \infty} \left(1 - \frac{3}{x} + \frac{4}{x^2}\right) = 1,$$

$f$ has a horizontal asymptote at $y = 1$. Finally, $f$ has a vertical asymptote at $x = 0$ with

$$\lim_{x \to 0^-} \left(1 - \frac{3}{x} + \frac{4}{x^2}\right) = -\infty \quad \text{and} \quad \lim_{x \to 0^+} \left(1 - \frac{3}{x} + \frac{4}{x^2}\right) = \infty.$$

65. $y = \frac{1}{x^2} - \frac{1}{(x - 2)^2}$

**SOLUTION** Let $f(x) = \frac{1}{x^2} - \frac{1}{(x - 2)^2}$. Then $f'(x) = -2x^{-3} + 2(x - 2)^{-3}$, so that $f$ is increasing for $x < 0$ and for $x > 2$ and is decreasing for $0 < x < 2$. Moreover,

$$f''(x) = 6x^{-4} - 6(x - 2)^{-4} = -\frac{48(x - 1)(x^2 - 2x + 2)}{x^4(x - 2)^4},$$

so that $f$ is concave up for $x < 0$ and for $0 < x < 1$, is concave down for $1 < x < 2$ and for $x > 2$, and has a point of inflection at $x = 1$. Because $\lim_{x \to \pm \infty} \left(\frac{1}{x^2} - \frac{1}{(x - 2)^2}\right) = 0$, $f$ has a horizontal asymptote at $y = 0$. Finally, $f$ has vertical asymptotes at $x = 0$ and $x = 2$ with

$$\lim_{x \to 0^-} \left(\frac{1}{x^2} - \frac{1}{(x - 2)^2}\right) = \infty \quad \text{and} \quad \lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{(x - 2)^2}\right) = \infty$$

and

$$\lim_{x \to 2^-} \left(\frac{1}{x^2} - \frac{1}{(x - 2)^2}\right) = -\infty \quad \text{and} \quad \lim_{x \to 2^+} \left(\frac{1}{x^2} - \frac{1}{(x - 2)^2}\right) = -\infty.$$

67. $y = \frac{1}{(x^2 + 1)^2}$

**SOLUTION** Let $f(x) = \frac{1}{(x^2 + 1)^2}$. Then $f'(x) = \frac{-4x}{(x^2 + 1)^3}$, so that $f$ is increasing for $x < 0$, is decreasing for $x > 0$ and has a local maximum at $x = 0$. Moreover,

$$f''(x) = \frac{-4(x^2 + 1)^3 + 4x \cdot 3(x^2 + 1)^2 \cdot 2x}{(x^2 + 1)^6} = \frac{20x^2 - 4}{(x^2 + 1)^4},$$

so that $f$ is concave up for $|x| > 1/\sqrt{3}$, is concave down for $|x| < 1/\sqrt{3}$, and has points of inflection at $x = \pm 1/\sqrt{3}$. Because $\lim_{x \to \pm \infty} \frac{1}{(x^2 + 1)^2} = 0$, $f$ has a horizontal asymptote at $y = 0$. Finally, $f$ has no vertical asymptotes.
69. \( y = \frac{1}{\sqrt{x^2 + 1}} \)

**SOLUTION** Let \( f(x) = \frac{1}{\sqrt{x^2 + 1}} \). Then

\[
f'(x) = -\frac{x}{(x^2 + 1)^{3/2}} = -x(x^2 + 1)^{-3/2},
\]

so that \( f \) is increasing for \( x < 0 \) and decreasing for \( x > 0 \). Moreover,

\[
f''(x) = -\frac{3}{2}x(x^2 + 1)^{-5/2}(-2x) - (x^2 + 1)^{-3/2} = (2x^2 - 1)(x^2 + 1)^{-5/2},
\]

so that \( f \) is concave down for \( |x| < \frac{\sqrt{2}}{2} \) and concave up for \( |x| > \frac{\sqrt{2}}{2} \). Because

\[
\lim_{x \to \pm\infty} \frac{1}{\sqrt{x^2 + 1}} = 0,
\]

\( f \) has a horizontal asymptote at \( y = 0 \). Finally, \( f \) has no vertical asymptotes.

---

**Further Insights and Challenges**

In Exercises 71–75, we explore functions whose graphs approach a nonhorizontal line as \( x \to \pm\infty \). A line \( y = ax + b \) is called a **slant asymptote** if

\[
\lim_{x \to \pm\infty} (f(x) - (ax + b)) = 0
\]

or

\[
\lim_{x \to -\infty} (f(x) - (ax + b)) = 0
\]

71. Let \( f(x) = \frac{x^2}{x - 1} \) (Figure 22). Verify the following:

(a) \( f(0) \) is a local max and \( f(2) \) a local min.

(b) \( f \) is concave down on \((-\infty, 1)\) and concave up on \((1, \infty)\).

(c) \( \lim_{x \to -1^-} f(x) = -\infty \) and \( \lim_{x \to -1^+} f(x) = \infty \).

(d) \( y = x + 1 \) is a slant asymptote of \( f(x) \) as \( x \to \pm\infty \).

(e) The slant asymptote lies above the graph of \( f(x) \) for \( x < 1 \) and below the graph for \( x > 1 \).
SIGN ANALYSES REVEAL THAT

(a) Sign analysis of $f''(x)$ reveals that $f''(x) < 0$ on $(-\infty, 1)$ and $f''(x) > 0$ on $(1, \infty)$.

(b) Critical points of $f'(x)$ occur at $x = 0$ and $x = 2$. $x = 1$ is not a critical point because it is not in the domain of $f$.

(c) Sign analyses reveal that $x = 2$ is a local minimum of $f$ and $x = 0$ is a local maximum.

(d) Note that using polynomial division, $f(x) = \frac{x^2}{x-1}$. Then

$$\lim_{x \to 1^-} \frac{1}{1-x} = -\infty \quad \text{and} \quad \lim_{x \to 1^+} \frac{1}{1-x} = \infty.$$ 

(e) For $x > 1$, $f(x) - (x + 1) = \frac{1}{x-1} > 0$, so $f(x)$ approaches $x + 1$ from above. Similarly, for $x < 1$, $f(x) - (x + 1) = \frac{1}{x-1} < 0$, so $f(x)$ approaches $x + 1$ from below.

73. Sketch the graph of

$$f(x) = \frac{x^2}{x+1}.$$ 

Proceed as in the previous exercise to find the slant asymptote.

**SOLUTION** Let $f(x) = \frac{x^2}{x+1}$. Then $f'(x) = \frac{x(x+2)}{(x+1)^2}$ and $f''(x) = \frac{2}{(x+1)^3}$. Thus, $f$ is increasing for $x < -2$ and for $x > 0$, is decreasing for $-2 < x < -1$ and for $-1 < x < 0$, has a local minimum at $x = 0$, has a local maximum at $x = -2$, is concave down on $(-\infty, -1)$ and concave up on $(-1, \infty)$. Limit analyses give a vertical asymptote at $x = -1$, with

$$\lim_{x \to -1^-} \frac{x^2}{x+1} = -\infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{x^2}{x+1} = \infty.$$ 

By polynomial division, $f(x) = x - 1 + \frac{1}{x+1}$ and

$$\lim_{x \to \pm \infty} \left( x - 1 + \frac{1}{x+1} - (x-1) \right) = 0,$$ 

which implies that the slant asymptote is $y = x - 1$. Notice that $f$ approaches the slant asymptote as in exercise 71.

75. Sketch the graph of $f(x) = \frac{1-x^2}{2-x}$.

**SOLUTION** Let $f(x) = \frac{1-x^2}{2-x}$. Using polynomial division, $f(x) = x + 2 + \frac{3}{x-2}$. Then

$$\lim_{x \to \pm \infty} \left( f(x) - (x+2) \right) = \lim_{x \to \pm \infty} \left( (x+2) + \frac{3}{x-2} - (x+2) \right) = \lim_{x \to \pm \infty} \frac{3}{x-2} = \frac{3}{1}, \quad \lim_{x \to \pm \infty} x^{-1} = 0$$

which implies that $y = x + 2$ is the slant asymptote of $f(x)$. Since $f(x) - (x+2) = \frac{3}{x-2} > 0$ for $x > 2$, $f(x)$ approaches the slant asymptote from above for $x > 2$; similarly, $\frac{3}{x-2} < 0$ for $x < 2$ so $f(x)$ approaches the slant asymptote from below for $x < 2$. Moreover, $f'(x) = \frac{x^2 - 4x + 1}{(2-x)^2}$ and $f''(x) = \frac{-6}{(2-x)^3}$. Sign analyses reveal a local minimum at $x = 2 + \sqrt{3}$, a local maximum at $x = 2 - \sqrt{3}$ and that $f$ is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$. Limit analyses give a vertical asymptote at $x = 2$. 

May 23, 2011
77. Assume that \( f'(x) \) exists and \( f''(x) > 0 \) for all \( x \). Show that \( f(x) \) cannot be negative for all \( x \). Hint: Show that \( f'(b) \neq 0 \) for some \( b \) and use the result of Exercise 64 in Section 4.4.

**Solution** Let \( f(x) \) be a function such that \( f''(x) \) exists and \( f''(x) > 0 \) for all \( x \). Since \( f''(x) > 0 \), there is at least one point \( x = b \) such that \( f'(b) \neq 0 \). If not, \( f'(x) = 0 \) for all \( x \), so \( f''(x) = 0 \). By the result of Exercise 64 in Section 4.4, \( f(x) \geq f(b) + f'(b)(x - b) \). Now, if \( f'(b) > 0 \), we find that \( f(b) + f'(b)(x - b) > 0 \) whenever
\[
x > \frac{bf'(b) - f(b)}{f'(b)},
\]
a condition that must be met for some \( x \) sufficiently large. For such \( x \), \( f(x) > f(b) + f'(b)(x - b) > 0 \). On the other hand, if \( f'(b) < 0 \), we find that \( f(b) + f'(b)(x - b) > 0 \) whenever
\[
x < \frac{bf'(b) - f(b)}{f'(b)}.
\]
For such an \( x \), \( f(x) > f(b) + f'(b)(x - b) > 0 \).

### 4.7 Applied Optimization

#### Preliminary Questions

1. The problem is to find the right triangle of perimeter 10 whose area is as large as possible. What is the constraint equation relating the base \( b \) and height \( h \) of the triangle?

**Solution** The perimeter of a right triangle is the sum of the lengths of the base, the height and the hypotenuse. If the base has length \( b \) and the height is \( h \), then the length of the hypotenuse is \( \sqrt{b^2 + h^2} \) and the perimeter of the triangle is \( P = b + h + \sqrt{b^2 + h^2} \). The requirement that the perimeter be 10 translates to the constraint equation
\[
b + h + \sqrt{b^2 + h^2} = 10.
\]

2. Describe a way of showing that a continuous function on an open interval \( (a, b) \) has a minimum value.

**Solution** If the function tends to infinity at the endpoints of the interval, then the function must take on a minimum value at a critical point.

3. Is there a rectangle of area 100 of largest perimeter? Explain.

**Solution** No. Even by fixing the area at 100, we can take one of the dimensions as large as we like thereby allowing the perimeter to become as large as we like.

#### Exercises

1. Find the dimensions \( x \) and \( y \) of the rectangle of maximum area that can be formed using 3 meters of wire.

   (a) What is the constraint equation relating \( x \) and \( y \)?
   (b) Find a formula for the area in terms of \( x \) alone.
   (c) What is the interval of optimization? Is it open or closed?
   (d) Solve the optimization problem.

**Solution**

(a) The perimeter of the rectangle is 3 meters, so \( 3 = 2x + 2y \), which is equivalent to \( y = \frac{3}{2} - x \).

(b) Using part (a), \( A = xy = x\left(\frac{3}{2} - x\right) = \frac{3}{2}x - x^2 \).

(c) This problem requires optimization over the closed interval \([0, \frac{3}{2}]\), since both \( x \) and \( y \) must be non-negative.

(d) \( A'(x) = \frac{3}{2} - 2x = 0 \), which yields \( x = \frac{3}{4} \) and consequently, \( y = \frac{3}{4} \). Because \( A(0) = A(3/2) = 0 \) and \( A(3/4) = 0.5625 \), the maximum area 0.5625 m² is achieved with \( x = y = \frac{3}{4} \) m.
3. Wire of length 12 m is divided into two pieces and the pieces are bent into a square and a circle. How should this be done in order to minimize the sum of their areas?

**Solution** Suppose the wire is divided into one piece of length \( x \) m that is bent into a circle and a piece of length \( 12 - x \) m that is bent into a square. Because the circle has circumference \( x \), it follows that the radius of the circle is \( x/2\pi \); therefore, the area of the circle is

\[
\pi \left( \frac{x}{2\pi} \right)^2 = \frac{x^2}{4\pi}.
\]

As for the square, because the perimeter is \( 12 - x \), the length of each side is \( 3 - x/4 \) and the area is \( (3 - x/4)^2 \). Then

\[
A(x) = \frac{x^2}{4\pi} + \left(3 - \frac{x}{4}\right)^2.
\]

Now

\[
A'(x) = \frac{x}{2\pi} - \frac{1}{2} \left(3 - \frac{x}{4}\right) = 0
\]

when

\[
x = \frac{12\pi}{4 + \pi} \approx 5.28 \text{ m}.
\]

Because \( A(0) = 9 \text{ m}^2 \), \( A(12) = \frac{36}{\pi} \approx 11.46 \text{ m}^2 \), and

\[
A \left( \frac{12\pi}{4 + \pi} \right) \approx 5.04 \text{ m}^2,
\]

we see that the sum of the areas is minimized when approximately 5.28 m of the wire is allotted to the circle.

5. A flexible tube of length 4 m is bent into an \( L \)-shape. Where should the bend be made to minimize the distance between the two ends?

**Solution** Let \( x, y > 0 \) be lengths of the side of the \( L \). Since \( x + y = 4 \) or \( y = 4 - x \), the distance between the ends of \( L \) is

\[
h(x) = \sqrt{x^2 + y^2} = \sqrt{x^2 + (4 - x)^2}.
\]

We may equivalently minimize the square of the distance,

\[
f(x) = x^2 + y^2 = x^2 + (4 - x)^2.
\]

This is easier computationally (when working by hand). Solve \( f'(x) = 4x - 8 = 0 \) to obtain \( x = 2 \) m. Now \( f(0) = f(4) = 16 \), whereas \( f(2) = 8 \). Hence the distance between the two ends of the \( L \) is minimized when the bend is made at the middle of the wire.

7. A rancher will use 600 m of fencing to build a corral in the shape of a semicircle on top of a rectangle (Figure 9). Find the dimensions that maximize the area of the corral.

**Solution** Let \( x \) be the width of the corral and therefore the diameter of the semicircle, and let \( y \) be the height of the rectangular section. Then the perimeter of the corral can be expressed by the equation

\[
2y + x + \frac{\pi}{2} \cdot x = 2y + (1 + \frac{\pi}{2})x = 600 \text{ m or equivalently, } y = \frac{1}{2} \left( 600 - (1 + \frac{\pi}{2})x \right).
\]

Since \( x \) and \( y \) must both be nonnegative, it follows that \( x \) must be restricted to the interval \( [0, \frac{600}{1 + \frac{\pi}{2}}] \). The area of the corral is the sum of the area of the rectangle and semicircle,

\[
A = xy + \frac{\pi}{8}x^2.
\]

Making the substitution for \( y \) from the constraint equation,

\[
A(x) = \frac{1}{2}x \left( 600 - (1 + \frac{\pi}{2})x \right) + \frac{\pi}{8}x^2 = 300x - \frac{1}{2} \left(1 + \frac{\pi}{2}\right) x^2 + \frac{\pi}{8}x^2.
\]

Now, \( A'(x) = 300 - (1 + \frac{\pi}{2})x + \frac{\pi}{4}x = 0 \) implies \( x = \frac{300}{1 + \frac{\pi}{4}} \approx 168.029746 \text{ m} \). With \( A(0) = 0 \text{ m}^2 \),

\[
A \left( \frac{300}{1 + \frac{\pi}{4}} \right) \approx 25204.5 \text{ m}^2 \quad \text{and} \quad A \left( \frac{600}{1 + \frac{\pi}{2}} \right) \approx 21390.8 \text{ m}^2,
\]

it follows that the corral of maximum area has dimensions

\[
x = \frac{300}{1 + \frac{\pi}{4}} \text{ m} \quad \text{and} \quad y = \frac{150}{1 + \frac{\pi}{4}} \text{ m}.
\]
9. Find the dimensions of the rectangle of maximum area that can be inscribed in a circle of radius \( r = 4 \) (Figure 11).

**SOLUTION** Place the center of the circle at the origin with the sides of the rectangle (of lengths \( 2x > 0 \) and \( 2y > 0 \)) parallel to the coordinate axes. By the Pythagorean Theorem, \( x^2 + y^2 = r^2 = 16 \), so that \( y = \sqrt{16 - x^2} \). To guarantee both \( x \) and \( y \) are real and nonnegative, we must restrict \( x \) to the interval \([0, 4]\). Solve

\[
A'(x) = 4\sqrt{16 - x^2} - \frac{4x^2}{\sqrt{16 - x^2}} = 0
\]

for \( x > 0 \) to obtain \( x = \frac{4}{\sqrt{2}} = 2\sqrt{2} \). Since \( A(0) = A(4) = 0 \) and \( A(2\sqrt{2}) = 32 \), the rectangle of maximum area has dimensions \( 2x = 2y = 4\sqrt{2} \).

11. Find the point on the line \( y = x \) closest to the point \((1, 0)\). *Hint:* It is equivalent and easier to minimize the square of the distance.

**SOLUTION** With \( y = x \), let’s equivalently minimize the square of the distance, \( f(x) = (x - 1)^2 + y^2 = 2x^2 - 2x + 1 \), which is computationally easier (when working by hand). Solve \( f'(x) = 4x - 2 = 0 \) to obtain \( x = \frac{1}{2} \). Since \( f(x) \rightarrow \infty \) as \( x \rightarrow \pm \infty \), \( (\frac{1}{2}, \frac{1}{2}) \) is the point on \( y = x \) closest to \((1, 0)\).

13. **CAS** Find a good numerical approximation to the coordinates of the point on the graph of \( y = \ln x - x \) closest to the origin (Figure 13).

**SOLUTION** The distance from the origin to the point \((x, \ln x - x)\) on the graph of \( y = \ln x - x \) is \( d = \sqrt{x^2 + (\ln x - x)^2} \). As usual, we will minimize \( d^2 \). Let \( d^2 = f(x) = x^2 + (\ln x - x)^2 \). Then

\[
f'(x) = 2x + 2(\ln x - x)\left(\frac{1}{x} - 1\right).
\]

To determine \( x \), we need to solve

\[
4x + \frac{2\ln x}{x} - 2\ln x - 2 = 0.
\]

This yields \( x \approx 0.632784 \). Thus, the point on the graph of \( y = \ln x - x \) that is closest to the origin is approximately \((0.632784, -1.090410)\).

15. Find the angle \( \theta \) that maximizes the area of the isosceles triangle whose legs have length \( \ell \) (Figure 14).

**SOLUTION** The area of the triangle is

\[
A(\theta) = \frac{1}{2} \ell^2 \sin \theta.
\]
where \( 0 \leq \theta \leq \pi \). Setting

\[
A'(\theta) = \frac{1}{2} \ell^2 \cos \theta = 0
\]
yields \( \theta = \frac{\pi}{2} \). Since \( A(0) = A(\pi) = 0 \) and \( A\left(\frac{\pi}{2}\right) = \frac{1}{4} \ell^2 \), the angle that maximizes the area of the isosceles triangle is \( \theta = \frac{\pi}{2} \).

17. Find the area of the largest isosceles triangle that can be inscribed in a circle of radius \( r \).

**Solution** Consider the following diagram:

The area of the isosceles triangle is

\[
A(\theta) = \frac{1}{2} r^2 \sin(\pi - \theta) + \frac{1}{2} r^2 \sin(2\theta) = r^2 \sin \theta + \frac{1}{2} r^2 \sin(2\theta),
\]
where \( 0 \leq \theta \leq \pi \). Solve

\[
A'(\theta) = r^2 \cos \theta + r^2 \cos(2\theta) = 0
\]
to obtain \( \theta = \frac{2\pi}{3} \). Since \( A(0) = A(\pi) = 0 \) and \( A\left(\frac{2\pi}{3}\right) = \frac{1}{4} \sqrt{3} r^2 \), the area of the largest isosceles triangle that can be inscribed in a circle of radius \( r \) is \( \frac{1}{4} \sqrt{3} r^2 \).

19. A poster of area 6000 cm\(^2\) has blank margins of width 10 cm on the top and bottom and 6 cm on the sides. Find the dimensions that maximize the printed area.

**Solution** Let \( x \) be the width of the printed region, and let \( y \) be the height. The total printed area is \( A = xy \). Because the total area of the poster is 6000 cm\(^2\), we have the constraint \((x + 12)(y + 20) = 6000\), so that \( xy + 12y + 20x + 240 = 6000 \), or \( y = \frac{5760-20x}{x+12} \). Therefore, \( A(x) = 20\frac{288x-x^2}{x+12} \), where \( 0 \leq x \leq 288 \).

\( A(0) = A(288) = 0 \), so we are looking for a critical point on the interval \([0, 288] \). Setting \( A'(x) = 0 \) yields

\[
\frac{20(x + 12)(288 - 2x) - (288x - x^2)}{(x + 12)^2} = 0
\]

\[
\frac{-x^2 - 24x + 3456}{(x + 12)^2} = 0
\]

\[
x^2 + 24x - 3456 = 0
\]

\[
(x - 48)(x + 72) = 0
\]

Therefore \( x = 48 \) or \( x = -72 \). \( x = 48 \) is the only critical point of \( A(x) \) in the interval \([0, 288] \), so \( A(48) = 3840 \) is the maximum value of \( A(x) \) in the interval \([0, 288] \). Now, \( y = \frac{20\cdot 288 - 48}{288 + 12} = 80 \) cm, so the poster with maximum printed area is 48 + 12 = 60 cm wide by 80 + 20 = 100 cm tall.

21. Kepler’s Wine Barrel Problem In his work *Nova stereometria doliorum vinariorum* (New Solid Geometry of a Wine Barrel), published in 1615, astronomer Johannes Kepler stated and solved the following problem: Find the dimensions of the cylinder of largest volume that can be inscribed in a sphere of radius \( R \). Hint: Show that an inscribed cylinder has volume \( 2\pi x(R^2 - x^2) \), where \( x \) is one-half the height of the cylinder.

**Solution** Place the center of the sphere at the origin in three-dimensional space. Let the cylinder be of radius \( y \) and half-height \( x \). The Pythagorean Theorem states, \( x^2 + y^2 = R^2 \), so that \( y^2 = R^2 - x^2 \). The volume of the cylinder is \( V(x) = \pi y^2 (2x) = 2\pi \left( R^2 - x^2 \right) x = 2\pi R^2 x - 2\pi x^3 \). Allowing for degenerate cylinders, we have \( 0 \leq x \leq R \).

Solve \( V'(x) = 2\pi R^2 - 6\pi x^2 = 0 \) for \( x \geq 0 \) to obtain \( x = \frac{R}{\sqrt{3}} \). Since \( V(0) = V(R) = 0 \), the largest volume is \( V\left(\frac{R}{\sqrt{3}}\right) = \frac{2\pi}{3}\sqrt{3} R^3 \) when \( x = \frac{R}{\sqrt{3}} \) and \( y = \sqrt{\frac{2}{3}} R \).
23. A landscape architect wishes to enclose a rectangular garden of area 1,000 m² on one side by a brick wall costing $90/m and on the other three sides by a metal fence costing $30/m. Which dimensions minimize the total cost?

**SOLUTION** Let $x$ be the length of the brick wall and $y$ the length of an adjacent side with $x, y > 0$. With $xy = 1000$ or $y = \frac{1000}{x}$, the total cost is

\[ C(x) = 90x + 30(2y) = 120x + 60000x^{-1}. \]

Solve $C'(x) = 120 - 60000x^{-2} = 0$ for $x > 0$ to obtain $x = 10\sqrt{2}$. Since $C(x) \to \infty$ as $x \to 0^+$ and as $x \to \infty$, the minimum cost is $C(10\sqrt{2}) = 2400\sqrt{2} \approx 5366.56$ when $x = 10\sqrt{2} \approx 22.36$ m and $y = 20\sqrt{2} \approx 44.72$ m.

25. Find the maximum area of a rectangle inscribed in the region bounded by the graph of $y = \frac{4-x}{2+x}$ and the axes (Figure 17).

**SOLUTION** Let $s$ be the width of the rectangle. The height of the rectangle is $h = \frac{4-s}{2+s}$, so that the area is

\[ A(s) = s \cdot h = \frac{4s - s^2}{2 + s}. \]

We are maximizing on the closed interval $[0, 4]$. It is obvious from the pictures that $A(0) = A(4) = 0$, so we look for critical points of $A$.

\[ A'(s) = \frac{(2 + s)(4 - 2s) - (4s - s^2)}{(2 + s)^2} = -s^2 + 4s - 8 \]

The only point where $A'(s)$ doesn’t exist is $s = -2$ which isn’t under consideration.

Setting $A'(s) = 0$ gives, by the quadratic formula,

\[ s = \frac{-4 \pm \sqrt{48}}{2} = -2 \pm 2\sqrt{3}. \]

Of these, only $-2 + 2\sqrt{3}$ is positive, so this is our lone critical point. $A(-2 + 2\sqrt{3}) \approx 1.0718 > 0$. Since we are finding the maximum over a closed interval and $-2 + 2\sqrt{3}$ is the only critical point, the maximum area is $A(-2 + 2\sqrt{3}) \approx 1.0718$.

27. Find the maximum area of a rectangle circumscribed around a rectangle of sides $L$ and $H$. **Hint:** Express the area in terms of the angle $\theta$ (Figure 18).

**SOLUTION** Position the $L \times H$ rectangle in the first quadrant of the $xy$-plane with its “northwest” corner at the origin. Let $\theta$ be the angle the base of the circumscribed rectangle makes with the positive $x$-axis, where $0 \leq \theta \leq \frac{\pi}{4}$. Then the area of the circumscribed rectangle is $A = LH + 2 \cdot \frac{1}{2}(H \sin \theta)(H \cos \theta) + 2 \cdot \frac{1}{2}(L \sin \theta)(L \cos \theta) = LH + \frac{1}{2}(L^2 + H^2) \sin 2\theta$, which has a maximum value of $LH + \frac{1}{2}(L^2 + H^2)$ when $\theta = \frac{\pi}{4}$ because $\sin 2\theta$ achieves its maximum when $\theta = \frac{\pi}{4}$.
29. Find the equation of the line through \( P = (4, 12) \) such that the triangle bounded by this line and the axes in the first quadrant has minimal area.

**Solution** Let \( P = (4, 12) \) be a point in the first quadrant and \( y - 12 = m(x - 4), -\infty < m < 0 \), be a line through \( P \) that cuts the positive \( x \) - and \( y \) -axes. Then \( y = L(x) = m(x - 4) + 12 \). The line \( L(x) \) intersects the \( y \)-axis at \( H(0, 12 - 4m) \) and the \( x \)-axis at \( W \left( 4 - \frac{12}{m}, 0 \right) \). Hence the area of the triangle is

\[
A(m) = \frac{1}{2} (12 - 4m) \left( 4 - \frac{12}{m} \right) = 48 - 8m - 72m^{-1}.
\]

Solve \( A'(m) = 72m - 8 = 0 \) for \( m < 0 \) to obtain \( m = -3 \). Since \( A \to \infty \) as \( m \to -\infty \) or \( m \to 0^- \), we conclude that the minimal triangular area is obtained when \( m = -3 \). The equation of the line through \( P = (4, 12) \) is

\[
y = -3(x - 4) + 12 = -3x + 24.
\]

31. Archimedes' Problem A spherical cap (Figure 20) of radius \( r \) and height \( h \) has volume \( V = \pi h^2 \left( r - \frac{1}{3} h \right) \) and surface area \( S = 2\pi rh \). Prove that the hemisphere encloses the largest volume among all spherical caps of fixed surface area \( S \).

**Solution** Consider all spherical caps of fixed surface area \( S \). Because \( S = 2\pi rh \), it follows that

\[
r = \frac{S}{2\pi h}
\]

and

\[
V(h) = \pi h^2 \left( \frac{S}{2\pi h} - \frac{1}{3} h \right) = \frac{S}{2} h - \frac{\pi}{3} h^3.
\]

Now

\[
V'(h) = \frac{S}{2} - \pi h^2 = 0
\]

when

\[
h^2 = \frac{S}{2\pi} \quad \text{or} \quad h = \frac{S}{2\pi} = r.
\]

Hence, the hemisphere encloses the largest volume among all spherical caps of fixed surface area \( S \).

33. A box of volume 72 m\(^3\) with square bottom and no top is constructed out of two different materials. The cost of the bottom is $40/m\(^2\) and the cost of the sides is $30/m\(^2\). Find the dimensions of the box that minimize total cost.

**Solution** Let \( s \) denote the length of the side of the square bottom of the box and \( h \) denote the height of the box. Then

\[
V = s^2h = 72 \quad \text{or} \quad h = \frac{72}{s^2}.
\]

The cost of the box is

\[
C = 40s^2 + 120sh = 40s^2 + \frac{8640}{s}.
\]

so

\[
C'(s) = 80s - \frac{8640}{s^2} = 0
\]

when \( s = 3\sqrt[4]{4} \) m and \( h = 2\sqrt[4]{4} \) m. Because \( C \to \infty \) as \( s \to 0^- \) and as \( s \to \infty \), we conclude that the critical point gives the minimum cost.
35. Your task is to design a rectangular industrial warehouse consisting of three separate spaces of equal size as in Figure 22. The wall materials cost $500 per linear meter and your company allocates $2,400,000 for the project.

(a) Which dimensions maximize the area of the warehouse?

(b) What is the area of each compartment in this case?

![Figure 22](image)

**SOLUTION**  Let one compartment have length $x$ and width $y$. Then total length of the wall of the warehouse is $P = 4x + 6y$ and the constraint equation is cost is $2,400,000 = 500(4x + 6y)$, which gives $y = 800 - \frac{5}{3}x$.

(a) Area is given by $A = 3xy = 3x \left(800 - \frac{5}{3}x\right) = 2400x - 2x^2$, where $0 \leq x \leq 1200$. Then $A'(x) = 2400 - 4x = 0$ yields $x = 600$ and consequently $y = 400$. Since $A(0) = A(1200) = 0$ and $A(600) = 720,000$, the area of the warehouse is maximized when each compartment has length of 600 m and width of 400 m.

(b) The area of one compartment is $600 \times 400 = 240,000$ square meters.

37. According to a model developed by economists E. Headly and J. Pesek, if fertilizer made from $N$ pounds of nitrogen and $P$ pounds of phosphate is used on an acre of farmland, then the yield of corn (in bushels per acre) is

$$Y = 7.5 + 0.6N + 0.7P - 0.001N^2 - 0.002P^2 + 0.001NP$$

A farmer intends to spend $30 per acre on fertilizer. If nitrogen costs 25 cents/lb and phosphate costs 20 cents/lb, which combination of $N$ and $L$ produces the highest yield of corn?

**SOLUTION**  The farmer’s budget for fertilizer is $30 per acre, so we have the constraint equation

$$0.25N + 0.2P = 30$$

Substituting for $P$ in the equation for $Y$, we find

$$Y(N) = 7.5 + 0.6N + 0.7(150 - 1.25N) - 0.001N^2 - 0.002(150 - 1.25N)^2 + 0.001N(150 - 1.25N)$$

$$= 67.5 + 0.625N - 0.005375N^2$$

Both $N$ and $P$ must be nonnegative. Since $P = 150 - 1.25N$ $\geq 0$, we require that $0 \leq N \leq 120$. Next, we find

$$\frac{dY}{dN} = 0.625 - 0.01075N = 0 \implies N = \frac{0.625}{0.01075} \approx 58.14$$

Now, $Y(0) = 67.5$, $Y(120) = 65.1$ and $Y(58.14) \approx 85.67$, so the maximum yield of corn occurs for $N \approx 58.14$ pounds and $P \approx 77.33$ pounds.

39. All units in a 100-unit apartment building are rented out when the monthly rent is set at $r = $900/month. Suppose that one unit becomes vacant with each $10 increase in rent and that each occupied unit costs $80/month in maintenance. Which rent $r$ maximizes monthly profit?

**SOLUTION**  Let $n$ denote the number of $10 in rent. Then the monthly profit is given by

$$P(n) = (100 - n)(900 + 10n - 80) = 82000 + 180n - 10n^2,$$

and

$$P'(n) = 180 - 20n = 0$$

when $n = 9$. We know this results in maximum profit because this gives the location of vertex of a downward opening parabola. Thus, monthly profit is maximized with a rent of $990.

41. The monthly output of a Spanish light bulb factory is $P = 2LK^2$ (in millions), where $L$ is the cost of labor and $K$ is the cost of equipment (in millions of euros). The company needs to produce 1.7 million units per month. Which values of $L$ and $K$ would minimize the total cost $L + K$?

**SOLUTION**  Since $P = 1.7$ and $P = 2LK^2$, we have $L = \frac{0.85}{K^2}$. Accordingly, the cost of production is

$$C(K) = L + K = K + \frac{0.85}{K^2}.$$ 

Solve $C'(K) = 1 - \frac{1.7}{K^3}$ for $K \geq 0$ to obtain $K = \sqrt[3]{1.7}$. Since $C(K) \to \infty$ as $K \to 0^+$ and as $K \to \infty$, the minimum cost of production is achieved for $K = \sqrt[3]{1.7} \approx 1.2$ and $L = 0.6$. The company should invest 1.2 million euros in equipment and 600,000 euros in labor.
43. Brandon is on one side of a river that is 50 m wide and wants to reach a point 200 m downstream on the opposite side as quickly as possible by swimming diagonally across the river and then running the rest of the way. Find the best route if Brandon can swim at 1.5 m/s and run at 4 m/s.

**Solution** Let lengths be in meters, times in seconds, and speeds in m/s. Suppose that Brandon swims diagonally to a point located \( x \) meters downstream on the opposite side. Then Brandon then swims a distance \( \sqrt{x^2 + 50^2} \) and runs a distance \( 200 - x \). The total time of the trip is

\[
T(x) = \frac{\sqrt{x^2 + 2500}}{1.5} + \frac{200 - x}{4}, \quad 0 \leq x \leq 200.
\]

Solve

\[
f'(x) = \frac{2x}{3\sqrt{x^2 + 2500}} - \frac{1}{4} = 0
\]

to obtain \( x = 30 \frac{5}{11} \approx 20.2 \) and \( f(20.2) \approx 80.9 \). Since \( f(0) \approx 83.3 \) and \( f(200) \approx 137.4 \), we conclude that the minimal time is 80.9 s. This occurs when Brandon swims diagonally to a point located 20.2 m downstream and then runs the rest of the way.

**In Exercises 45–47, a box (with no top) is to be constructed from a piece of cardboard of sides \( A \) and \( B \) by cutting out squares of length \( h \) from the corners and folding up the sides (Figure 26).**

**45.** Find the value of \( h \) that maximizes the volume of the box if \( A = 15 \) and \( B = 24 \). What are the dimensions of this box?

**Solution** Once the sides have been folded up, the base of the box will have dimensions \((A - 2h) \times (B - 2h)\) and the height of the box will be \(h\). Thus

\[
V(h) = h(A - 2h)(B - 2h) = 4h^3 - 2(A + B)h^2 + ABh.
\]

When \(A = 15\) and \(B = 24\), this gives

\[
V(h) = 4h^3 - 78h^2 + 360h,
\]

and we need to maximize over \(0 \leq h \leq 15\). Now,

\[
V'(h) = 12h^2 - 156h + 360 = 0
\]
yields \(h = 3\) and \(h = 10\). Because \(h = 10\) is not in the domain of the problem and \(V(0) = V(15/2) = 0\) and \(V(3) = 486\), volume is maximized when \(h = 3\). The corresponding dimensions are \(9 \times 18 \times 3\).

**47.** Which values of \( A \) and \( B \) maximize the volume of the box if \( h = 10 \) cm and \( AB = 900 \) cm.

**Solution** With \(h = 10\) and \(AB = 900\) (which means that \(B = 900/A\)), the volume of the box is

\[
V(A) = 10(A - 20) \left( \frac{900}{A} - 20 \right) = 13,000 - 200A - \frac{180,000}{A},
\]

where \(20 \leq A \leq 45\). Now, solving

\[
V'(A) = -200 + \frac{180,000}{A^2} = 0
\]
yields \(A = 30\). Because \(V(20) = V(45) = 0\) and \(V(30) = 1000 \text{ cm}^3\), maximum volume is achieved with \(A = B = 30 \) cm.
49. A billboard of height $b$ is mounted on the side of a building with its bottom edge at a distance $h$ from the street as in Figure 27. At what distance $x$ should an observer stand from the wall to maximize the angle of observation $\theta$?

![FIGURE 27](image)

**SOLUTION** From the upper diagram in Figure 27 and the addition formula for the cotangent function, we see that

$$
\cot \theta = \frac{1 + \frac{x}{b+h}}{\frac{x}{b+h}} = \frac{x^2 + h(b + h)}{bx},
$$

where $b$ and $h$ are constant. Now, differentiate with respect to $x$ and solve

$$
-\csc^2 \theta \frac{d\theta}{dx} = \frac{x^2 - h(b + h)}{bx^2} = 0
$$

to obtain $x = \sqrt{bh + h^2}$. Since this is the only critical point, and since $\theta \to 0$ as $x \to 0+$ and $\theta \to 0$ as $x \to \infty$, $\theta(x)$ reaches its maximum at $x = \sqrt{bh + h^2}$.

51. **Optimal Delivery Schedule** A gas station sells $Q$ gallons of gasoline per year, which is delivered $N$ times per year in equal shipments of $Q/N$ gallons. The cost of each delivery is $d$ dollars and the yearly storage costs are $sQ/N$. Let $T$ be the length of time (a fraction of a year) between shipments and $s$ is a constant. Show that costs are minimized for $N = \sqrt{sQT}$. (Hint: $T = 1/N$.) Find the optimal number of deliveries if $Q = 2$ million gal, $d = $8000, and $s = 30$ cents/gal-yr. Your answer should be a whole number, so compare costs for the two integer values of $N$ nearest the optimal value.

**SOLUTION** There are $N$ shipments per year, so the time interval between shipments is $T = 1/N$ years. Hence, the total storage costs per year are $sQ/N$. The yearly delivery costs are $dN$ and the total costs is $C(N) = dN + sQ/N$. Solving,

$$
C'(N) = d - \frac{sQ}{N^2} = 0
$$

for $N$ yields $N = \sqrt{sQT}$. For the specific case $Q = 2,000,000$, $d = 8000$ and $s = 0.30$,

$$
N = \sqrt{\frac{0.30(2,000,000)}{8000}} = 8.66.
$$

With $C(8) = $139,000 and $C(9) = $138,667, the optimal number of deliveries per year is $N = 9$.

53. Let $(a, b)$ be a fixed point in the first quadrant and let $S(d)$ be the sum of the distances from $(d, 0)$ to the points $(0, 0)$, $(a, b)$, and $(a, -b)$.

(a) Find the value of $d$ for which $S(d)$ is minimal. The answer depends on whether $b < \sqrt{3}a$ or $b \geq \sqrt{3}a$. Hint: Show that $d = 0$ when $b \geq \sqrt{3}a$.

(b) Let $a = 1$. Plot $S(d)$ for $b = 0.5$, $\sqrt{3}$, and 3 and describe the position of the minimum.

**SOLUTION** (a) If $d < 0$, then the distance from $(d, 0)$ to the other three points can all be reduced by increasing the value of $d$. Similarly, if $d > a$, then the distance from $(d, 0)$ to the other three points can all be reduced by decreasing the value of $d$. It follows that the minimum of $S(d)$ must occur for $0 \leq d \leq a$. Restricting attention to this interval, we find

$$
S(d) = d + 2\sqrt{(d-a)^2 + b^2}.
$$

Solving

$$
S'(d) = 1 + \frac{2(d-a)}{\sqrt{(d-a)^2 + b^2}} = 0
$$

yields the critical point $d = a - b/\sqrt{3}$. If $b < \sqrt{3}a$, then $d = a - b/\sqrt{3} > 0$ and the minimum occurs at this value of $d$. On the other hand, if $b \geq \sqrt{3}a$, then the minimum occurs at the endpoint $d = 0$. 

May 23, 2011
(b) Let $a = 1$. Plots of $S(d)$ for $b = 0.5, b = \sqrt{3}$ and $b = 3$ are shown below. For $b = 0.5$, the results of (a) indicate that the minimum should occur for $d = 1 - 0.5/\sqrt{3} \approx 0.711$, and this is confirmed in the plot. For both $b = \sqrt{3}$ and $b = 3$, the results of (a) indicate that the minimum should occur at $d = 0$, and both of these conclusions are confirmed in the plots.

55. In the setting of Exercise 54, show that for any $f$ the minimal force required is proportional to $1/\sqrt{1 + f^2}$.

**Solution** We minimize $F(\theta)$ by finding the maximum value $g(\theta) = \cos \theta + f \sin \theta$. The angle $\theta$ is restricted to the interval $[0, \pi/2]$. We solve for the critical points:

$$g'(\theta) = -\sin \theta + f \cos \theta = 0$$

We obtain

$$f \cos \theta = \sin \theta \Rightarrow \tan \theta = f$$

From the figure below we find that $\cos \theta = 1/\sqrt{1 + f^2}$ and $\sin \theta = f/\sqrt{1 + f^2}$. Hence

$$g(\theta) = \frac{1}{f} + \frac{f^2}{\sqrt{1 + f^2}} = \frac{1 + f^2}{\sqrt{1 + f^2}} = \sqrt{1 + f^2}$$

The values at the endpoints are

$$g(0) = 1, \quad g\left(\frac{\pi}{2}\right) = f$$

Both of these values are less than $\sqrt{1 + f^2}$. Therefore the maximum value of $g(\theta)$ is $\sqrt{1 + f^2}$ and the minimum value of $F(\theta)$ is

$$F = \frac{fmg}{g(\theta)} = \frac{fmg}{\sqrt{1 + f^2}}$$

57. The problem is to put a “roof” of side $s$ on an attic room of height $h$ and width $b$. Find the smallest length $s$ for which this is possible if $b = 27$ and $h = 8$ (Figure 31).

**Solution** Consider the right triangle formed by the right half of the rectangle and its “roof”. This triangle has hypotenuse $s$. 

![Figure 31](image-url)
As shown, let \( y \) be the height of the roof, and let \( x \) be the distance from the right base of the rectangle to the base of the roof. By similar triangles applied to the smaller right triangles at the top and right of the larger triangle, we get:

\[
\frac{y - 8}{27/2} = \frac{8}{x} \quad \text{or} \quad y = \frac{108}{x} + 8.
\]

\( s, y, \) and \( x \) are related by the Pythagorean Theorem:

\[
s^2 = \left(\frac{27}{2} + x\right)^2 + y^2 = \left(\frac{27}{2} + x\right)^2 + \left(\frac{108}{x} + 8\right)^2.
\]

Since \( s > 0 \), \( s^2 \) is least whenever \( s \) is least, so we can minimize \( s^2 \) instead of \( s \). Setting the derivative equal to zero yields

\[
2 \left(\frac{27}{2} + x\right) \cdot 2 \left(\frac{108}{x} + 8\right) - 8 \cdot x \cdot \left(\frac{27}{2} + x\right) = 0
\]

\[
2 \left(\frac{27}{2} + x\right) \cdot \left(1 - \frac{864}{x^3}\right) = 0
\]

The zeros are \( x = -\frac{27}{2} \) (irrelevant) and \( x = 6\sqrt[3]{4} \). Since this is the only critical point of \( s \) with \( x > 0 \), and since \( s \to \infty \) as \( x \to 0 \) and \( s \to \infty \) as \( x \to \infty \), this is the point where \( s \) attains its minimum. For this value of \( x \),

\[
s^2 = \left(\frac{27}{2} + 6\sqrt[3]{4}\right)^2 + \left(9\sqrt[3]{2} + 8\right)^2 \approx 904.13,
\]

so the smallest roof length is

\[ s \approx 30.07. \]

59. Find the maximum length of a pole that can be carried horizontally around a corner joining corridors of widths \( a = 24 \) and \( b = 3 \) (Figure 32).

\[ \text{FIGURE 32} \]

**SOLUTION** In order to find the length of the longest pole that can be carried around the corridor, we have to find the shortest length from the left wall to the top wall touching the corner of the inside wall. Any pole that does not fit in this shortest space cannot be carried around the corner, so an exact fit represents the longest possible pole.

Let \( \theta \) be the angle between the pole and a horizontal line to the right. Let \( c_1 \) be the length of pole in the corridor of width 24 and let \( c_2 \) be the length of pole in the corridor of width 3. By the definitions of sine and cosine,

\[
\frac{3}{c_2} = \sin \theta \quad \text{and} \quad \frac{24}{c_1} = \cos \theta,
\]

so that \( c_1 = \frac{24}{\cos \theta} \), \( c_2 = \frac{3}{\sin \theta} \). What must be minimized is the total length, given by

\[
f(\theta) = \frac{24}{\cos \theta} + \frac{3}{\sin \theta}
\]

Setting \( f'(\theta) = 0 \) yields

\[
\frac{24 \sin \theta}{\cos^2 \theta} - \frac{3 \cos \theta}{\sin^2 \theta} = 0
\]

\[
\frac{24 \sin \theta}{\cos^2 \theta} = \frac{3 \cos \theta}{\sin^2 \theta}
\]

\[
24 \sin^3 \theta = 3 \cos^3 \theta
\]

As \( \theta < \frac{\pi}{2} \) (the pole is being turned around a corner, after all), we can divide both sides by \( \cos^3 \theta \), getting \( \tan^3 \theta = \frac{1}{8} \). This implies that \( \tan \theta = \frac{1}{2} \) (tan \( \theta > 0 \) as the angle is acute).
Since \( f(\theta) \to \infty \) as \( \theta \to 0^+ \) and as \( \theta \to \frac{\pi}{2}^− \), we can tell that the minimum is attained at \( \theta_0 = \frac{1}{2} \). Because

\[
\tan \theta_0 = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{5}}.
\]

we draw a triangle with opposite side 1 and adjacent side 2. By Pythagoras, \( c = \sqrt{5} \), so

\[
\sin \theta_0 = \frac{1}{\sqrt{5}} \quad \text{and} \quad \cos \theta_0 = \frac{2}{\sqrt{5}}.
\]

From this, we get

\[
f(\theta_0) = \frac{24}{\cos \theta_0} + \frac{3}{\sin \theta_0} = \frac{24}{2\sqrt{5}} + \frac{3}{\sqrt{5}} = 15\sqrt{5}.
\]

61. Find the minimum length \( \ell \) of a beam that can clear a fence of height \( h \) and touch a wall located \( b \) ft behind the fence (Figure 33).

\[
\text{FIGURE 33}
\]

**SOLUTION**  

Let \( y \) be the height of the point where the beam touches the wall in feet. By similar triangles,

\[
\frac{y - h}{b} = \frac{h}{x} \quad \text{or} \quad y = \frac{bh}{x} + h
\]

and by Pythagoras:

\[
\ell^2 = (b + x)^2 + \left(\frac{bh}{x} + h\right)^2.
\]

We can minimize \( \ell^2 \) rather than \( \ell \), so setting the derivative equal to zero gives:

\[
2(b + x) + 2 \left(\frac{bh}{x} + h\right) \left(-\frac{bh}{x^2}\right) = 2(b + x) \left(1 - \frac{b^2h}{x^3}\right) = 0.
\]

The zeroes are \( b = -x \) (irrelevant) and \( x = \sqrt{bh} \). Since \( \ell^2 \to \infty \) as \( x \to 0^+ \) and as \( x \to \infty \), \( x = \sqrt{bh} \) corresponds to a minimum for \( \ell^2 \). For this value of \( x \), we have

\[
\ell^2 = (b + h^{2/3}b^{1/3})^2 + (h + h^{1/3}b^{2/3})^2
\]

\[
= b^{2/3}(b^{2/3} + b^{1/3})^2 + h^{2/3}(h^{2/3} + b^{2/3})^2
\]

and so

\[
\ell = (b^{2/3} + h^{2/3})^{3/2}.
\]

A beam that clears a fence of height \( h \) feet and touches a wall \( b \) feet behind the fence must have length at least \( \ell = (b^{2/3} + h^{2/3})^{3/2} \) ft.

63. A basketball player stands \( d \) feet from the basket. Let \( h \) and \( \alpha \) be as in Figure 34. Using physics, one can show that if the player releases the ball at an angle \( \theta \), then the initial velocity required to make the ball go through the basket satisfies

\[
v^2 = \frac{16d}{\cos^2 \theta (\tan \theta - \tan \alpha)}
\]

(a) Explain why this formula is meaningful only for \( \alpha < \theta < \frac{\pi}{2} \). Why does \( v \) approach infinity at the endpoints of this interval?
(b) Take $\alpha = \frac{\pi}{6}$ and plot $v^2$ as a function of $\theta$ for $\frac{\pi}{6} < \theta < \frac{\pi}{2}$. Verify that the minimum occurs at $\theta = \frac{\pi}{3}$.

(c) Set $F(\theta) = \cos^2 \theta (\tan \theta - \tan \alpha)$. Explain why $v$ is minimized for $\theta$ such that $F(\theta)$ is maximized.

(d) Verify that $F'(\theta) = \cos(\alpha - 2\theta) \sec \alpha$ (you will need to use the addition formula for cosine) and show that the maximum value of $F(\theta)$ on $[\alpha, \frac{\pi}{2}]$ occurs at $\theta_0 = \frac{\alpha}{2} + \frac{\pi}{4}$.

(e) For a given $\alpha$, the optimal angle for shooting the basket is $\theta_0$ because it minimizes $v^2$ and therefore minimizes the energy required to make the shot (energy is proportional to $v^2$). Show that the velocity $v_{opt}$ at the optimal angle $\theta_0$ satisfies

$$v_{opt}^2 = \frac{32d \cos \alpha}{1 - \sin \alpha} = \frac{32d^2}{-h + \sqrt{d^2 + h^2}}$$

(f) Show with a graph that for fixed $d$ (say, $d = 15$ ft, the distance of a free throw), $v_{opt}^2$ is an increasing function of $h$. Use this to explain why taller players have an advantage and why it can help to jump while shooting.

**Solution**

(a) $\alpha = 0$ corresponds to shooting the ball directly at the basket while $\alpha = \pi/2$ corresponds to shooting the ball directly upward. In neither case is it possible for the ball to go into the basket.

If the angle $\alpha$ is extremely close to 0, the ball is shot almost directly at the basket, so that it must be launched with great speed, as it can only fall an extremely short distance on the way to the basket.

On the other hand, if the angle $\alpha$ is extremely close to $\pi/2$, the ball is launched almost vertically. This requires the ball to travel a great distance upward in order to travel the horizontal distance. In either one of these cases, the ball has to travel at an enormous speed.

(b)

The minimum clearly occurs where $\theta = \pi/3$.

(c) If $F(\theta) = \cos^2 \theta (\tan \theta - \tan \alpha)$,

$$v^2 = \frac{16d}{\cos^2 \theta (\tan \theta - \tan \alpha)} = \frac{16d}{F(\theta)}.$$ Since $\alpha \leq \theta$, $F(\theta) > 0$, hence $v^2$ is smallest whenever $F(\theta)$ is greatest.

(d) $F'(\theta) = -2 \sin \theta \cos \theta (\tan \theta - \tan \alpha) + \cos^2 \theta \left( \sec^2 \theta \right) = -2 \sin \theta \cos \theta \tan \theta + 2 \sin \theta \cos \theta \tan \alpha + 1$. We will apply all the double angle formulas:

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta = 1 - 2 \sin^2 \theta; \sin 2\theta = 2 \sin \theta \cos \theta,$$

getting:

$$F'(\theta) = 2 \sin \theta \cos \theta \tan \alpha - 2 \sin \theta \cos \theta \tan \theta + 1$$

$$= 2 \sin \theta \cos \theta \frac{\sin \alpha}{\cos \alpha} - 2 \sin \theta \cos \theta \frac{\sin \theta}{\cos \theta} + 1$$

$$= \sec \alpha \left( -2 \sin^2 \theta \cos \alpha + 2 \sin \theta \cos \theta \sin \alpha + \cos \alpha \right)$$

$$= \sec \alpha \left( \cos \alpha (1 - 2 \sin^2 \theta) + \sin \alpha (2 \sin \theta \cos \theta) \right)$$

$$= \sec \alpha (\cos \alpha (\cos 2\theta) + \sin \alpha (\sin 2\theta))$$

$$= \sec \alpha \cos(\alpha - 2\theta)$$
A critical point of \( F(\theta) \) occurs where \( \cos(\alpha - 2\theta) = 0 \), so that \( \alpha - 2\theta = -\frac{\pi}{2} \) (negative because \( 2\theta > \theta > \alpha \)), and this gives us \( \theta = \alpha/2 + \pi/4 \). The minimum value \( F(\theta_0) \) takes place at \( \theta_0 = \alpha/2 + \pi/4 \).

(c) Plug in \( \theta_0 = \alpha/2 + \pi/4 \). To find \( v_{\text{opt}}^2 \) we must simplify

\[
\cos^2 \theta_0 (\tan \theta_0 - \tan \alpha) = \frac{\cos \theta_0 (\sin \theta_0 \cos \alpha - \cos \theta_0 \sin \alpha)}{\cos \alpha}
\]

By the addition law for sine:

\[
\sin \theta_0 \cos \alpha - \cos \theta_0 \sin \alpha = \sin(\theta_0 - \alpha) = \sin(-\alpha/2 + \pi/4)
\]

and so

\[
\cos \theta_0 (\sin \theta_0 \cos \alpha - \cos \theta_0 \sin \alpha) = \cos(\alpha/2 + \pi/4) \sin(-\alpha/2 + \pi/4)
\]

Now use the identity (that follows from the addition law):

\[
\sin x \cos y = \frac{1}{2} (\sin(x + y) + \sin(x - y))
\]

to get

\[
\cos(\alpha/2 + \pi/4) \sin(-\alpha/2 + \pi/4) = (1/2)(1 - \sin \alpha)
\]

So we finally get

\[
\cos^2 \theta_0 (\tan \theta_0 - \tan \alpha) = \frac{(1/2)(1 - \sin \alpha)}{\cos \alpha}
\]

and therefore

\[
v_{\text{opt}}^2 = \frac{32d \cos \alpha}{1 - \sin \alpha}
\]

as claimed. From Figure 34 we see that

\[
\cos \alpha = \frac{d}{\sqrt{d^2 + h^2}} \quad \text{and} \quad \sin \alpha = \frac{h}{\sqrt{d^2 + h^2}}
\]

Substituting these values into the expression for \( v_{\text{opt}}^2 \) yields

\[
v_{\text{opt}}^2 = \frac{32d^2}{-h + \sqrt{d^2 + h^2}}.
\]

(f) A sketch of the graph of \( v_{\text{opt}}^2 \) versus \( h \) for \( d = 15 \) feet is given below: \( v_{\text{opt}}^2 \) increases with respect to basket height relative to the shooter. This shows that the minimum velocity required to launch the ball to the basket drops as shooter height increases. This shows one of the ways height is an advantage in free throws; a taller shooter need not shoot the ball as hard to reach the basket.

---

**Further Insights and Challenges**

65. Tom and Ali drive along a highway represented by the graph of \( f(x) \) in Figure 36. During the trip, Ali views a billboard represented by the segment \( \overline{BC} \) along the \( y \)-axis. Let \( Q \) be the \( y \)-intercept of the tangent line to \( y = f(x) \). Show that \( \theta \) is maximized at the value of \( x \) for which the angles \( \angle QPB \) and \( \angle QC P \) are equal. This generalizes Exercise 50 (c) (which corresponds to the case \( f(x) = 0 \)). Hints:

(a) Show that \( d\theta/dx \) is equal to

\[
(b - c) \cdot \frac{(x^2 + (xf'(x))^2) - (b - (f(x) - xf'(x))(c - (f(x) - xf'(x)))}{(x^2 + (b - f(x))^2)(x^2 + (c - f(x))^2)}
\]
(b) Show that the y-coordinate of \( Q \) is \( f(x) - xf'(x) \).

(c) Show that the condition \( \frac{d\theta}{dx} = 0 \) is equivalent to

\[
PQ^2 = BQ \cdot CQ
\]

(d) Conclude that \( \triangle QPB \) and \( \triangle QCP \) are similar triangles.

**SOLUTION**

(a) From the figure, we see that

\[
\theta(x) = \tan^{-1} \frac{c - f(x)}{x} - \tan^{-1} \frac{b - f(x)}{x}.
\]

Then

\[
\theta'(x) = \frac{b - (f(x) - xf'(x))}{x^2 + (b - f(x))^2} - \frac{c - (f(x) - xf'(x))}{x^2 + (c - f(x))^2}
\]

\[
= (b - c) \frac{x^2 - bc + (b + c)(f(x) - xf'(x)) - (f(x))^2 + 2xf(x)f'(x)}{(x^2 + (b - f(x))^2)(x^2 + (c - f(x))^2)}
\]

\[
= (b - c) \frac{(x^2 + (xf'(x))^2) - (bc - (b + c)(f(x) - xf'(x))) + (f(x) - xf'(x))^2}{(x^2 + (b - f(x))^2)(x^2 + (c - f(x))^2)}
\]

\[
= (b - c) \frac{(x^2 + (xf'(x))^2) - (bc - (b + c)(f(x) - xf'(x))) + (f(x) - xf'(x))^2}{(x^2 + (b - f(x))^2)(x^2 + (c - f(x))^2)}.
\]

(b) The point \( Q \) is the y-intercept of the line tangent to the graph of \( f(x) \) at point \( P \). The equation of this tangent line is

\[
Y - f(x) = f'(x)(X - x).
\]

The y-coordinate of \( Q \) is then \( f(x) - xf'(x) \).

(c) From the figure, we see that

\[
BQ = b - (f(x) - xf'(x)),
\]

\[
CQ = c - (f(x) - xf'(x))
\]

and

\[
PQ = \sqrt{x^2 + (f(x) - (f(x) - xf'(x)))^2} = \sqrt{x^2 + (xf'(x))^2}.
\]

Comparing these expressions with the numerator of \( \frac{d\theta}{dx} \), it follows that \( \frac{d\theta}{dx} = 0 \) is equivalent to

\[
PQ^2 = BQ \cdot CQ.
\]

(d) The equation \( PQ^2 = BQ \cdot CQ \) is equivalent to

\[
\frac{PQ}{BQ} = \frac{CQ}{PQ}
\]

In other words, the sides \( CQ \) and \( PQ \) from the triangle \( \triangle QCP \) are proportional in length to the sides \( PQ \) and \( BQ \) from the triangle \( \triangle QPB \). As \( \angle PQB = \angle CQP \), it follows that triangles \( \triangle QCP \) and \( \triangle QPB \) are similar.
Seismic Prospecting  Exercises 66–68 are concerned with determining the thickness $d$ of a layer of soil that lies on top of a rock formation. Geologists send two sound pulses from point $A$ to point $D$ separated by a distance $s$. The first pulse travels directly from $A$ to $D$ along the surface of the earth. The second pulse travels down to the rock formation, then along its surface, and then back up to $D$ (path ABCD), as in Figure 37. The pulse travels with velocity $v_1$ in the soil and $v_2$ in the rock.

![FIGURE 37](image)

67. In this exercise, assume that $v_2/v_1 \geq \sqrt{1 + 4(d/s)^2}$.

(a) Show that inequality (2) holds if $\sin \theta = v_1/v_2$.

(b) Show that the time required for the second pulse is

$$t_2 = \frac{2d}{v_1} \left(1 - k^2\right)^{1/2} + \frac{s}{v_2}$$

where $k = v_1/v_2$.

(c) Conclude that $t_2/t_1 = \frac{2d(1 - k^2)^{1/2}}{s} + k$.

SOLUTION

(a) If $\sin \theta = \frac{v_1}{v_2}$, then

$$\tan \theta = \frac{v_1}{\sqrt{v_2^2 - v_1^2}} = \frac{1}{\sqrt{v_1^2 - 1}}.$$

Because $\frac{v_2}{v_1} \geq \sqrt{1 + 4\left(\frac{d}{s}\right)^2}$, it follows that

$$\sqrt{\left(\frac{v_2}{v_1}\right)^2 - 1} \geq \sqrt{1 + 4\left(\frac{d}{s}\right)^2} - 1 = \frac{2d}{s}.$$

Hence, $\tan \theta \leq \frac{4d}{s}$ as required.

(b) For the time-minimizing choice of $\theta$, we have $\sin \theta = \frac{v_1}{v_2}$ from which $\sec \theta = \frac{v_2}{\sqrt{v_2^2 - v_1^2}}$ and $\tan \theta = \frac{v_1}{\sqrt{v_2^2 - v_1^2}}$.

Thus

$$t_2 = \frac{2d}{v_1} \sec \theta + \frac{s}{v_2} - 2d \frac{\tan \theta}{v_2} = \frac{2d}{v_1} \frac{v_2}{\sqrt{v_2^2 - v_1^2}} + \frac{s - 2d \frac{v_1}{\sqrt{v_2^2 - v_1^2}}}{v_2}$$

$$= \frac{2d}{v_1} \left(\frac{\sqrt{v_2^2 - v_1^2}}{v_2} - \frac{v_1^2}{\sqrt{v_2^2 - v_1^2}}\right) + \frac{s}{v_2}$$

$$= \frac{2d}{v_1} \left(\frac{v_2^2 - v_1^2}{v_2 \sqrt{v_2^2 - v_1^2}}\right) + \frac{s}{v_2} = \frac{2d}{v_1} \left(\frac{\sqrt{v_2^2 - v_1^2}}{\sqrt{v_2^2}}\right) + \frac{s}{v_2}$$

$$= \frac{2d}{v_1} \left(1 - \frac{v_1^2}{v_2^2}\right) + \frac{s}{v_2} = \frac{2d}{v_1} \left(1 - k^2\right)^{1/2} + \frac{s}{v_2}.$$

(e) Recall that $t_1 = \frac{s}{v_1}$. We therefore have

$$\frac{t_2}{t_1} = \frac{2d(1 - k^2)^{1/2}}{s} + \frac{s}{v_1 v_2} = \frac{2d}{s} \left(1 - k^2\right)^{1/2} + \frac{v_1}{v_2} + \frac{2d}{s} \left(1 - k^2\right)^{1/2} + k.$$
In this exercise we use Figure 38 to prove Heron’s principle of Example 6 without calculus. By definition, $C$ is the reflection of $B$ across the line $MN$ (so that $BC$ is perpendicular to $MN$ and $BN = CN$). Let $P$ be the intersection of $AC$ and $MN$. Use geometry to justify:

(a) $\triangle PNB$ and $\triangle PNC$ are congruent and $\theta_1 = \theta_2$.

(b) The paths $APB$ and $APC$ have equal length.

(c) Similarly $AQB$ and $AQC$ have equal length.

(d) The path $APC$ is shorter than $AQC$ for all $Q \neq P$.

Conclude that the shortest path $AQB$ occurs for $Q = P$.

**SOLUTION**

(a) By definition, $BC$ is orthogonal to $QM$, so triangles $\triangle PNB$ and $\triangle PNC$ are congruent by side–angle–side. Therefore $\theta_1 = \theta_2$.

(b) Because $\triangle PNB$ and $\triangle PNC$ are congruent, it follows that $\triangle ABP$ and $\triangle ACP$ are of equal length. Thus, paths $APB$ and $APC$ have equal length.

(c) The same reasoning used in parts (a) and (b) lead us to conclude that $\triangle QNB$ and $\triangle QNC$ are congruent and that $\triangle ABP$ and $\triangle ACP$ are of equal length. Thus, paths $AQB$ and $AQC$ are of equal length.

(d) Consider triangle $\triangle AQC$. By the triangle inequality, the length of side $\overline{AC}$ is less than or equal to the sum of the lengths of the sides $\overline{AQ}$ and $\overline{QC}$. Thus, the path $APC$ is shorter than $AQC$ for all $Q \neq P$.

Finally, the shortest path $AQB$ occurs for $Q = P$.

---

**4.8 Newton’s Method**

**Preliminary Questions**

1. How many iterations of Newton’s Method are required to compute a root if $f(x)$ is a linear function?

**SOLUTION** Remember that Newton’s Method uses the linear approximation of a function to estimate the location of a root. If the original function is linear, then only one iteration of Newton’s Method will be required to compute the root.

2. What happens in Newton’s Method if your initial guess happens to be a zero of $f$?

**SOLUTION** If $x_0$ happens to be a zero of $f$, then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - 0 = x_0;$$

in other words, every term in the Newton’s Method sequence will remain $x_0$.

3. What happens in Newton’s Method if your initial guess happens to be a local min or max of $f$?

**SOLUTION** Assuming that the function is differentiable, then the derivative is zero at a local maximum or a local minimum. If Newton’s Method is started with an initial guess such that $f'(x_0) = 0$, then Newton’s Method will fail in the sense that $x_1$ will not be defined. That is, the tangent line will be parallel to the $x$-axis and will never intersect it.

4. Is the following a reasonable description of Newton’s Method: "A root of the equation of the tangent line to $f(x)$ is used as an approximation to a root of $f(x)$ itself"? Explain.

**SOLUTION** Yes, that is a reasonable description. The iteration formula for Newton’s Method was derived by solving the equation of the tangent line to $y = f(x)$ at $x_0$ for its $x$-intercept.


### Exercises

In this exercise set, all approximations should be carried out using Newton’s Method.

In Exercises 1–6, apply Newton’s Method to $f(x)$ and initial guess $x_0$ to calculate $x_1, x_2, x_3$.

1. $f(x) = x^2 - 6, \quad x_0 = 2$

**Solution** Let $f(x) = x^2 - 6$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 6}{2x_n}.$$ 

With $x_0 = 2$, we compute

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_n$</td>
<td>2.5</td>
<td>2.45</td>
<td>2.4494980</td>
</tr>
</tbody>
</table>

3. $f(x) = x^3 - 10, \quad x_0 = 2$

**Solution** Let $f(x) = x^3 - 10$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 10}{3x_n^2}.$$ 

With $x_0 = 2$ we compute

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_n$</td>
<td>2.16666667</td>
<td>2.15450362</td>
<td>2.15443469</td>
</tr>
</tbody>
</table>

5. $f(x) = \cos x - 4x, \quad x_0 = 1$

**Solution** Let $f(x) = \cos x - 4x$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\cos x_n - 4x_n}{-\sin x_n - 4}.$$ 

With $x_0 = 1$ we compute

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_n$</td>
<td>0.28540361</td>
<td>0.24288009</td>
<td>0.24267469</td>
</tr>
</tbody>
</table>

7. Use Figure 6 to choose an initial guess $x_0$ to the unique real root of $x^3 + 2x + 5 = 0$ and compute the first three Newton iterates.

![Figure 6 Graph of $y = x^3 + 2x + 5$.](image)

**Solution** Let $f(x) = x^3 + 2x + 5$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + 2x_n + 5}{3x_n^2 + 2}.$$ 

We take $x_0 = -1.4$, based on the figure, and then calculate

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_n$</td>
<td>-1.330964467</td>
<td>-1.328272820</td>
<td>-1.328268856</td>
</tr>
</tbody>
</table>
SECTION 4.8 Newton’s Method

9. Approximate both solutions of \( e^x = 5x \) to three decimal places (Figure 7).

SOLUTION We need to solve \( e^x - 5x = 0 \), so let \( f(x) = e^x - 5x \). Then \( f'(x) = e^x - 5 \). With an initial guess of \( x_0 = 0.2 \), we calculate

\[
\begin{array}{c|c}
\text{Newton’s Method (First root)} & x_0 = 0.2 \text{ (guess)} \\
\hline
x_1 = 0.2 - \frac{f(0.2)}{f'(0.2)} & x_1 \approx 0.25859 \\
x_2 = 0.25859 - \frac{f(0.25859)}{f'(0.25859)} & x_2 \approx 0.25917 \\
x_3 = 0.25917 - \frac{f(0.25917)}{f'(0.25917)} & x_3 \approx 0.25917 \\
\end{array}
\]

For the second root, we use an initial guess of \( x_0 = 2.5 \).

\[
\begin{array}{c|c}
\text{Newton’s Method (Second root)} & x_0 = 2.5 \text{ (guess)} \\
\hline
x_1 = 2.5 - \frac{f(2.5)}{f'(2.5)} & x_1 \approx 2.54421 \\
x_2 = 2.54421 - \frac{f(2.54421)}{f'(2.54421)} & x_2 \approx 2.54264 \\
x_3 = 2.54264 - \frac{f(2.54264)}{f'(2.54264)} & x_3 \approx 2.54264 \\
\end{array}
\]

Thus the two solutions of \( e^x = 5x \) are approximately \( r_1 \approx 0.25917 \) and \( r_2 \approx 2.54264 \).

11. \( \sqrt{11} \)

SOLUTION Let \( f(x) = x^2 - 11 \), and let \( x_0 = 3 \). Newton’s Method yields:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_n )</td>
<td>3.33333333</td>
<td>3.31666667</td>
<td>3.31662479</td>
</tr>
</tbody>
</table>

A calculator yields 3.31662479.

13. \( 27/3 \)

SOLUTION Note that \( 27/3 = 4 \cdot 21/3 \). Let \( f(x) = x^3 - 2 \), and let \( x_0 = 1 \). Newton’s Method yields:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_n )</td>
<td>1.33333333</td>
<td>1.26388889</td>
<td>1.25993349</td>
</tr>
</tbody>
</table>

Thus, \( 27/3 \approx 4 \cdot 1.25993349 = 5.03973397 \). A calculator yields 5.0396842.

15. Approximate the largest positive root of \( f(x) = x^4 - 6x^2 + x + 5 \) to within an error of at most \( 10^{-4} \). Refer to Figure 5.

SOLUTION Figure 5 from the text suggests the largest positive root of \( f(x) = x^4 - 6x^2 + x + 5 \) is near 2. So let \( f(x) = x^4 - 6x^2 + x + 5 \) and take \( x_0 = 2 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_n )</td>
<td>2.11111111</td>
<td>2.093568458</td>
<td>2.093064768</td>
<td>2.093064358</td>
</tr>
</tbody>
</table>

The largest positive root of \( x^4 - 6x^2 + x + 5 \) is approximately 2.093064358.
In Exercises 16–19, approximate the root specified to three decimal places using Newton’s Method. Use a plot to choose an initial guess.

17. Negative root of \( f(x) = x^5 - 20x + 10 \).

**SOLUTION** Let \( f(x) = x^5 - 20x + 10 \). The graph of \( f(x) \) shown below suggests taking \( x_0 = -2.2 \). Starting from \( x_0 = -2.2 \), the first three iterates of Newton’s Method are:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_n )</td>
<td>-2.22536529</td>
<td>-2.22468998</td>
<td>-2.22468949</td>
</tr>
</tbody>
</table>

Thus, to three decimal places, the negative root of \( f(x) = x^5 - 20x + 10 \) is \(-2.225\).

19. Solution of \( \ln(x + 4) = x \).

**SOLUTION** From the graph below, we see that the positive solution to the equation \( \ln(x + 4) = x \) is approximately \( x = 2 \). Now, let \( f(x) = \ln(x + 4) - x \) and define

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\ln(x_n + 4) - x_n}{\frac{1}{x_n+4} - 1}.
\]

With \( x_0 = 2 \) we find

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_n )</td>
<td>1.750111363</td>
<td>1.749031407</td>
<td>1.749031386</td>
</tr>
</tbody>
</table>

Thus, to three decimal places, the positive solution to the equation \( \ln(x + 4) = x \) is 1.749.

21. **GU** Find the smallest positive value of \( x \) at which \( y = x \) and \( y = \tan x \) intersect. **Hint:** Draw a plot.

**SOLUTION** Here is a plot of \( \tan x \) and \( x \) on the same axes:

The first intersection with \( x > 0 \) lies on the second “branch” of \( y = \tan x \), between \( x = \frac{5\pi}{4} \) and \( x = \frac{7\pi}{4} \). Let \( f(x) = \tan x - x \). The graph suggests an initial guess \( x_0 = \frac{5\pi}{4} \), from which we get the following table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_n )</td>
<td>6.85398</td>
<td>21.921</td>
<td>4480.8</td>
<td>7456.27</td>
</tr>
</tbody>
</table>

This is clearly leading nowhere, so we need to try a better initial guess. **Note:** This happens with Newton’s Method—it is sometimes difficult to choose an initial guess. We try the point directly between \( \frac{5\pi}{4} \) and \( \frac{7\pi}{4} \), \( x_0 = \frac{11\pi}{8} \).
Newton’s Method

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_n$</td>
<td>4.64662</td>
<td>4.60091</td>
<td>4.54662</td>
<td>4.50658</td>
<td>4.49422</td>
<td>4.49341</td>
<td>4.49341</td>
</tr>
</tbody>
</table>

The first point where $y = x$ and $y = \tan x$ cross is at approximately $x = 4.49341$, which is approximately $1.4303\pi$.

23. Find (to two decimal places) the coordinates of the point $P$ in Figure 9 where the tangent line to $y = \cos x$ passes through the origin.

**SOLUTION**  Let $(x_r, \cos(x_r))$ be the coordinates of the point $P$. The slope of the tangent line is $-\sin(x_r)$, so we are looking for a tangent line:

$$y = \sin(x_r)(x - x_r) + \cos(x_r)$$

such that $y = 0$ when $x = 0$. This gives us the equation:

$$-\sin(x_r)(-x_r) + \cos(x_r) = 0.$$

Let $f(x) = \cos x + x \sin x$. We are looking for the first point $x = r$ where $f(r) = 0$. The sketch given indicates that $x_0 = 3\pi/4$ would be a good initial guess. The following table gives successive Newton’s Method approximations:

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_n$</td>
<td>2.931781309</td>
<td>2.803636974</td>
<td>2.798395826</td>
<td>2.798386046</td>
</tr>
</tbody>
</table>

The point $P$ has approximate coordinates $(2.7984, -0.941684)$.

Newton’s Method is often used to determine interest rates in financial calculations. In Exercises 24–26, $r$ denotes a yearly interest rate expressed as a decimal (rather than as a percent).

25. If you borrow $L$ dollars for $N$ years at a yearly interest rate $r$, your monthly payment of $P$ dollars is calculated using the equation

$$L = P \left( \frac{1 - b^{-12N}}{b - 1} \right)$$

where $b = 1 + \frac{r}{12}$

(a) Find $P$ if $L = 55000$, $N = 3$, and $r = 0.08$ (8%).

(b) You are offered a loan of $L = 55000$ to be paid back over 3 years with monthly payments of $P = 200$. Use Newton’s Method to compute $b$ and find the implied interest rate $r$ of this loan. Hint: Show that $(L/P)b^{12N+1} - (1 + L/P)b^{12N} + 1 = 0$.

**SOLUTION**

(a) $b = (1 + 0.08/12) = 1.00667$

$$P = L \left( \frac{b - 1}{1 - b^{-12N}} \right) = 55000 \left( \frac{1.00667 - 1}{1 - 1.00667^{-36}} \right) \approx \$156.69$$

(b) Starting from

$$L = P \left( \frac{1 - b^{-12N}}{b - 1} \right),$$

divide by $P$, multiply by $b - 1$, multiply by $b^{12N}$ and collect like terms to arrive at

$$(L/P)b^{12N+1} - (1 + L/P)b^{12N} + 1 = 0.$$

Since $L/P = 5000/200 = 25$, we must solve

$$25b^{37} - 26b^{36} + 1 = 0.$$

Newton’s Method gives $b \approx 1.02121$ and

$$r = 12(b - 1) = 12(0.02121) \approx 0.25452$$

So the interest rate is around 25.45%.
27. There is no simple formula for the position at time $t$ of a planet $P$ in its orbit (an ellipse) around the sun. Introduce the auxiliary circle and angle $\theta$ in Figure 10 (note that $P$ determines $\theta$ because it is the central angle of point $B$ on the circle). Let $a = OA$ and $e = OS/OA$ (the eccentricity of the orbit).

(a) Show that sector $BSA$ has area $(a^2/2)(\theta - e \sin \theta)$.

(b) By Kepler's Second Law, the area of sector $BSA$ is proportional to the time $t$ elapsed since the planet passed point $A$, and because the circle has area $\pi a^2$, $BSA$ has area $(\pi a^2)(t/T)$, where $T$ is the period of the orbit. Deduce Kepler's Equation:

$$\frac{2\pi t}{T} = \theta - e \sin \theta$$

(c) The eccentricity of Mercury's orbit is approximately $e = 0.2$. Use Newton's Method to find $\theta$ after a quarter of Mercury's year has elapsed ($t = T/4$). Convert $\theta$ to degrees. Has Mercury covered more than a quarter of its orbit at $t = T/4$?

![FIGURE 10]

**SOLUTION**

(a) The sector $SAB$ is the slice $OAB$ with the triangle $OPS$ removed. $OAB$ is a central sector with arc $\theta$ and radius $OA = a$, and therefore has area $a^2/2$; $OPS$ is a triangle with height $a \sin \theta$ and base length $OS = ea$. Hence, the area of the sector is

$$\frac{a^2}{2} \theta - \frac{1}{2} ea^2 \sin \theta = \frac{a^2}{2} (\theta - e \sin \theta).$$

(b) Since Kepler's second law indicates that the area of the sector is proportional to the time $t$ since the planet passed point $A$, we get

$$\pi a^2 \left( \frac{t}{T} \right) = \frac{a^2}{2} (\theta - e \sin \theta)$$

$$\frac{2\pi}{T} \frac{t}{T} = \theta - e \sin \theta.$$ 

(c) If $t = T/4$, the last equation in (b) gives:

$$\frac{\pi}{2} = \theta - e \sin \theta = \theta - 0.2 \sin \theta.$$ 

Let $f(\theta) = \theta - 0.2 \sin \theta - \frac{\pi}{2}$. We will use Newton's Method to find the point where $f(\theta) = 0$. Since a quarter of the year on Mercury has passed, a good first estimate $\theta_0$ would be $\frac{\pi}{2}$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_n$</td>
<td>1.7708</td>
<td>1.76696</td>
<td>1.76696</td>
<td>1.76696</td>
</tr>
</tbody>
</table>

From the point of view of the Sun, Mercury has traversed an angle of approximately 1.76696 radians = 101.24°. Mercury has therefore traveled more than one fourth of the way around (from the point of view of central angle) during this time.

29. What happens when you apply Newton's Method to find a zero of $f(x) = x^{1/3}$? Note that $x = 0$ is the only zero.

**SOLUTION** Let $f(x) = x^{1/3}$. Define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - \frac{3}{2}x_n = -\frac{1}{2}x_n.$$ 

Take $x_0 = 0.5$. Then the sequence of iterates is $-1, 2, -4, 8, -16, 32, -64, \ldots$. That is, for any nonzero starting value, the sequence of iterates diverges spectacularly, since $x_n = (-2)^n x_0$. Thus $\lim_{n \to \infty} |x_n| = \lim_{n \to \infty} 2^n |x_0| = \infty.$
Further Insights and Challenges

31. Newton’s Method can be used to compute reciprocals without performing division. Let \( c > 0 \) and set \( f(x) = x^{-1} - c \).

(a) Show that \( x - \frac{f(x)}{f'(x)} = 2x - cx^2 \).

(b) Calculate the first three iterates of Newton’s Method with \( c = 10.3 \) and the two initial guesses \( x_0 = 0.1 \) and \( x_0 = 0.5 \).

(c) Explain graphically why \( x_0 = 0.5 \) does not yield a sequence converging to \( 1/10.3 \).

Solution

(a) Let \( f(x) = \frac{1}{x} - c \). Then

\[
x - \frac{f(x)}{f'(x)} = x - \frac{\frac{1}{x} - c}{-\frac{1}{x^2}} = 2x - cx^2.
\]

(b) For \( c = 10.3 \), we have \( f(x) = \frac{1}{x} - 10.3 \) and thus \( x_{n+1} = 2x_n - 10.3x_n^2 \).

1. Take \( x_0 = 0.1 \).
   
   \[
   \begin{array}{|c|c|c|c|}
   \hline
   n & 1 & 2 & 3 \\
   x_n & 0.097 & 0.0970873 & 0.09708738 \\
   \hline
   \end{array}
   \]

2. Take \( x_0 = 0.5 \).

   \[
   \begin{array}{|c|c|c|c|}
   \hline
   n & 1 & 2 & 3 \\
   x_n & -1.575 & -28.7004375 & -8541.66654 \\
   \hline
   \end{array}
   \]

(c) The graph is disconnected. If \( x_0 = .5 \), \((x_1, f(x_1))\) is on the other portion of the graph, which will never converge to any point under Newton’s Method.

In Exercises 32 and 33, consider a metal rod of length \( L \) fastened at both ends. If you cut the rod and weld on an additional segment of length \( m \), leaving the ends fixed, the rod will bow up into a circular arc of radius \( R \) (unknown), as indicated in Figure 12.

![Figure 12](image)

33. Let \( L = 3 \) and \( m = 1 \). Apply Newton’s Method to Eq. (2) to estimate \( \theta \), and use this to estimate \( h \).

Solution

We let \( L = 3 \) and \( m = 1 \). We want the solution of:

\[
\frac{\sin \theta}{\theta} = \frac{L}{L + m}
\]

\[
\frac{\sin \theta}{\theta} - \frac{L}{L + m} = 0
\]

\[
\frac{\sin \theta}{\theta} - \frac{3}{4} = 0
\]

Let \( f(\theta) = \frac{\sin \theta}{\theta} - \frac{3}{4} \).

The figure above suggests that \( \theta_0 = 1.5 \) would be a good initial guess. The Newton’s Method approximations for the solution follow:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta )</td>
<td>1.2854388</td>
<td>1.2757223</td>
<td>1.2756981</td>
<td>1.2756981</td>
</tr>
</tbody>
</table>
The angle where $\tan \frac{\theta}{L} = \frac{L}{L+M}$ is approximately 1.2757. Hence

$$h = L \frac{1 - \cos \theta}{2 \sin \theta} \approx 1.11181.$$ 

In Exercises 35–37, a flexible chain of length $L$ is suspended between two poles of equal height separated by a distance $2M$ (Figure 13). By Newton’s laws, the chain describes a catenary $y = a \cosh \left( \frac{x}{a} \right)$, where $a$ is the number such that $L = 2a \sinh \left( \frac{M}{a} \right)$. The sag $s$ is the vertical distance from the highest to the lowest point on the chain.

Figure 13 Chain hanging between two poles.

35. Suppose that $L = 120$ and $M = 50$.
   (a) Use Newton’s Method to find a value of $a$ (to two decimal places) satisfying $L = 2a \sinh(M/a)$.
   (b) Compute the sag $s$.

**SOLUTION**
   (a) Let

   $f(a) = 2a \sinh \left( \frac{50}{a} \right) - 120.$

   The graph of $f$ shown below suggests $a \approx 47$ is a root of $f$. Starting with $a_0 = 47$, we find the following approximations using Newton’s method:

   $a_1 = 46.95408$ and $a_2 = 46.95415$

   Thus, to two decimal places, $a = 46.95$.

   (b) The sag is given by

   $$s = y(M) - y(0) = \left( a \cosh \frac{M}{a} + C \right) - \left( a \cosh \frac{0}{a} + C \right) = a \cosh \frac{M}{a} - a.$$

   Using $M = 50$ and $a = 46.95$, we find $s = 29.24$.

37. Suppose that $L = 160$ and $M = 50$.
   (a) Use Newton’s Method to find a value of $a$ (to two decimal places) satisfying $L = 2a \sinh(M/a)$.
   (b) Use Eq. (3) and the Linear Approximation to estimate the increase in sag $\Delta s$ for changes in length $\Delta L = 1$ and $\Delta L = 5$.
   (c) Compute $s(161) - s(160)$ and $s(165) - s(160)$ directly and compare with your estimates in (b).

**SOLUTION**
   (a) Let $f(x) = 2x \sinh(50/x) - 160$. Using the graph below, we select an initial guess of $x_0 = 30$. Newton’s Method then yields:

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_n$</td>
<td>28.30622107</td>
<td>28.45653356</td>
<td>28.45797517</td>
</tr>
</tbody>
</table>

   Thus, to two decimal places, $a \approx 28.46$.
(b) With \( M = 50 \) and \( a \approx 28.46 \), we find using Eq. (3) that
\[
\frac{ds}{dL} = 0.61.
\]
By the Linear Approximation,
\[
\Delta s \approx \frac{ds}{dL} \cdot \Delta L.
\]
If \( L \) increases from 160 to 161, then \( \Delta L = 1 \) and \( \Delta s \approx 0.61 \); if \( L \) increases from 160 to 165, then \( \Delta L = 5 \) and \( \Delta s \approx 3.05 \).

(c) When \( L = 160, a \approx 28.46 \) and
\[
s(160) = 28.46 \cosh \left( \frac{50}{28.46} \right) - 28.46 \approx 56.45;
\]
whereas, when \( L = 161, a \approx 28.25 \) and
\[
s(161) = 28.25 \cosh \left( \frac{50}{28.25} \right) - 28.25 \approx 57.07.
\]
Therefore, \( s(161) - s(160) = 0.62 \), very close to the approximation obtained from the Linear Approximation. Moreover, when \( L = 165, a \approx 27.49 \) and
\[
s(165) = 27.49 \cosh \left( \frac{50}{27.49} \right) - 27.49 \approx 59.47;
\]
thus, \( s(165) - s(160) = 3.02 \), again very close to the approximation obtained from the Linear Approximation.

4.9 Antiderivatives

**Preliminary Questions**

1. Find an antiderivative of the function \( f(x) = 0 \).

**SOLUTION** Since the derivative of any constant is zero, any constant function is an antiderivative for the function \( f(x) = 0 \).

2. Is there a difference between finding the general antiderivative of a function \( f(x) \) and evaluating \( \int f(x) \, dx \)?

**SOLUTION** No difference. The indefinite integral is the symbol for denoting the general antiderivative.

3. Jacques was told that \( f(x) \) and \( g(x) \) have the same derivative, and he wonders whether \( f(x) = g(x) \). Does Jacques have sufficient information to answer his question?

**SOLUTION** No. Knowing that the two functions have the same derivative is only good enough to tell Jacques that the functions may differ by at most an additive constant. To determine whether the functions are equal for all \( x \), Jacques needs to know the value of each function for a single value of \( x \). If the two functions produce the same output value for a single input value, they must take the same value for all input values.

4. Suppose that \( F'(x) = f(x) \) and \( G'(x) = g(x) \). Which of the following statements are true? Explain.
   (a) If \( f = g \), then \( F = G \).
   (b) If \( F \) and \( G \) differ by a constant, then \( f = g \).
   (c) If \( f \) and \( g \) differ by a constant, then \( F = G \).

**SOLUTION**
   (a) False. Even if \( f(x) = g(x) \), the antiderivatives \( F \) and \( G \) may differ by an additive constant.
   (b) True. This follows from the fact that the derivative of any constant is 0.
   (c) False. If the functions \( f \) and \( g \) are different, then the antiderivatives \( F \) and \( G \) differ by a linear function: \( F(x) - G(x) = ax + b \) for some constants \( a \) and \( b \).
5. Is \( y = x \) a solution of the following Initial Value Problem?

\[
\frac{dy}{dx} = 1, \quad y(0) = 1
\]

**SOLUTION** Although \( \frac{d}{dx} x = 1 \), the function \( f(x) = x \) takes the value 0 when \( x = 0 \), so \( y = x \) is not a solution of the indicated initial value problem.

**Exercises**

In Exercises 1–8, find the general antiderivative of \( f(x) \) and check your answer by differentiating.

1. \( f(x) = 18x^2 \)

**SOLUTION**

\[
\int 18x^2 \, dx = 18 \int x^2 \, dx = 18 \cdot \frac{1}{3} x^3 + C = 6x^3 + C.
\]

As a check, we have

\[
\frac{d}{dx} (6x^3 + C) = 18x^2
\]

as needed.

3. \( f(x) = 2x^4 - 24x^2 + 12x^{-1} \)

**SOLUTION**

\[
\int (2x^4 - 24x^2 + 12x^{-1}) \, dx = 2 \int x^4 \, dx - 24 \int x^2 \, dx + 12 \int x^{-1} \, dx
\]

\[
= 2 \cdot \frac{1}{5} x^5 - 24 \cdot \frac{1}{3} x^3 + 12 \ln |x| + C
\]

\[
= \frac{2}{5} x^5 - 8x^3 + 12 \ln |x| + C.
\]

As a check, we have

\[
\frac{d}{dx} \left( \frac{2}{5} x^5 - 8x^3 + 12 \ln |x| + C \right) = 2x^4 - 24x^2 + 12x^{-1}
\]

as needed.

5. \( f(x) = 2 \cos x - 9 \sin x \)

**SOLUTION**

\[
\int (2 \cos x - 9 \sin x) \, dx = 2 \int \cos x \, dx - 9 \int \sin x \, dx
\]

\[
= 2 \sin x - 9 (-\cos x) + C = 2 \sin x + 9 \cos x + C
\]

As a check, we have

\[
\frac{d}{dx} (2 \sin x + 9 \cos x + C) = 2 \cos x + 9 (-\sin x) = 2 \cos x - 9 \sin x
\]

as needed.

7. \( f(x) = 12e^x - 5x^{-2} \)

**SOLUTION**

\[
\int (12e^x - 5x^{-2}) \, dx = 12 \int e^x \, dx - 5 \int x^{-2} \, dx = 12e^x - 5(-x^{-1}) + C = 12e^x + 5x^{-1} + C.
\]

As a check, we have

\[
\frac{d}{dx} (12e^x + 5x^{-1} + C) = 12e^x + 5(-x^{-2}) = 12e^x - 5x^{-2}
\]

as needed.
9. Match functions (a)–(d) with their antiderivatives (i)–(iv).

(a) \(f(x) = \sin x\)  \quad (i) \(F(x) = \cos(1 - x)\)
(b) \(f(x) = x \sin(x^2)\)  \quad (ii) \(F(x) = -\cos x\)
(c) \(f(x) = \sin(1 - x)\)  \quad (iii) \(F(x) = -\frac{1}{2} \cos(x^2)\)
(d) \(f(x) = x \sin x\)  \quad (iv) \(F(x) = \sin x - x \cos x\)

**SOLUTION**

(a) An antiderivative of \(\sin x\) is \(-\cos x\), which is (ii). As a check, we have \(\frac{d}{dx}(-\cos x) = -(-\sin x) = \sin x\).

(b) An antiderivative of \(x \sin(x^2)\) is \(-\frac{1}{2} \cos(x^2)\), which is (iii). This is because, by the Chain Rule, we have \(\frac{d}{dx}(-\frac{1}{2} \cos(x^2)) = -\frac{1}{2} (-\sin(x^2)) \cdot 2x = x \sin(x^2)\).

(c) An antiderivative of \(\sin(1 - x)\) is \(\cos(1 - x)\) or (i). As a check, we have \(\frac{d}{dx}\cos(1 - x) = -\sin(1 - x) \cdot (-1) = \sin(1 - x)\).

(d) An antiderivative of \(x \sin x\) is \(x - \cos x\), which is (iv). This is because

\[
\frac{d}{dx}(\sin x - x \cos x) = \cos x - (x (-\sin x) + \cos x \cdot 1) = x \sin x
\]

In Exercises 10–39, evaluate the indefinite integral.

11. \(\int (4 - 18x) \, dx\)

**SOLUTION** \(\int (4 - 18x) \, dx = 4x - 9x^2 + C\).

13. \(\int t^{-6/11} \, dt\)

**SOLUTION** \(\int t^{-6/11} \, dt = \frac{t^{5/11}}{5/11} + C = \frac{11}{5} t^{5/11} + C\).

15. \(\int (18t^5 - 10t^4 - 28t) \, dt\)

**SOLUTION** \(\int (18t^5 - 10t^4 - 28t) \, dt = 3t^6 - 2t^5 - 14t^2 + C\).

17. \(\int (z^{-4/5} - z^{2/3} + z^{5/4}) \, dz\)

**SOLUTION** \(\int (z^{-4/5} - z^{2/3} + z^{5/4}) \, dz = \frac{z^{1/5}}{1/5} - \frac{z^{5/3}}{5/3} + \frac{z^{9/4}}{9/4} + C = 5z^{1/5} - \frac{3}{5} z^{5/3} + \frac{4}{9} z^{9/4} + C\).

19. \(\int \frac{1}{\sqrt{x}} \, dx\)

**SOLUTION** \(\int \frac{1}{\sqrt{x}} \, dx = \int x^{-1/2} \, dx = \frac{x^{2/3}}{2/3} + C = \frac{3}{2} x^{2/3} + C\).

21. \(\int \frac{36\, dt}{t^3}\)

**SOLUTION** \(\int \frac{36\, dt}{t^3} = \int 36t^{-3} \, dt = 36\frac{t^{-2}}{-2} + C = -18t^{-2} + C\).

23. \(\int (t^{1/2} + 1)(t + 1) \, dt\)

**SOLUTION**

\[
\int (t^{1/2} + 1)(t + 1) \, dt = \int (t^{3/2} + t + t^{1/2} + 1) \, dt = \frac{t^{5/2}}{5/2} + \frac{1}{2} t^2 + \frac{3}{2} t^{3/2} + t + C = \frac{2}{5} t^{5/2} + \frac{1}{2} t^2 + \frac{2}{3} t^{3/2} + t + C
\]
25. \( \int \frac{x^3 + 3x - 4}{x^2} \, dx \)

**SOLUTION**

\[ \int \frac{x^3 + 3x - 4}{x^2} \, dx = \int (x + 3x^{-1} - 4x^{-2}) \, dx \]
\[ = \frac{1}{2}x^2 + 3 \ln |x| + 4x^{-1} + C \]

27. \( \int 12 \sec x \tan x \, dx \)

**SOLUTION**

\[ \int 12 \sec x \tan x \, dx = 12 \sec x + C. \]

29. \( \int (\csc t \cot t) \, dt \)

**SOLUTION**

\[ \int (\csc t \cot t) \, dt = - \csc t + C. \]

31. \( \int \sec^2(7 - 3\theta) \, d\theta \)

**SOLUTION**

\[ \int \sec^2(7 - 3\theta) \, d\theta = -\frac{1}{3} \tan(7 - 3\theta) + C. \]

33. \( \int 25 \sec^2(3z + 1) \, dz \)

**SOLUTION**

\[ \int 25 \sec^2(3z + 1) \, dz = \frac{25}{3} \tan(3z + 1) + C. \]

35. \( \int \left( \cos(3\theta) - \frac{1}{2} \sec^2 \left( \frac{\theta}{4} \right) \right) \, d\theta \)

**SOLUTION**

\[ \int \left( \cos(3\theta) - \frac{1}{2} \sec^2 \left( \frac{\theta}{4} \right) \right) \, d\theta = \frac{1}{3} \sin(3\theta) - 2 \tan \left( \frac{\theta}{4} \right) + C. \]

37. \( \int (3e^{5x}) \, dx \)

**SOLUTION**

\[ \int (3e^{5x}) \, dx = \frac{3}{5}e^{5x} + C. \]

39. \( \int (8x - 4e^{5-2x}) \, dx \)

**SOLUTION**

\[ \int (8x - 4e^{5-2x}) \, dx = 4x^2 + 2e^{5-2x} + C. \]

41. In Figure 4, which of graphs (A), (B), and (C) is *not* the graph of an antiderivative of \( f(x) \)? Explain.

**SOLUTION**

Let \( F(x) \) be an antiderivative of \( f(x) \). Notice that \( f(x) = F'(x) \) changes sign from \(-\) to \(+\) to \(-\) to \(+\). Hence, \( F(x) \) must transition from decreasing to increasing to decreasing to increasing.

- Both graph (A) and graph (C) meet the criteria discussed above and only differ by an additive constant. Thus either could be an antiderivative of \( f(x) \).
- Graph (B) does not have the same local extrema as indicated by \( f(x) \) and therefore is *not* an antiderivative of \( f(x) \).
In Exercises 43–46, verify by differentiation.

43. $\int (x + 13)^6 \, dx = \frac{1}{7} (x + 13)^7 + C$

**SOLUTION** $\frac{d}{dx} \left( \frac{1}{7} (x + 13)^7 + C \right) = (x + 13)^6$ as required.

45. $\int (4x + 13)^2 \, dx = \frac{1}{12} (4x + 13)^3 + C$

**SOLUTION** $\frac{d}{dx} \left( \frac{1}{12} (4x + 13)^3 + C \right) = \frac{1}{4} (4x + 13)^2 (4) = (4x + 13)^2$ as required.

In Exercises 47–62, solve the initial value problem.

47. $\frac{dy}{dx} = x^3$, $y(0) = 4$

**SOLUTION** Since $\frac{dy}{dx} = x^3$, we have

$$y = \int x^3 \, dx = \frac{1}{4} x^4 + C.$$

Thus,

$$4 = y(0) = \frac{1}{4} (0)^4 + C = C,$$

so that $C = 4$. Therefore, $y = \frac{1}{4} x^4 + 4$.

49. $\frac{dy}{dt} = 2t + 9t^2$, $y(1) = 2$

**SOLUTION** Since $\frac{dy}{dt} = 2t + 9t^2$, we have

$$y = \int (2t + 9t^2) \, dt = t^2 + 3t^3 + C.$$

Thus,

$$2 = y(1) = 1^2 + 3(1)^3 + C,$$

so that $C = -2$. Therefore $y = t^2 + 3t^3 - 2$.

51. $\frac{dy}{dt} = \sqrt{t}$, $y(1) = 1$

**SOLUTION** Since $\frac{dy}{dt} = \sqrt{t} = t^{1/2}$, we have

$$y = \int t^{1/2} \, dt = \frac{2}{3} t^{3/2} + C.$$

Thus

$$1 = y(1) = \frac{2}{3} + C,$$

so that $C = \frac{1}{3}$. Therefore, $y = \frac{2}{3} t^{3/2} + \frac{1}{3}$.

53. $\frac{dy}{dx} = (3x + 2)^3$, $y(0) = 1$

**SOLUTION** Since $\frac{dy}{dx} = (3x + 2)^3$, we have

$$y = \int (3x + 2)^3 \, dx = \frac{1}{4} \cdot \frac{1}{3} (3x + 2)^4 + C = \frac{1}{12} (3x + 2)^4 + C.$$

Thus,

$$1 = y(0) = \frac{1}{12} (2)^4 + C,$$

so that $C = 1 - \frac{4}{3} = -\frac{1}{3}$. Therefore, $y = \frac{1}{12} (3x + 2)^4 - \frac{1}{3}$. 

May 23, 2011
Applications of the Derivative

55. \( \frac{dy}{dx} = \sin x \), \( y \left( \frac{\pi}{2} \right) = 1 \)

**Solution** Since \( \frac{dy}{dx} = \sin x \), we have

\[
y = \int \sin x \, dx = -\cos x + C.
\]

Thus, \[ 1 = y \left( \frac{\pi}{2} \right) = 0 + C, \]
so that \( C = 1 \). Therefore, \( y = 1 - \cos x \).

57. \( \frac{dy}{dx} = \cos 5x \), \( y(\pi) = 3 \)

**Solution** Since \( \frac{dy}{dx} = \cos 5x \), we have

\[
y = \int \cos 5x \, dx = \frac{1}{5} \sin 5x + C.
\]

Thus \( 3 = y(\pi) = 0 + C \), so that \( C = 3 \). Therefore, \( y = 3 + \frac{1}{5} \sin 5x \).

59. \( \frac{dy}{dx} = e^x \), \( y(2) = 0 \)

**Solution** Since \( \frac{dy}{dx} = e^x \), we have

\[
y = \int e^x \, dx = e^x + C.
\]

Thus, \[ 0 = y(2) = e^2 + C, \]
so that \( C = -e^2 \). Therefore, \( y = e^x - e^2 \).

61. \( \frac{dy}{dt} = 9e^{12-3t} \), \( y(4) = 7 \)

**Solution** Since \( \frac{dy}{dt} = 9e^{12-3t} \), we have

\[
y = \int 9e^{12-3t} \, dt = -3e^{12-3t} + C.
\]

Thus, \[ 7 = y(4) = -3e^{0} + C, \]
so that \( C = 10 \). Therefore, \( y = -3e^{12-3t} + 10 \).

In Exercises 63–69, first find \( f' \) and then find \( f \).

63. \( f''(x) = 12x \), \( f'(0) = 1 \), \( f(0) = 2 \)

**Solution** Let \( f''(x) = 12x \). Then \( f'(x) = 6x^2 + C \). Given \( f'(0) = 1 \), it follows that \( 1 = 6(0)^2 + C \) and \( C = 1 \). Thus, \( f'(x) = 6x^2 + 1 \). Next, \( f(x) = 2x^3 + x + C \). Given \( f(0) = 2 \), it follows that \( 2 = 2(0)^3 + 0 + C \) and \( C = 2 \). Finally, \( f(x) = 2x^3 + x + 2 \).

65. \( f''(x) = x^3 - 2x + 1 \), \( f'(0) = 1 \), \( f(0) = 0 \)

**Solution** Let \( g(x) = f'(x) \). The statement gives us \( g'(x) = x^3 - 2x + 1 \). \( g(0) = 1 \). From this, we get \( g(x) = \frac{1}{4}x^4 - x^2 + x + C \). \( g(0) = 1 \) gives us \( 1 = C \), so \( f'(x) = g(x) = \frac{1}{4}x^4 - x^2 + x + 1 \). \( f'(x) = \frac{1}{6}x^3 - \frac{1}{2}x^2 + x + C \). \( f(0) = 0 \) gives \( C = 0 \), so \( f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + x \).

67. \( f''(t) = t^{-3/2} \), \( f'(4) = 1 \), \( f(4) = 4 \)

**Solution** Let \( g(t) = f'(t) \). The problem statement is \( g'(t) = t^{-3/2} \). \( g(4) = 1 \). From \( g'(t) \) we get \( g(t) = -\frac{1}{4\sqrt{t}}t^{-1/2} + C = -2t^{-1/2} + C \). From \( g(4) = 1 \) we get \(-1 + C = 1 \) so that \( C = 2 \). Hence \( f'(t) = g(t) = -2t^{-1/2} + 2 \). From \( f'(t) \) we get \( f(t) = -2t^{1/2} + 2t + C = -4t^{1/2} + 2t + C \). From \( f(4) = 4 \) we get \(-8 + 4 + C = 4 \), so that \( C = 4 \). Hence, \( f(t) = -4t^{1/2} + 2t + 4 \).
69. \( f''(t) = t - \cos t, \quad f'(0) = 2, \quad f(0) = -2 \)

**SOLUTION** Let \( g(t) = f'(t) \). The problem statement gives
\[
g'(t) = t - \cos t, \quad g(0) = 2.
\]
From \( g'(t) \), we get \( g(t) = \frac{1}{2}t^2 - \sin t + C \). From \( g(0) = 2 \), we get \( C = 2 \). Hence \( f'(t) = g(t) = \frac{1}{2}t^2 - \sin t + 2 \). From \( f'(t) \), we get
\[
f(t) = \frac{1}{6}t^3 + \cos t + 2t - 3.
\]

71. A particle located at the origin at \( t = 1 \) s moves along the \( x \)-axis with velocity \( v(t) = (6t^2 - t) \) m/s. State the differential equation with initial condition satisfied by the position \( s(t) \) of the particle, and find \( s(t) \).

**SOLUTION** The differential equation satisfied by \( s(t) \) is
\[
\frac{ds}{dt} = v(t) = 6t^2 - t,
\]
and the associated initial condition is \( s(1) = 0 \). From the differential equation, we find
\[
s(t) = \int (6t^2 - t) \, dt = 2t^3 - \frac{1}{2}t^2 + C.
\]
Using the initial condition, it follows that
\[
0 = s(1) = 2 - \frac{1}{2} + C \quad \text{so} \quad C = -\frac{3}{2}.
\]

Finally,
\[
s(t) = 2t^3 - \frac{1}{2}t^2 - \frac{3}{2}.
\]

73. A mass oscillates at the end of a spring. Let \( s(t) \) be the displacement of the mass from the equilibrium position at time \( t \). Assuming that the mass is located at the origin at \( t = 0 \) and has velocity \( v(t) = \sin(\pi t/2) \) m/s, state the differential equation with initial condition satisfied by \( s(t) \), and find \( s(t) \).

**SOLUTION** The differential equation satisfied by \( s(t) \) is
\[
\frac{ds}{dt} = v(t) = \sin(\pi t/2),
\]
and the associated initial condition is \( s(0) = 0 \). From the differential equation, we find
\[
s(t) = \int \sin(\pi t/2) \, dt = -\frac{2}{\pi} \cos(\pi t/2) + C.
\]
Using the initial condition, it follows that
\[
0 = s(0) = -\frac{2}{\pi} + C \quad \text{so} \quad C = \frac{2}{\pi}.
\]

Finally,
\[
s(t) = \frac{2}{\pi} (1 - \cos(\pi t/2)).
\]

75. A car traveling 25 m/s begins to decelerate at a constant rate of 4 m/s\(^2\). After how many seconds does the car come to a stop and how far will the car have traveled before stopping?

**SOLUTION** Since the acceleration of the car is a constant \(-4\) m/s\(^2\), \( v \) is given by the differential equation:
\[
\frac{dv}{dt} = -4, \quad v(0) = 25.
\]
From \( \frac{dv}{dt} \), we get \( v(t) = \int -4 \, dt = -4t + C \). Since \( v(0) = 25 \), \( C = 25 \). From this, \( v(t) = -4t + 25 \frac{25}{4} \). To find the time until the car stops, we must solve \( v(t) = 0 \):
\[
-4t + 25 = 0
\]
\[
t = \frac{25}{4} = 6.25 \text{ s}.
\]

May 23, 2011
Now we have a differential equation for \( s(t) \). Since we want to know how far the car has traveled from the beginning of its deceleration at time \( t = 0 \), we have \( s(0) = 0 \) by definition, so:

\[
\frac{ds}{dt} = v(t) = -4t + 25, \quad s(0) = 0.
\]

From this, \( s(t) = \int (-4t + 25) \, dt = -2t^2 + 25t + C. \) Since \( s(0) = 0 \), we have \( C = 0 \), and

\[
s(t) = -2t^2 + 25t.
\]

At stopping time \( t = 0.25 \) s, the car has traveled

\[
s(0.25) = -2(0.25)^2 + 25(0.25) = 78.125 \, \text{m}.
\]

77. A 900-kg rocket is released from a space station. As it burns fuel, the rocket’s mass decreases and its velocity increases. Let \( v(m) \) be the velocity (in meters per second) as a function of mass \( m \). Find the velocity when \( m = 729 \) if \( \frac{dv}{dm} = -50m^{-1/2} \). Assume that \( v(900) = 0 \).

**Solution** Since \( \frac{dv}{dm} = -50m^{-1/2} \), we have \( v(m) = \int -50m^{-1/2} \, dm = -100\sqrt{m} + C. \) Thus \( 0 = v(900) = -100\sqrt{900} + C = -3000 + C, \) and \( C = 3000 \). Therefore, \( v(m) = 3000 - 100\sqrt{m}. \) Accordingly,

\[
v(729) = 3000 - 100\sqrt{729} = 3000 - 100(27) = 300 \, \text{meters/sec}.
\]

79. Verify the linearity properties of the indefinite integral stated in Theorem 4.

**Solution** To verify the Sum Rule, let \( f(x) \) and \( g(x) \) be any antiderivatives of \( f(x) \) and \( g(x) \), respectively. Because

\[
\frac{d}{dx} (F(x) + G(x)) = \frac{d}{dx} F(x) + \frac{d}{dx} G(x) = f(x) + g(x),
\]

it follows that \( F(x) + G(x) \) is an antiderivative of \( f(x) + g(x) \); i.e.,

\[
\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx.
\]

To verify the Multiples Rule, again let \( f(x) \) be any antiderivative of \( f(x) \) and let \( c \) be a constant. Because

\[
\frac{d}{dx} (cF(x)) = c \frac{d}{dx} F(x) = cf(x),
\]

it follows that \( cF(x) \) is an antiderivative of \( cf(x) \); i.e.,

\[
\int (cf(x)) \, dx = c \int f(x) \, dx.
\]

**Further Insights and Challenges**

81. Find constants \( c_1 \) and \( c_2 \) such that \( F(x) = c_1 xe^x + c_2 e^x \) is an antiderivative of \( f(x) = xe^x \).

**Solution** Let \( F(x) = c_1 xe^x + c_2 e^x \). If \( F(x) \) is to be an antiderivative of \( f(x) = xe^x \), we must have \( F'(x) = f(x) \) for all \( x \). Hence,

\[
c_1 xe^x + (c_1 + c_2)e^x = xe^x + 0 \cdot e^x.
\]

Equating coefficients of like terms we have \( c_1 = 1 \) and \( c_1 + c_2 = 0. \) Thus, \( c_1 = 1 \) and \( c_2 = -1 \).

83. Suppose that \( F'(x) = f(x) \).

(a) Show that \( \frac{1}{2} F(2x) \) is an antiderivative of \( f(2x) \).

(b) Find the general antiderivative of \( f(kx) \) for \( k \neq 0 \).

**Solution** Let \( F'(x) = f(x) \).

(a) By the Chain Rule, we have

\[
\frac{d}{dx} \left( \frac{1}{2} F(2x) \right) = \frac{1}{2} F'(2x) \cdot 2 = F'(2x) = f(2x).
\]

Thus \( \frac{1}{2} F(2x) \) is an antiderivative of \( f(2x) \).
(b) For nonzero constant $k$, the Chain Rules gives

$$\frac{d}{dx} \left( \frac{1}{k} F(kx) \right) = \frac{1}{k} F'(kx) \cdot k = F'(kx).$$

Thus $\frac{1}{k} F(kx)$ is an antiderivative of $f(kx)$. Hence the general antiderivative of $f(kx)$ is $\frac{1}{k} F(kx) + C$, where $C$ is a constant.

85. Using Theorem 1, prove that $F'(x) = f(x)$ where $f(x)$ is a polynomial of degree $n - 1$, then $F(x)$ is a polynomial of degree $n$. Then prove that if $g(x)$ is any function such that $g^{(n)}(x) = 0$, then $g(x)$ is a polynomial of degree at most $n$.

**Solution** Suppose $F'(x) = f(x)$ where $f(x)$ is a polynomial of degree $n - 1$. Now, we know that the derivative of a polynomial of degree $n$ is a polynomial of degree $n - 1$, and hence an antiderivative of a polynomial of degree $n - 1$ is a polynomial of degree $n$. Thus, by Theorem 1, $F(x)$ can differ from a polynomial of degree $n$ by at most a constant term, which is still a polynomial of degree $n$. Now, suppose that $g(x)$ is any function such that $g^{(n+1)}(x) = 0$. We know that the $n + 1$-st derivative of any polynomial of degree at most $n$ is zero, so by repeated application of Theorem 1, $g(x)$ can differ from a polynomial of degree at most $n$ by at most a constant term. Hence, $g(x)$ is a polynomial of degree at most $n$.

---

**CHAPTER REVIEW EXERCISES**

---

**In Exercises 1–6, estimate using the Linear Approximation or linearization, and use a calculator to estimate the error:**

1. $8.11^{1/3} - 2$

**Solution** Let $f(x) = x^{1/3}$, $a = 8$ and $\Delta x = 0.1$. Then $f'(x) = \frac{1}{3}x^{-2/3}$, $f'(a) = \frac{1}{12}$ and, by the Linear Approximation,

$$\Delta f = 8.11^{1/3} - 2 \approx f'(a)\Delta x = \frac{1}{12}(0.1) = 0.00833333.$$

Using a calculator, $8.11^{1/3} - 2 = 0.00829885$. The error in the Linear Approximation is therefore

$$|0.00829885 - 0.00833333| = 3.445 \times 10^{-5}.$$

3. $625^{1/4} - 624^{1/4}$

**Solution** Let $f(x) = x^{1/4}$, $a = 625$ and $\Delta x = -1$. Then $f'(x) = \frac{1}{4}x^{-3/4}$, $f'(a) = \frac{1}{500}$ and, by the Linear Approximation,

$$\Delta f = 625^{1/4} - 624^{1/4} \approx f'(a)\Delta x = \frac{1}{500}(-1) = -0.002.$$

Thus $625^{1/4} - 624^{1/4} \approx 0.002$. Using a calculator,

$$625^{1/4} - 624^{1/4} = 0.00200120.$$

The error in the Linear Approximation is therefore

$$|0.00200120 - (0.002)| = 1.201 \times 10^{-6}.$$

5. $\frac{1}{1.02}$

**Solution** Let $f(x) = x^{-1}$ and $a = 1$. Then $f(a) = 1$, $f'(x) = -x^{-2}$ and $f'(a) = -1$. The linearization of $f(x)$ at $a = 1$ is therefore

$$L(x) = f(a) + f'(a)(x - a) = 1 - (x - 1) = 2 - x,$$

and $\frac{1}{1.02} \approx L(1.02) = 0.98$. Using a calculator, $\frac{1}{1.02} = 0.980392$, so the error in the Linear Approximation is

$$|0.980392 - 0.98| = 3.922 \times 10^{-4}.$$

---

**In Exercises 7–12, find the linearization at the point indicated.**

7. $y = \sqrt{x}$, $a = 25$

**Solution** Let $y = \sqrt{x}$ and $a = 25$. Then $y(a) = 5$, $y'(x) = \frac{1}{2}x^{-1/2}$ and $y'(a) = \frac{1}{10}$. The linearization of $y$ at $a = 25$ is therefore

$$L(x) = y(a) + y'(a)(x - a) = 5 + \frac{1}{10}(x - 25).$$
9. \( A(r) = \frac{4}{3} \pi r^3, \ a = 3 \)

**Solution** Let \( A(r) = \frac{4}{3} \pi r^3 \) and \( a = 3 \). Then \( A(a) = 36 \pi \), \( A'(r) = 4 \pi r^2 \) and \( A'(a) = 36 \pi \). The linearization of \( A(r) \) at \( a = 3 \) is therefore

\[
L(r) = A(a) + A'(a)(r - a) = 36 \pi + 36 \pi (r - 3) = 36 \pi (r - 2).
\]

11. \( P(x) = e^{-x^2/2}, \ a = 1 \)

**Solution** Let \( P(x) = e^{-x^2/2} \) and \( a = 1 \). Then \( P(a) = e^{-1/2} \), \( P'(x) = -xe^{-x^2/2} \), and \( P'(a) = -e^{-1/2} \). The linearization of \( P(x) \) at \( a = 1 \) is therefore

\[
L(x) = P(a) + P'(a)(x - a) = e^{-1/2} - e^{-1/2}(x - 1) = \frac{1}{\sqrt{e}}(2 - x).
\]

In Exercises 13–18, use the Linear Approximation.

13. The position of an object in linear motion at time \( t \) is \( s(t) = 0.4t^2 + (t + 1)^{-1} \). Estimate the distance traveled over the time interval \([4, 4.2]\).

**Solution** Let \( s(t) = 0.4t^2 + (t + 1)^{-1}, \ a = 4 \) and \( \Delta t = 0.2 \). Then \( s'(t) = 0.8t - (t + 1)^{-2} \) and \( s'(a) = 3.16 \). Using the Linear Approximation, the distance traveled over the time interval \([4, 4.2]\) is approximately

\[
\Delta s = s(4.2) - s(4) = s'(a)\Delta t = 3.16(0.2) = 0.632.
\]

15. When a bus pass from Albuquerque to Los Alamos is priced at \( p \) dollars, a bus company takes in a monthly revenue of \( R(p) = 1.5p - 0.01p^2 \) (in thousands of dollars).

(a) Estimate \( \Delta R \) if the price rises from \( \$50 \) to \( \$53 \).

(b) If \( p = 80 \), how will revenue be affected by a small increase in price? Explain using the Linear Approximation.

**Solution**

(a) If the price is raised from \( \$50 \) to \( \$53 \), then \( \Delta p = 3 \) and

\[
\Delta R \approx R'(50)\Delta p = (1.5 - 0.02(50))(3) = 1.5
\]

We therefore estimate an increase of \( \$1500 \) in revenue.

(b) Because \( R'(80) = 1.5 - 0.02(80) = -0.1 \), the Linear Approximation gives \( \Delta R \approx -0.1\Delta p \). A small increase in price would thus result in a decrease in revenue.

17. The circumference of a sphere is measured at \( C = 100 \) cm. Estimate the maximum percentage error in \( V \) if the error in \( C \) is at most 3 cm.

**Solution** The volume of a sphere is \( V = \frac{4}{3} \pi r^3 \) and the circumference is \( C = 2\pi r \), where \( r \) is the radius of the sphere. Thus, \( r = \frac{1}{2\pi} C \) and

\[
V = \frac{4}{3} \pi \left( \frac{C}{2\pi} \right)^3 = \frac{1}{6\pi^2} C^3.
\]

Using the Linear Approximation,

\[
\Delta V \approx \frac{dV}{dC}\Delta C = \frac{1}{2\pi^2} C^2 \Delta C,
\]

so

\[
\frac{\Delta V}{V} \approx \frac{\frac{1}{2\pi^2} C^2 \Delta C}{\frac{1}{6\pi^2} C^3} = \frac{3\Delta C}{C}.
\]

With \( C = 100 \) cm and \( \Delta C \) at most 3 cm, we estimate that the maximum percentage error in \( V \) is \( 3\frac{3}{100} = 0.09 \), or 9%.

19. Use the Intermediate Value Theorem to prove that \( \sin x - \cos x = 3x \) has a solution, and use Rolle’s Theorem to show that this solution is unique.

**Solution** Let \( f(x) = \sin x - \cos x - 3x \), and observe that each root of this function corresponds to a solution of the equation \( \sin x - \cos x = 3x \). Now,

\[
f(-\frac{\pi}{2}) = -1 + \frac{3\pi}{2} > 0 \quad \text{and} \quad f(0) = -1 < 0.
\]
Because \( f \) is continuous on \((-\frac{\pi}{2}, 0)\) and \( f(-\frac{\pi}{2}) \) and \( f(0) \) are of opposite sign, the Intermediate Value Theorem guarantees there exists a \( c \in (-\frac{\pi}{2}, 0) \) such that \( f(c) = 0 \). Thus, the equation \( \sin x - \cos x = 3x \) has at least one solution.

Next, suppose that the equation \( \sin x - \cos x = 3x \) has two solutions, and therefore \( f(x) \) has two roots, say \( a \) and \( b \). Because \( f \) is continuous on \([a, b]\), differentiable on \((a, b)\) and \( f(a) = f(b) = 0 \), Rolle’s Theorem guarantees there exists \( c \in (a, b) \) such that \( f'(c) = 0 \). However,

\[
f'(x) = \cos x + \sin x - 3 \leq -1
\]

for all \( x \). We have reached a contradiction. Consequently, \( f(x) \) has a unique root and the equation \( \sin x - \cos x = 3x \) has a unique solution.

21. Verify the MVT for \( f(x) = \ln x \) on \([1, 4]\).

**Solution** Let \( f(x) = \ln x \). On the interval \([1, 4]\), this function is continuous and differentiable, so the MVT applies. Now, \( f'(x) = \frac{1}{x} \), so

\[
\frac{1}{c} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\ln 4 - \ln 1}{4 - 1} = \frac{1}{3} \ln 4,
\]

or

\[
c = \frac{3}{\ln 4} \approx 2.164 \in (1, 4).
\]

23. Use the MVT to prove that if \( f'(x) \leq 2 \) for \( x > 0 \) and \( f(0) = 4 \), then \( f(x) \leq 2x + 4 \) for all \( x \geq 0 \).

**Solution** Let \( x > 0 \). Because \( f \) is continuous on \([0, x]\) and differentiable on \((0, x)\), the Mean Value Theorem guarantees there exists a \( c \in (0, x) \) such that

\[
f'(c) = \frac{f(x) - f(0)}{x - 0} \quad \text{or} \quad f(x) = f(0) + xf'(c).
\]

Now, we are given that \( f(0) = 4 \) and that \( f'(x) \leq 2 \) for \( x > 0 \). Therefore, for all \( x \geq 0 \),

\[
f(x) \leq 4 + x(2) = 2x + 4.
\]

In Exercises 25–30, find the critical points and determine whether they are minima, maxima, or neither.

25. \( f(x) = x^3 - 4x^2 + 4x \)

**Solution** Let \( f(x) = x^3 - 4x^2 + 4x \). Then \( f'(x) = 3x^2 - 8x + 4 = (3x - 2)(x - 2) \), so that \( x = \frac{2}{3} \) and \( x = 2 \) are critical points. Next, \( f''(x) = 6x - 8 \), so \( f''(\frac{2}{3}) = -4 < 0 \) and \( f''(2) = 4 > 0 \). Therefore, by the Second Derivative Test, \( f(\frac{2}{3}) \) is a local maximum while \( f(2) \) is a local minimum.

27. \( f(x) = x^2(x + 2)^3 \)

**Solution** Let \( f(x) = x^2(x + 2)^3 \). Then

\[
f'(x) = 3x^2(x + 2)^2 + 2x(x + 2)^3 = x(x + 2)^2(3x + 2x + 4) = x(x + 2)^2(5x + 4),
\]

so that \( x = 0, x = -2 \) and \( x = -\frac{4}{3} \) are critical points. The sign of the first derivative on the intervals surrounding the critical points is indicated in the table below. Based on this information, \( f(-2) \) is neither a local maximum nor a local minimum, \( f(-\frac{4}{3}) \) is a local maximum and \( f(0) \) is a local minimum.

<table>
<thead>
<tr>
<th>Interval</th>
<th>((-\infty, -2))</th>
<th>((-2, -\frac{4}{3}))</th>
<th>((-\frac{4}{3}, 0))</th>
<th>((0, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sign of ( f' )</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

29. \( g(\theta) = \sin^2 \theta + \theta \)

**Solution** Let \( g(\theta) = \sin^2 \theta + \theta \). Then

\[
g'(\theta) = 2 \sin \theta \cos \theta + 1 = 2 \sin 2\theta + 1,
\]

so the critical points are

\[
\theta = \frac{3\pi}{4} + n\pi
\]

for all integers \( n \). Because \( g'(\theta) \geq 0 \) for all \( \theta \), it follows that \( g\left(\frac{3\pi}{4} + n\pi\right) \) is neither a local maximum nor a local minimum for all integers \( n \).
In Exercises 31–38, find the extreme values on the interval.

31. \( f(x) = x(10 - x), \quad [-1, 3] \)

**Solution** Let \( f(x) = x(10 - x) = 10x - x^2 \). Then \( f'(x) = 10 - 2x \), so that \( x = 5 \) is the only critical point. As this critical point is not in the interval \([-1, 3]\), we only need to check the value of \( f \) at the endpoints to determine the extreme values. Because \( f(-1) = -11 \) and \( f(3) = 21 \), the maximum value of \( f(x) = x(10 - x) \) on the interval \([-1, 3]\) is 21 while the minimum value is -11.

33. \( g(\theta) = \sin^2 \theta - \cos \theta, \quad [0, 2\pi] \)

**Solution** Let \( g(\theta) = \sin^2 \theta - \cos \theta \). Then

\[
g'(\theta) = 2 \sin \theta \cos \theta + \sin \theta = \sin \theta(2 \cos \theta + 1) = 0
\]

when \( \theta = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi \). The table below lists the value of \( g \) at each of the critical points and the endpoints of the interval \( [0, 2\pi] \). Based on this information, the minimum value of \( g(\theta) \) on the interval \([0, 2\pi]\) is -1 and the maximum value is \( \frac{21}{2} \).

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0</th>
<th>2\pi/3</th>
<th>( \pi )</th>
<th>4\pi/3</th>
<th>2\pi</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(\theta) )</td>
<td>-1</td>
<td>5/4</td>
<td>1</td>
<td>5/4</td>
<td>-1</td>
</tr>
</tbody>
</table>

35. \( f(x) = x^{2/3} - 2x^{1/3}, \quad [-1, 3] \)

**Solution** Let \( f(x) = x^{2/3} - 2x^{1/3} \). Then \( f'(x) = \frac{2}{3}x^{-1/3} - \frac{2}{3}x^{-2/3} = \frac{2}{3}x^{-2/3}(x^{1/3} - 1) \), so that the critical points are \( x = 0 \) and \( x = 1 \). With \( f(-1) = 3, f(0) = 0, f(1) = -1 \) and \( f(3) = \frac{2}{3}\sqrt[3]{2} - 2\sqrt[3]{3} \approx -0.804 \), it follows that the minimum value of \( f(x) \) on the interval \([-1, 3]\) is -1 and the maximum value is 3.

37. \( f(x) = x - 12 \ln x, \quad [5, 40] \)

**Solution** Let \( f(x) = x - 12 \ln x \). Then \( f'(x) = 1 - \frac{12}{x} \), whence \( x = 12 \) is the only critical point. The minimum value of \( f \) is then \( 12 - 12 \ln 12 \approx -17.818880 \), and the maximum value is \( 40 - 12 \ln 40 \approx -4.266553 \). Note that \( f(5) = 5 - 12 \ln 5 \approx -14.313255 \).

39. Find the critical points and extreme values of \( f(x) = |x - 1| + |2x - 6| \) in \([0, 8]\).

**Solution** Let

\[
f(x) = |x - 1| + |2x - 6| = \begin{cases} 7 - 3x, & x < 1 \\ 5 - x, & 1 \leq x < 3 \\ 3x - 7, & x \geq 3 \end{cases}
\]

The derivative of \( f(x) \) is never zero but does not exist at the transition points \( x = 1 \) and \( x = 3 \). Thus, the critical points of \( f \) are \( x = 1 \) and \( x = 3 \). With \( f(0) = 7, f(1) = 4, f(3) = 2 \) and \( f(8) = 17 \), it follows that the minimum value of \( f(x) \) on the interval \([0, 8]\) is 2 and the maximum value is 17.

In Exercises 41–46, find the points of inflection.

41. \( y = x^3 - 4x^2 + 4x \)

**Solution** Let \( y = x^3 - 4x^2 + 4x \). Then \( y' = 3x^2 - 8x + 4 \) and \( y'' = 6x - 8 \). Thus, \( y'' > 0 \) and \( y \) is concave up for \( x > \frac{4}{3} \), while \( y'' < 0 \) and \( y \) is concave down for \( x < \frac{4}{3} \). Hence, there is a point of inflection at \( x = \frac{4}{3} \).

43. \( y = \frac{x^2}{x^2 + 4} \)

**Solution** Let \( y = \frac{x^2}{x^2 + 4} = 1 - \frac{4}{x^2 + 4} \). Then \( y' = \frac{8x}{(x^2 + 4)^2} \) and

\[
y'' = \frac{(x^2 + 4)^2(8) - 8x(2)(2x)(x^2 + 4)}{(x^2 + 4)^4} = \frac{8(4 - x^2)}{(x^2 + 4)^3}.
\]

Thus, \( y'' > 0 \) and \( y \) is concave up for

\[
-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}.
\]
while \( y'' < 0 \) and \( y \) is concave down for
\[
|x| \geq \frac{2}{\sqrt{3}}.
\]

Hence, there are points of inflection at
\[
x = \pm \frac{2}{\sqrt{3}}.
\]

45. \( f(x) = (x^2 - x)e^{-x} \)

**Solution** Let \( f(x) = (x^2 - x)e^{-x} \). Then
\[
y' = -(x^2 - x)e^{-x} + (2x - 1)e^{-x} = -(x^2 - 3x + 1)e^{-x},
\]
and
\[
y'' = (x^2 - 3x + 1)e^{-x} - (2x - 3)e^{-x} = e^{-x}(x^2 - 5x + 4) = e^{-x}(x - 1)(x - 4).
\]
Thus, \( y'' > 0 \) and \( y \) is concave up for \( x < 1 \) and for \( x > 4 \), while \( y'' < 0 \) and \( y \) is concave down for \( 1 < x < 4 \). Hence, there are points of inflection at \( x = 1 \) and \( x = 4 \).

In Exercises 47–56, sketch the graph, noting the transition points and asymptotic behavior.

47. \( y = 12x - 3x^2 \)

**Solution** Let \( y = 12x - 3x^2 \). Then \( y' = 12 - 6x \) and \( y'' = -6 \). It follows that the graph of \( y = 12x - 3x^2 \) is increasing for \( x < 2 \), decreasing for \( x > 2 \), has a local maximum at \( x = 2 \) and is concave down for all \( x \). Because
\[
\lim_{x \to -\infty} (12x - 3x^2) = -\infty,
\]
the graph has no horizontal asymptotes. There are also no vertical asymptotes. The graph is shown below.

49. \( y = x^3 - 2x^2 + 3 \)

**Solution** Let \( y = x^3 - 2x^2 + 3 \). Then \( y' = 3x^2 - 4x \) and \( y'' = 6x - 4 \). It follows that the graph of \( y = x^3 - 2x^2 + 3 \) is increasing for \( x < 0 \) and \( x > \frac{2}{3} \), is decreasing for \( 0 < x < \frac{4}{3} \), has a local maximum at \( x = 0 \), has a local minimum at \( x = \frac{2}{3} \), is concave up for \( x > \frac{2}{3} \), is concave down for \( x < \frac{2}{3} \) and has a point of inflection at \( x = \frac{2}{3} \). Because
\[
\lim_{x \to -\infty} (x^3 - 2x^2 + 3) = -\infty \quad \text{and} \quad \lim_{x \to \infty} (x^3 - 2x^2 + 3) = \infty,
\]
the graph has no horizontal asymptotes. There are also no vertical asymptotes. The graph is shown below.

51. \( y = \frac{x}{x^3 + 1} \)

**Solution** Let \( y = \frac{x}{x^3 + 1} \). Then
\[
y' = \frac{x^3 + 1 - x(3x^2)}{(x^3 + 1)^2} = \frac{1 - 2x^3}{(x^3 + 1)^2},
\]
and
\[
y'' = \frac{(x^3 + 1)^2(-6x^2) - (1 - 2x^3)(2)(x^3 + 1)(3x^2)}{(x^3 + 1)^4} = \frac{6x^2(2 - x^3)}{(x^3 + 1)^3}.
\]
It follows that the graph of \( y = \frac{x}{x^3 + 1} \) is increasing for \( x < -1 \) and \(-1 < x < \sqrt[3]{\frac{1}{2}} \), is decreasing for \( x > \sqrt[3]{\frac{1}{2}} \), has a local maximum at \( x = \sqrt[3]{\frac{1}{2}} \), is concave up for \( x < -1 \) and \( x > \sqrt[3]{\frac{3}{2}} \), is concave down for \(-1 < x < 0 \) and \( 0 < x < \sqrt[3]{\frac{3}{2}} \) and has a point of inflection at \( x = \sqrt[3]{\frac{3}{2}} \). Note that \( x = -1 \) is not an inflection point because \( x = -1 \) is not in the domain of the function. Now,

\[
\lim_{x \to \pm \infty} \frac{x}{x^3 + 1} = 0,
\]

so \( y = 0 \) is a horizontal asymptote. Moreover,

\[
\lim_{x \to -1^-} \frac{x}{x^3 + 1} = \infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{x}{x^3 + 1} = -\infty,
\]

so \( x = -1 \) is a vertical asymptote. The graph is shown below.

Let \( y = \frac{1}{|x + 2| + 1} \).

**SOLUTION** Let \( y = \frac{1}{|x + 2| + 1} \). Because

\[
\lim_{x \to \pm \infty} \frac{1}{|x + 2| + 1} = 0,
\]

the graph of this function has a horizontal asymptote of \( y = 0 \). The graph has no vertical asymptotes as \(|x + 2| + 1 \geq 1\) for all \( x \). The graph is shown below. From this graph we see there is a local maximum at \( x = -2 \).

\( y = \sqrt{3} \sin x - \cos x \) on \([0, 2\pi]\)

**SOLUTION** Let \( y = \sqrt{3} \sin x - \cos x \). Then \( y' = \sqrt{3} \cos x + \sin x \) and \( y'' = -\sqrt{3} \sin x + \cos x \). It follows that the graph of \( y = \sqrt{3} \sin x - \cos x \) is increasing for \( 0 < x < 5\pi/6 \) and \( 11\pi/6 < x < 2\pi \), is decreasing for \( 5\pi/6 < x < 11\pi/6 \), has a local maximum at \( x = 5\pi/6 \), has a local minimum at \( x = 11\pi/6 \), is concave up for \( 0 < x < \pi/3 \) and \( 4\pi/3 < x < 2\pi \), is concave down for \( \pi/3 < x < 4\pi/3 \) and has points of inflection at \( x = \pi/3 \) and \( x = 4\pi/3 \). The graph is shown below.

57. Draw a curve \( y = f(x) \) for which \( f' \) and \( f'' \) have signs as indicated in Figure 2.

**SOLUTION** The figure below depicts a curve for which \( f'(x) \) and \( f''(x) \) have the required signs.
59. A rectangular box of height $h$ with square base of side $b$ has volume $V = 4 \, \text{m}^3$. Two of the side faces are made of material costing $40/\text{m}^2$. The remaining sides cost $20/\text{m}^2$. Which values of $b$ and $h$ minimize the cost of the box?

**Solution** Because the volume of the box is

$$V = b^2h = 4$$

it follows that

$$h = \frac{4}{b^2}.$$

Now, the cost of the box is

$$C = 40(2bh) + 20(2b^2) = 120bh + 20b^2 = \frac{480}{b} + 20b^2.$$

Thus,

$$C'(b) = -\frac{480}{b^2} + 40b = 0$$

when $b = \sqrt[3]{12}$ meters. Because $C(b) \to \infty$ as $b \to 0^+$ and as $b \to \infty$, it follows that cost is minimized when $b = \sqrt[3]{12}$ meters and $h = \frac{1}{4}\sqrt[3]{12}$ meters.

61. Let $N(t)$ be the size of a tumor (in units of $10^6$ cells) at time $t$ (in days). According to the **Gompertz Model**, $dN/dt = N(a - b \ln N)$ where $a, b$ are positive constants. Show that the maximum value of $N$ is $e^{a/b}$ and that the tumor increases most rapidly when $N = e^{a/b}$.

**Solution** Given $dN/dt = N(a - b \ln N)$, the critical points of $N$ occur when $N = 0$ and when $N = e^{a/b}$. The sign of $N'(t)$ changes from positive to negative at $N = e^{a/b}$ so the maximum value of $N$ is $e^{a/b}$. To determine when $N$ changes most rapidly, we calculate

$$N''(t) = N \left( -\frac{b}{N} \right) + a - b \ln N = (a - b) - b \ln N.$$

Thus, $N''(t)$ is increasing for $N < e^{a/b-1}$, is decreasing for $N > e^{a/b-1}$ and is therefore maximum when $N = e^{a/b-1}$. Therefore, the tumor increases most rapidly when $N = e^{a/b-1}$.

63. Find the maximum volume of a right-circular cone placed upside-down in a right-circular cone of radius $R = 3$ and height $H = 4$ as in Figure 3. A cone of radius $r$ and height $h$ has volume $\frac{1}{3}\pi r^2h$.

**Solution** Let $r$ denote the radius and $h$ the height of the upside down cone. By similar triangles, we obtain the relation

$$\frac{4-h}{r} = \frac{4}{3} \quad \text{so} \quad h = 4 \left( \frac{1 - r}{3} \right)$$

and the volume of the upside down cone is

$$V(r) = \frac{1}{3}\pi r^2h = \frac{4}{3}\pi \left( r^2 - \frac{r^3}{3} \right)$$

for $0 \leq r \leq 3$. Thus,

$$\frac{dV}{dr} = \frac{4}{3} \pi \left( 2r - r^2 \right).$$
and the critical points are \( r = 0 \) and \( r = 2 \). Because \( V(0) = V(3) = 0 \) and
\[
V(2) = \frac{4}{3} \pi \left(4 - \frac{8}{3}\right) = \frac{16}{9} \pi,
\]
the maximum volume of a right-circular cone placed upside down in a right-circular cone of radius 3 and height 4 is
\[
\frac{16}{9} \pi.
\]

65. Show that the maximum area of a parallelogram \( ADE F \) that is inscribed in a triangle \( ABC \), as in Figure 4, is equal to one-half the area of \( \triangle ABC \).

![Figure 4](image)

**Solution** Let \( \theta \) denote the measure of angle \( BAC \). Then the area of the parallelogram is given by \( \overline{AD} \cdot \overline{AF} \sin \theta \).

Now, suppose that \( \overline{BE}/\overline{BC} = x \).

Then, by similar triangles, \( \overline{AD} = (1 - x)\overline{AB} \), \( \overline{AF} = \overline{DE} = x\overline{AC} \), and the area of the parallelogram becomes \( \overline{AB} \cdot \overline{AC}x(1 - x) \sin \theta \). The function \( x(1 - x) \) achieves its maximum value of \( \frac{1}{4} \) when \( x = \frac{1}{2} \). Thus, the maximum area of a parallelogram inscribed in a triangle \( \triangle ABC \) is
\[
\frac{1}{4} \overline{AB} \cdot \overline{AC} \sin \theta = \frac{1}{2} \left( \frac{1}{2} \overline{AB} \cdot \overline{AC} \sin \theta \right) = \frac{1}{2} \text{(area of } \triangle ABC \).
\]

67. Let \( f(x) \) be a function whose graph does not pass through the \( x \)-axis and let \( Q = (a, 0) \). Let \( P = (x_0, f(x_0)) \) be the point on the graph closest to \( Q \) (Figure 5). Prove that \( \overline{PQ} \) is perpendicular to the tangent line to the graph of \( x_0 \). Hint: Find the minimum value of the square of the distance from \( (x, f(x)) \) to \( (a, 0) \).

![Figure 5](image)

**Solution** Let \( P = (a, 0) \) and let \( Q = (x_0, f(x_0)) \) be the point on the graph of \( y = f(x) \) closest to \( P \). The slope of the segment joining \( P \) and \( Q \) is then
\[
\frac{f(x_0)}{x_0 - a}.
\]

Now, let
\[
q(x) = \sqrt{(x - a)^2 + (f(x))}\)
\]

the distance from the arbitrary point \( (x, f(x)) \) on the graph of \( y = f(x) \) to the point \( P \). As \( (x_0, f(x_0)) \) is the point closest to \( P \), we must have
\[
q'(x_0) = \frac{2(x_0 - a) + 2f(x_0)f'(x_0)}{\sqrt{(x_0 - a)^2 + (f(x_0))^2}} = 0.
\]

Thus,
\[
f'(x_0) = -\frac{x_0 - a}{f(x_0)} = -\left( \frac{f(x_0)}{x_0 - a} \right)^{-1}.
\]

In other words, the slope of the segment joining \( P \) and \( Q \) is the negative reciprocal of the slope of the line tangent to the graph of \( y = f(x) \) at \( x = x_0 \); hence; the two lines are perpendicular.
69. Use Newton’s Method to estimate $\sqrt[3]{25}$ to four decimal places.

**Solution** Let $f(x) = x^3 - 25$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 25}{3x_n^2}.$$

With $x_0 = 3$, we find

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_n$</td>
<td>2.925925926</td>
<td>2.924018982</td>
<td>2.924017738</td>
</tr>
</tbody>
</table>

Thus, to four decimal places $\sqrt[3]{25} = 2.9240$.

In Exercises 71–84, calculate the indefinite integral.

71. $\int (4x^3 - 2x^2) \, dx$

**Solution**

$$\int (4x^3 - 2x^2) \, dx = x^4 - \frac{2}{3}x^3 + C.$$

73. $\int \sin(\theta - 8) \, d\theta$

**Solution**

$$\int \sin(\theta - 8) \, d\theta = -\cos(\theta - 8) + C.$$

75. $\int (4r^{-3} - 12r^{-4}) \, dt$

**Solution**

$$\int (4r^{-3} - 12r^{-4}) \, dt = -2r^{-2} + 4r^{-3} + C.$$

77. $\int \sec^2 x \, dx$

**Solution**

$$\int \sec^2 x \, dx = \tan x + C.$$

79. $\int (y + 2)^4 \, dy$

**Solution**

$$\int (y + 2)^4 \, dy = \frac{1}{5}(y + 2)^5 + C.$$

81. $\int (e^x - x) \, dx$

**Solution**

$$\int (e^x - x) \, dx = e^x - \frac{1}{2}x^2 + C.$$

83. $\int 4x^{-1} \, dx$

**Solution**

$$\int 4x^{-1} \, dx = 4 \ln |x| + C.$$

In Exercises 85–90, solve the differential equation with the given initial condition.

85. $\frac{dy}{dx} = 4x^3, \quad y(1) = 4$

**Solution** Let $\frac{dy}{dx} = 4x^3$. Then

$$y(x) = \int 4x^3 \, dx = x^4 + C.$$

Using the initial condition $y(1) = 4$, we find $y(1) = 1^4 + C = 4$, so $C = 3$. Thus, $y(x) = x^4 + 3$.

87. $\frac{dy}{dx} = x^{-1/2}, \quad y(1) = 1$

**Solution** Let $\frac{dy}{dx} = x^{-1/2}$. Then

$$y(x) = \int x^{-1/2} \, dx = 2x^{1/2} + C.$$

Using the initial condition $y(1) = 1$, we find $y(1) = 2\sqrt{1} + C = 1$, so $C = -1$. Thus, $y(x) = 2x^{1/2} - 1$. 
89. \( \frac{dy}{dx} = e^{-x} \), \( y(0) = 3 \)

**SOLUTION** Let \( \frac{dy}{dx} = e^{-x} \). Then

\[
y(x) = \int e^{-x} \, dx = -e^{-x} + C.
\]

Using the initial condition \( y(0) = 3 \), we find \( y(0) = -e^0 + C = 3 \), so \( C = 4 \). Thus, \( y(x) = 4 - e^{-x} \).

91. Find \( f(t) \) if \( f''(t) = 1 - 2t \), \( f(0) = 2 \), and \( f'(0) = -1 \).

**SOLUTION** Suppose \( f''(t) = 1 - 2t \). Then

\[
f'(t) = \int f''(t) \, dt = \int (1 - 2t) \, dt = t - t^2 + C.
\]

Using the initial condition \( f'(0) = -1 \), we find \( f'(0) = 0 - 0^2 + C = -1 \), so \( C = -1 \). Thus, \( f'(t) = t - t^2 - 1 \). Now,

\[
f(t) = \int f'(t) \, dt = \int (t - t^2 - 1) \, dt = \frac{1}{2}t^2 - \frac{1}{3}t^3 - t + C.
\]

Using the initial condition \( f(0) = 2 \), we find \( f(0) = \frac{1}{2}0^2 - \frac{1}{3}0^3 - 0 + C = 2 \), so \( C = 2 \). Thus,

\[
f(t) = \frac{1}{2}t^2 - \frac{1}{3}t^3 - t + 2.
\]

93. Find the local extrema of \( f(x) = \frac{e^{2x} + 1}{e^{x^2} + 1} \).

**SOLUTION** To simplify the differentiation, we first rewrite \( f(x) = \frac{e^{2x} + 1}{e^{x^2} + 1} \) using the Laws of Exponents:

\[
f(x) = \frac{e^{2x}}{e^{x^2} + 1} + \frac{1}{e^{x^2} + 1} = e^{2x-x^2+1} + e^{-x^2+1} = e^{x-1} + e^{-x-1}.
\]

Now,

\[
f'(x) = e^{x-1} - e^{-x-1}.
\]

Setting the derivative equal to zero yields

\[
e^{x-1} - e^{-x-1} = 0 \quad \text{or} \quad e^{x-1} = e^{-x-1}.
\]

Thus,

\[
x - 1 = -x - 1 \quad \text{or} \quad x = 0.
\]

Next, we use the Second Derivative Test. With \( f''(x) = e^{x-1} + e^{-x-1} \), it follows that

\[
f''(x) = e^{x-1} + e^{-x-1} = \frac{2}{e} > 0.
\]

Hence, \( x = 0 \) is a local minimum. Since \( f(0) = e^{0-1} + e^{0-1} = \frac{2}{e} \), we conclude that the point \((0, \frac{2}{e})\) is a local minimum.

In Exercises 95–98, find the local extrema and points of inflection, and sketch the graph. Use L'Hôpital's Rule to determine the limits as \( x \to 0^+ \) or \( x \to \pm\infty \) if necessary.

95. \( y = x \ln x \) \( (x > 0) \)

**SOLUTION** Let \( y = x \ln x \). Then

\[
y' = \ln x + x \left( \frac{1}{x} \right) = 1 + \ln x,
\]

and \( y'' = \frac{1}{x} \). Solving \( y' = 0 \) yields the critical point \( x = e^{-1} \). Since \( y''(e^{-1}) = e > 0 \), the function has a local minimum at \( x = e^{-1} \). \( y'' \) is positive for \( x > 0 \), hence the function is concave up for \( x > 0 \) and there are no points of inflection. As \( x \to 0^+ \) and as \( x \to \infty \), we find

\[
\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{x}{-x^2} = \lim_{x \to 0^+} (-x) = 0;
\]

\[
\lim_{x \to \infty} x \ln x = \infty.
\]
The graph is shown below:

97. \( y = x(\ln x)^2 \quad (x > 0) \)

**SOLUTION** Let \( y = x(\ln x)^2 \). Then

\[
y' = x \frac{2\ln x}{x} + (\ln x)^2 = 2 \ln x + (\ln x)^2 = \ln x(2 + \ln x),
\]
and

\[
y'' = \frac{2}{x} + \frac{2 \ln x}{x} = \frac{2}{x}(1 + \ln x).
\]

Solving \( y' = 0 \) yields the critical points \( x = e^{-2} \) and \( x = 1 \). Since \( y''(e^{-2}) = -2e^2 < 0 \) and \( y''(1) = 2 > 0 \), the function has a local maximum at \( x = e^{-1} \), whereas \( y'' > 0 \) and the function is concave up for \( x > e^{-1} \); hence, there is a point of inflection at \( x = e^{-1} \).

As \( x \to 0^+ \) and as \( x \to \infty \), we find

\[
\lim_{x \to 0^+} x(\ln x)^2 = \lim_{x \to 0^+} \frac{2 \ln x}{x} = \lim_{x \to 0^+} \ln x \cdot \frac{2}{x} = \lim_{x \to 0^+} \frac{2}{x} = 0;
\]

\[
\lim_{x \to \infty} x(\ln x)^2 = \infty.
\]

The graph is shown below:

99. **Explain why L'Hôpital's Rule gives no information about \( \lim_{x \to \infty} \frac{2x - \sin x}{3x + \cos 2x} \). Evaluate the limit by another method.**

**SOLUTION** As \( x \to \infty \), both \( 2x - \sin x \) and \( 3x + \cos 2x \) tend toward infinity, so L'Hôpital's Rule applies to \( \lim_{x \to \infty} \frac{2x - \sin x}{3x + \cos 2x} \); however, the resulting limit, \( \lim_{x \to \infty} \frac{2 - \cos x}{3 + 2 \sin 2x} \), does not exist due to the oscillation of \( \sin x \) and \( \cos x \) and further applications of L'Hôpital's rule will not change this situation.

To evaluate the limit, we note

\[
\lim_{x \to \infty} \frac{2x - \sin x}{3x + \cos 2x} = \lim_{x \to \infty} \frac{2 - \sin x}{3 + 2 \sin 2x} = \frac{2}{3}.
\]

In Exercises 101–112, verify that L'Hôpital's Rule applies and evaluate the limit.

101. \( \lim_{x \to 3} \frac{4x - 12}{x^2 - 5x + 6} \)

**SOLUTION** The given expression is an indeterminate form of type \( \frac{0}{0} \), therefore L'Hôpital's Rule applies. We find

\[
\lim_{x \to 3} \frac{4x - 12}{x^2 - 5x + 6} = \lim_{x \to 3} \frac{4}{2x - 5} = 4.
\]
103. \( \lim_{x \to 0^+} x^{1/2} \ln x \)

**Solution** First rewrite

\( x^{1/2} \ln x = \frac{\ln x}{x^{-1/2}} \).

The rewritten expression is an indeterminate form of type \( \frac{\infty}{\infty} \), therefore L'Hôpital's Rule applies. We find

\[
\lim_{x \to 0^+} x^{1/2} \ln x = \lim_{x \to 0^+} \frac{\ln x}{x^{-1/2}} = \lim_{x \to 0^+} \frac{1/x}{-1/2x^{-3/2}} = \lim_{x \to 0^+} -\frac{x^{1/2}}{2} = 0.
\]

105. \( \lim_{\theta \to 0} \frac{2\sin \theta - \sin 2\theta}{\sin \theta - \theta \cos \theta} \)

**Solution** The given expression is an indeterminate form of type \( \frac{0}{0} \); hence, we may apply L'Hôpital's Rule. We find

\[
\lim_{\theta \to 0} \frac{2\sin \theta - \sin 2\theta}{\sin \theta - \theta \cos \theta} = \lim_{\theta \to 0} \frac{2\cos \theta - 2\cos 2\theta}{\cos \theta - (\cos \theta - \theta \sin \theta)} = \lim_{\theta \to 0} \frac{2\cos \theta - 2\cos 2\theta}{\theta \sin \theta} = \lim_{\theta \to 0} \frac{-2 + 2\cos 2\theta}{\theta \sin \theta + \theta \cos \theta} = \lim_{\theta \to 0} \frac{-2 + 8\cos 2\theta}{\theta \cos \theta + \cos \theta - \theta \sin \theta} = -2 + 8 = 3.
\]

107. \( \lim_{t \to \infty} \frac{\ln(t + 2)}{\log_2 t} \)

**Solution** The limit is an indeterminate form of type \( \frac{\infty}{\infty} \); hence, we may apply L'Hôpital's Rule. We find

\[
\lim_{t \to \infty} \frac{\ln(t + 2)}{\log_2 t} = \lim_{t \to \infty} \frac{t}{t + 2} = \lim_{t \to \infty} \frac{1}{1 + 2} = \ln 2.
\]

109. \( \lim_{y \to 0} \frac{\sin^{-1} y - y}{y^3} \)

**Solution** The limit is an indeterminate form of type \( \frac{0}{0} \); hence, we may apply L'Hôpital's Rule. We find

\[
\lim_{y \to 0} \frac{\sin^{-1} y - y}{y^3} = \lim_{y \to 0} \frac{\frac{1}{\sqrt{1-y^2}} - 1}{3y^2} = \lim_{y \to 0} \frac{y(1 - y^2)^{-3/2}}{6y} = \lim_{y \to 0} \frac{(1 - y^2)^{-3/2}}{6} = \frac{1}{6}.
\]

111. \( \lim_{x \to 0} \frac{\sinh(x^2)}{\cosh x - 1} \)

**Solution** The limit is an indeterminate form of type \( \frac{0}{0} \); hence, we may apply L'Hôpital's Rule. We find

\[
\lim_{x \to 0} \frac{\sinh(x^2)}{\cosh x - 1} = \lim_{x \to 0} \frac{2x \cosh(x^2)}{\sinh x} = \lim_{x \to 0} \frac{2\cosh(x^2) + 4x^2 \sinh(x^2)}{\cosh x} = \frac{2 + 0}{1} = 2.
\]

113. Let \( f(x) = e^{-Ax^2/2} \), where \( A > 0 \). Given any \( n \) numbers \( a_1, a_2, \ldots, a_n \), set

\[ \Phi(x) = f(x - a_1)f(x - a_2) \cdots f(x - a_n) \]

(a) Assume \( n = 2 \) and prove that \( \Phi(x) \) attains its maximum value at the average \( x = \frac{1}{2}(a_1 + a_2) \). Hint: Calculate \( \Phi'(x) \) using logarithmic differentiation.

(b) Show that for any \( n \), \( \Phi(x) \) attains its maximum value at \( x = \frac{1}{n}(a_1 + a_2 + \cdots + a_n) \). This fact is related to the role of \( f(x) \) (whose graph is a bell-shaped curve) in statistics.

**Solution**

(a) For \( n = 2 \) we have,

\[ \Phi(x) = f(x - a_1)f(x - a_2) = e^{-\frac{1}{2}(x-a_1)^2}e^{-\frac{1}{2}(x-a_2)^2} = e^{-\frac{1}{2}(x-a_1)^2+(x-a_2)^2}. \]

Since \( e^{-\frac{1}{2}y^2} \) is a decreasing function of \( y \), it attains its maximum value where \( y \) is minimum. Therefore, we must find the minimum value of

\[ y = (x - a_1)^2 + (x - a_2)^2 = 2x^2 - 2(a_1 + a_2)x + a_1^2 + a_2^2. \]
Now, \( y' = 4x - 2(a_1 + a_2) = 0 \) when

\[ x = \frac{a_1 + a_2}{2}. \]

We conclude that \( \Phi(x) \) attains a maximum value at this point.

(b) We have

\[ \Phi(x) = e^{-\frac{A}{2}(x-a_1)^2} \cdot e^{-\frac{A}{2}(x-a_2)^2} \cdots e^{-\frac{A}{2}(x-a_n)^2} = e^{-\frac{A}{2}(x-a_1)^2+\cdots+(x-a_n)^2}. \]

Since the function \( e^{-\frac{A}{2}y} \) is a decreasing function of \( y \), it attains a maximum value where \( y \) is minimum. Therefore we must minimize the function

\[ y = (x - a_1)^2 + (x - a_2)^2 + \cdots + (x - a_n)^2. \]

We find the critical points by solving:

\[ y' = 2(x - a_1) + 2(x - a_2) + \cdots + 2(x - a_n) = 0 \]

\[ 2nx = 2(a_1 + a_2 + \cdots + a_n) \]

\[ x = \frac{a_1 + \cdots + a_n}{n}. \]

We verify that this point corresponds the minimum value of \( y \) by examining the sign of \( y'' \) at this point: \( y'' = 2n > 0 \). We conclude that \( y \) attains a minimum value at the point \( x = \frac{a_1 + \cdots + a_n}{n} \), hence \( \Phi(x) \) attains a maximum value at this point.