10 INFINITE SERIES

10.1 Sequences

Preliminary Questions

1. What is a_4 for the sequence $a_n = n^2 - n$?

SOLUTION Substituting $n = 4$ in the expression for a_n gives

$$a_4 = 4^2 - 4 = 12.$$

2. Which of the following sequences converge to zero?

(a) $\dfrac{n^2}{n^2+1}$ **(b)** 2^n **(c)** $\left(\dfrac{-1}{2}\right)^n$

SOLUTION

(a) This sequence does not converge to zero:

$$\lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{x\to\infty} \frac{x^2}{x^2+1} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = \frac{1}{1+0} = 1.$$

(b) This sequence does not converge to zero: this is a geometric sequence with $r = 2 > 1$; hence, the sequence diverges to ∞.

(c) Recall that if $|a_n|$ converges to 0, then a_n must also converge to zero. Here,

$$\left|\left(-\frac{1}{2}\right)^n\right| = \left(\frac{1}{2}\right)^n,$$

which is a geometric sequence with $0 < r < 1$; hence, $(\frac{1}{2})^n$ converges to zero. It therefore follows that $(-\frac{1}{2})^n$ converges to zero.

3. Let a_n be the nth decimal approximation to $\sqrt{2}$. That is, $a_1 = 1$, $a_2 = 1.4$, $a_3 = 1.41$, etc. What is $\lim_{n\to\infty} a_n$?

SOLUTION $\lim_{n\to\infty} a_n = \sqrt{2}$.

4. Which of the following sequences is defined recursively?

(a) $a_n = \sqrt{4+n}$ **(b)** $b_n = \sqrt{4+b_{n-1}}$

SOLUTION

(a) a_n can be computed directly, since it depends on n only and not on preceding terms. Therefore a_n is defined explicitly and not recursively.

(b) b_n is computed in terms of the preceding term b_{n-1}, hence the sequence $\{b_n\}$ is defined recursively.

5. Theorem 5 says that every convergent sequence is bounded. Determine if the following statements are true or false and if false, give a counterexample.

(a) If $\{a_n\}$ is bounded, then it converges.

(b) If $\{a_n\}$ is not bounded, then it diverges.

(c) If $\{a_n\}$ diverges, then it is not bounded.

SOLUTION

(a) This statement is false. The sequence $a_n = \cos \pi n$ is bounded since $-1 \le \cos \pi n \le 1$ for all n, but it does not converge: since $a_n = \cos n\pi = (-1)^n$, the terms assume the two values 1 and -1 alternately, hence they do not approach one value.

(b) By Theorem 5, a converging sequence must be bounded. Therefore, if a sequence is not bounded, it certainly does not converge.

(c) The statement is false. The sequence $a_n = (-1)^n$ is bounded, but it does not approach one limit.

Exercises

1. Match each sequence with its general term:

$a_1, a_2, a_3, a_4, \dots$	**General term**
(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$	(i) $\cos \pi n$
(b) $-1, 1, -1, 1, \dots$	(ii) $\frac{n!}{2^n}$
(c) $1, -1, 1, -1, \dots$	(iii) $(-1)^{n+1}$
(d) $\frac{1}{2}, \frac{2}{4}, \frac{6}{8}, \frac{24}{16} \dots$	(iv) $\frac{n}{n+1}$

SOLUTION

(a) The numerator of each term is the same as the index of the term, and the denominator is one more than the numerator; hence $a_n = \frac{n}{n+1}$, $n = 1, 2, 3, \dots$.

(b) The terms of this sequence are alternating between -1 and 1 so that the positive terms are in the even places. Since $\cos \pi n = 1$ for even n and $\cos \pi n = -1$ for odd n, we have $a_n = \cos \pi n$, $n = 1, 2, \dots$.

(c) The terms a_n are 1 for odd n and -1 for even n. Hence, $a_n = (-1)^{n+1}$, $n = 1, 2, \dots$

(d) The numerator of each term is $n!$, and the denominator is 2^n; hence, $a_n = \frac{n!}{2^n}$, $n = 1, 2, 3, \dots$.

In Exercises 3–12, calculate the first four terms of the sequence, starting with $n = 1$.

3. $c_n = \dfrac{3^n}{n!}$

SOLUTION Setting $n = 1, 2, 3, 4$ in the formula for c_n gives

$$c_1 = \frac{3^1}{1!} = \frac{3}{1} = 3, \qquad c_2 = \frac{3^2}{2!} = \frac{9}{2},$$
$$c_3 = \frac{3^3}{3!} = \frac{27}{6} = \frac{9}{2}, \qquad c_4 = \frac{3^4}{4!} = \frac{81}{24} = \frac{27}{8}.$$

5. $a_1 = 2, \quad a_{n+1} = 2a_n^2 - 3$

SOLUTION For $n = 1, 2, 3$ we have:

$$a_2 = a_{1+1} = 2a_1^2 - 3 = 2 \cdot 4 - 3 = 5;$$
$$a_3 = a_{2+1} = 2a_2^2 - 3 = 2 \cdot 25 - 3 = 47;$$
$$a_4 = a_{3+1} = 2a_3^2 - 3 = 2 \cdot 2209 - 3 = 4415.$$

The first four terms of $\{a_n\}$ are 2, 5, 47, 4415.

7. $b_n = 5 + \cos \pi n$

SOLUTION For $n = 1, 2, 3, 4$ we have

$$b_1 = 5 + \cos \pi = 4;$$
$$b_2 = 5 + \cos 2\pi = 6;$$
$$b_3 = 5 + \cos 3\pi = 4;$$
$$b_4 = 5 + \cos 4\pi = 6.$$

The first four terms of $\{b_n\}$ are 4, 6, 4, 6.

9. $c_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$

SOLUTION

$$c_1 = 1;$$
$$c_2 = 1 + \frac{1}{2} = \frac{3}{2};$$
$$c_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{3}{2} + \frac{1}{3} = \frac{11}{6};$$
$$c_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{11}{6} + \frac{1}{4} = \frac{25}{12}.$$

11. $b_1 = 2, \quad b_2 = 3, \quad b_n = 2b_{n-1} + b_{n-2}$

SOLUTION We need to find b_3 and b_4. Setting $n = 3$ and $n = 4$ and using the given values for b_1 and b_2 we obtain:

$$b_3 = 2b_{3-1} + b_{3-2} = 2b_2 + b_1 = 2 \cdot 3 + 2 = 8;$$
$$b_4 = 2b_{4-1} + b_{4-2} = 2b_3 + b_2 = 2 \cdot 8 + 3 = 19.$$

The first four terms of the sequence $\{b_n\}$ are 2, 3, 8, 19.

13. Find a formula for the nth term of each sequence.

(a) $\frac{1}{1}, \frac{-1}{8}, \frac{1}{27}, \dots$ **(b)** $\frac{2}{6}, \frac{3}{7}, \frac{4}{8}, \dots$

SOLUTION

(a) The denominators are the third powers of the positive integers starting with $n = 1$. Also, the sign of the terms is alternating with the sign of the first term being positive. Thus,

$$a_1 = \frac{1}{1^3} = \frac{(-1)^{1+1}}{1^3}; \quad a_2 = -\frac{1}{2^3} = \frac{(-1)^{2+1}}{2^3}; \quad a_3 = \frac{1}{3^3} = \frac{(-1)^{3+1}}{3^3}.$$

This rule leads to the following formula for the nth term:

$$a_n = \frac{(-1)^{n+1}}{n^3}.$$

(b) Assuming a starting index of $n = 1$, we see that each numerator is one more than the index and the denominator is four more than the numerator. Thus, the general term a_n is

$$a_n = \frac{n+1}{n+5}.$$

In Exercises 15–26, use Theorem 1 to determine the limit of the sequence or state that the sequence diverges.

15. $a_n = 12$

SOLUTION We have $a_n = f(n)$ where $f(x) = 12$; thus,

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} 12 = 12.$$

17. $b_n = \frac{5n-1}{12n+9}$

SOLUTION We have $b_n = f(n)$ where $f(x) = \frac{5x-1}{12x+9}$; thus,

$$\lim_{n\to\infty} \frac{5n-1}{12n+9} = \lim_{x\to\infty} \frac{5x-1}{12x+9} = \frac{5}{12}.$$

19. $c_n = -2^{-n}$

SOLUTION We have $c_n = f(n)$ where $f(x) = -2^{-x}$; thus,

$$\lim_{n\to\infty} \left(-2^{-n}\right) = \lim_{x\to\infty} -2^{-x} = \lim_{x\to\infty} -\frac{1}{2^x} = 0.$$

21. $c_n = 9^n$

SOLUTION We have $c_n = f(n)$ where $f(x) = 9^x$; thus,

$$\lim_{n\to\infty} 9^n = \lim_{x\to\infty} 9^x = \infty$$

Thus, the sequence 9^n diverges.

23. $a_n = \frac{n}{\sqrt{n^2+1}}$

SOLUTION We have $a_n = f(n)$ where $f(x) = \frac{x}{\sqrt{x^2+1}}$; thus,

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{\sqrt{x^2+1}}{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{\frac{x^2+1}{x^2}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = \frac{1}{\sqrt{1+0}} = 1.$$

25. $a_n = \ln\left(\dfrac{12n+2}{-9+4n}\right)$

SOLUTION We have $a_n = f(n)$ where $f(x) = \ln\left(\dfrac{12x+2}{-9+4x}\right)$; thus,

$$\lim_{n\to\infty} \ln\left(\frac{12n+2}{-9+4n}\right) = \lim_{x\to\infty} \ln\left(\frac{12x+2}{-9+4x}\right) = \ln \lim_{x\to\infty}\left(\frac{12x+2}{-9+4x}\right) = \ln 3$$

In Exercises 27–30, use Theorem 4 to determine the limit of the sequence.

27. $a_n = \sqrt{4+\dfrac{1}{n}}$

SOLUTION We have

$$\lim_{n\to\infty} 4 + \frac{1}{n} = \lim_{x\to\infty} 4 + \frac{1}{x} = 4$$

Since $\sqrt{x}$ is a continuous function for $x > 0$, Theorem 4 tells us that

$$\lim_{n\to\infty} \sqrt{4+\frac{1}{n}} = \sqrt{\lim_{n\to\infty} 4 + \frac{1}{n}} = \sqrt{4} = 2$$

29. $a_n = \cos^{-1}\left(\dfrac{n^3}{2n^3+1}\right)$

SOLUTION We have

$$\lim_{n\to\infty} \frac{n^3}{2n^3+1} = \frac{1}{2}$$

Since $\cos^{-1}(x)$ is continuous for all x, Theorem 4 tells us that

$$\lim_{n\to\infty} \cos^{-1}\left(\frac{n^3}{2n^3+1}\right) = \cos^{-1}\left(\lim_{n\to\infty} \frac{n^3}{2n^3+1}\right) = \cos^{-1}(1/2) = \frac{\pi}{3}$$

31. Let $a_n = \dfrac{n}{n+1}$. Find a number M such that:

(a) $|a_n - 1| \le 0.001$ for $n \ge M$.

(b) $|a_n - 1| \le 0.00001$ for $n \ge M$.

Then use the limit definition to prove that $\lim_{n\to\infty} a_n = 1$.

SOLUTION

(a) We have

$$|a_n - 1| = \left|\frac{n}{n+1} - 1\right| = \left|\frac{n-(n+1)}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1}.$$

Therefore $|a_n - 1| \le 0.001$ provided $\frac{1}{n+1} \le 0.001$, that is, $n \ge 999$. It follows that we can take $M = 999$.

(b) By part (a), $|a_n - 1| \le 0.00001$ provided $\frac{1}{n+1} \le 0.00001$, that is, $n \ge 99999$. It follows that we can take $M = 99999$.

We now prove formally that $\lim_{n\to\infty} a_n = 1$. Using part (a), we know that

$$|a_n - 1| = \frac{1}{n+1} < \epsilon,$$

provided $n > \frac{1}{\epsilon} - 1$. Thus, Let $\epsilon > 0$ and take $M = \frac{1}{\epsilon} - 1$. Then, for $n > M$, we have

$$|a_n - 1| = \frac{1}{n+1} < \frac{1}{M+1} = \epsilon.$$

33. Use the limit definition to prove that $\lim_{n\to\infty} n^{-2} = 0$.

SOLUTION We see that

$$|n^{-2} - 0| = \left|\frac{1}{n^2}\right| = \frac{1}{n^2} < \epsilon$$

provided

$$n > \frac{1}{\sqrt{\epsilon}}.$$

Thus, let $\epsilon > 0$ and take $M = \frac{1}{\sqrt{\epsilon}}$. Then, for $n > M$, we have

$$|n^{-2} - 0| = \left|\frac{1}{n^2}\right| = \frac{1}{n^2} < \frac{1}{M^2} = \epsilon.$$

In Exercises 35–62, use the appropriate limit laws and theorems to determine the limit of the sequence or show that it diverges.

35. $a_n = 10 + \left(-\frac{1}{9}\right)^n$

SOLUTION By the Limit Laws for Sequences we have:

$$\lim_{n\to\infty}\left(10 + \left(-\frac{1}{9}\right)^n\right) = \lim_{n\to\infty} 10 + \lim_{n\to\infty}\left(-\frac{1}{9}\right)^n = 10 + \lim_{n\to\infty}\left(-\frac{1}{9}\right)^n.$$

Now,

$$-\left(\frac{1}{9}\right)^n \le \left(-\frac{1}{9}\right)^n \le \left(\frac{1}{9}\right)^n.$$

Because

$$\lim_{n\to\infty}\left(\frac{1}{9}\right)^n = 0,$$

by the Limit Laws for Sequences,

$$\lim_{n\to\infty} -\left(\frac{1}{9}\right)^n = -\lim_{n\to\infty}\left(\frac{1}{9}\right)^n = 0.$$

Thus, we have

$$\lim_{n\to\infty}\left(-\frac{1}{9}\right)^n = 0,$$

and

$$\lim_{n\to\infty}\left(10 + \left(-\frac{1}{9}\right)^n\right) = 10 + 0 = 10.$$

37. $c_n = 1.01^n$

SOLUTION Since $c_n = f(n)$ where $f(x) = 1.01^x$, we have

$$\lim_{n\to\infty} 1.01^n = \lim_{x\to\infty} 1.01^x = \infty$$

so that the sequence diverges.

39. $a_n = 2^{1/n}$

SOLUTION Because 2^x is a continuous function,

$$\lim_{n\to\infty} 2^{1/n} = \lim_{x\to\infty} 2^{1/x} = 2^{\lim_{x\to\infty}(1/x)} = 2^0 = 1.$$

41. $c_n = \frac{9^n}{n!}$

SOLUTION For $n \ge 9$, write

$$c_n = \frac{9^n}{n!} = \underbrace{\frac{9}{1}\cdot\frac{9}{2}\cdots\frac{9}{9}}_{\text{call this } C}\cdot\underbrace{\frac{9}{10}\cdot\frac{9}{11}\cdots\frac{9}{n-1}\cdot\frac{9}{n}}_{\text{Each factor is less than 1}}$$

Then clearly

$$0 \le \frac{9^n}{n!} \le C\frac{9}{n}$$

since each factor after the first nine is < 1. The squeeze theorem tells us that

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{9^n}{n!} \le \lim_{n\to\infty} C\frac{9}{n} = C \lim_{n\to\infty} \frac{9}{n} = C \cdot 0 = 0$$

so that $\lim_{n\to\infty} c_n = 0$ as well.

43. $a_n = \dfrac{3n^2+n+2}{2n^2-3}$

SOLUTION

$$\lim_{n\to\infty} \frac{3n^2+n+2}{2n^2-3} = \lim_{x\to\infty} \frac{3x^2+x+2}{2x^2-3} = \frac{3}{2}.$$

45. $a_n = \dfrac{\cos n}{n}$

SOLUTION Since $-1 \le \cos n \le 1$ the following holds:

$$-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}.$$

We now apply the Squeeze Theorem for Sequences and the limits

$$\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

to conclude that $\lim\limits_{n\to\infty} \frac{\cos n}{n} = 0$.

47. $d_n = \ln 5^n - \ln n!$

SOLUTION Note that

$$d_n = \ln \frac{5^n}{n!}$$

so that

$$e^{d_n} = \frac{5^n}{n!} \quad \text{so} \quad \lim_{n\to\infty} e^{d_n} = \lim_{n\to\infty} \frac{5^n}{n!} = 0$$

by the method of Exercise 41. If d_n converged, we could, since $f(x) = e^x$ is continuous, then write

$$\lim_{n\to\infty} e^{d_n} = e^{\lim_{n\to\infty} d_n} = 0$$

which is impossible. Thus $\{d_n\}$ diverges.

49. $a_n = \left(2 + \dfrac{4}{n^2}\right)^{1/3}$

SOLUTION Let $a_n = \left(2 + \frac{4}{n^2}\right)^{1/3}$. Taking the natural logarithm of both sides of this expression yields

$$\ln a_n = \ln\left(2 + \frac{4}{n^2}\right)^{1/3} = \frac{1}{3}\ln\left(2 + \frac{4}{n^2}\right).$$

Thus,

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{1}{3}\ln\left(2 + \frac{4}{n^2}\right)^{1/3} = \frac{1}{3}\lim_{x\to\infty} \ln\left(2 + \frac{4}{x^2}\right) = \frac{1}{3}\ln\left(\lim_{x\to\infty}\left(2 + \frac{4}{x^2}\right)\right)$$
$$= \frac{1}{3}\ln(2+0) = \frac{1}{3}\ln 2 = \ln 2^{1/3}.$$

Because $f(x) = e^x$ is a continuous function, it follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln a_n} = e^{\lim_{n\to\infty}(\ln a_n)} = e^{\ln 2^{1/3}} = 2^{1/3}.$$

51. $c_n = \ln\left(\dfrac{2n+1}{3n+4}\right)$

SOLUTION Because $f(x) = \ln x$ is a continuous function, it follows that

$$\lim_{n\to\infty} c_n = \lim_{x\to\infty} \ln\left(\frac{2x+1}{3x+4}\right) = \ln\left(\lim_{x\to\infty} \frac{2x+1}{3x+4}\right) = \ln\frac{2}{3}.$$

53. $y_n = \dfrac{e^n}{2^n}$

SOLUTION $\frac{e^n}{2^n} = \left(\frac{e}{2}\right)^n$ and $\frac{e}{2} > 1$. By the Limit of Geometric Sequences,we conclude that $\lim_{n\to\infty} \left(\frac{e}{2}\right)^n = \infty$. Thus, the given sequence diverges.

55. $y_n = \dfrac{e^n + (-3)^n}{5^n}$

SOLUTION

$$\lim_{n\to\infty} \frac{e^n + (-3)^n}{5^n} = \lim_{n\to\infty} \left(\frac{e}{5}\right)^n + \lim_{n\to\infty} \left(\frac{-3}{5}\right)^n$$

assuming both limits on the right-hand side exist. But by the Limit of Geometric Sequences, since

$$-1 < \frac{-3}{5} < 0 < \frac{e}{5} < 1$$

both limits on the right-hand side are 0, so that y_n converges to 0.

57. $a_n = n \sin \dfrac{\pi}{n}$

SOLUTION By the Theorem on Sequences Defined by a Function, we have

$$\lim_{n\to\infty} n \sin \frac{\pi}{n} = \lim_{x\to\infty} x \sin \frac{\pi}{x}.$$

Now,

$$\lim_{x\to\infty} x \sin \frac{\pi}{x} = \lim_{x\to\infty} \frac{\sin \frac{\pi}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\left(\cos \frac{\pi}{x}\right)\left(-\frac{\pi}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \left(\pi \cos \frac{\pi}{x}\right)$$

$$= \pi \lim_{x\to\infty} \cos \frac{\pi}{x} = \pi \cos 0 = \pi \cdot 1 = \pi.$$

Thus,

$$\lim_{n\to\infty} n \sin \frac{\pi}{n} = \pi.$$

59. $b_n = \dfrac{3 - 4^n}{2 + 7 \cdot 4^n}$

SOLUTION Divide the numerator and denominator by 4^n to obtain

$$a_n = \frac{3 - 4^n}{2 + 7 \cdot 4^n} = \frac{\frac{3}{4^n} - \frac{4^n}{4^n}}{\frac{2}{4^n} + \frac{7 \cdot 4^n}{4^n}} = \frac{\frac{3}{4^n} - 1}{\frac{2}{4^n} + 7}.$$

Thus,

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{\frac{3}{4^x} - 1}{\frac{2}{4^x} + 7} = \frac{\lim_{x\to\infty}\left(\frac{3}{4^x} - 1\right)}{\lim_{x\to\infty}\left(\frac{2}{4^x} + 7\right)} = \frac{3 \lim_{x\to\infty} \frac{1}{4^x} - \lim_{x\to\infty} 1}{2 \lim_{x\to\infty} \frac{1}{4^x} - \lim_{x\to\infty} 7} = \frac{3 \cdot 0 - 1}{2 \cdot 0 + 7} = -\frac{1}{7}.$$

61. $a_n = \left(1 + \dfrac{1}{n}\right)^n$

SOLUTION Taking the natural logarithm of both sides of this expression yields

$$\ln a_n = \ln \left(1 + \frac{1}{n}\right)^n = n \ln \left(1 + \frac{1}{n}\right) = \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}}.$$

Thus,

$$\lim_{n\to\infty} (\ln a_n) = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{d}{dx}\left(\ln\left(1 + \frac{1}{x}\right)\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1 + \frac{1}{x}} = \frac{1}{1+0} = 1.$$

Because $f(x) = e^x$ is a continuous function, it follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln a_n} = e^{\lim_{n\to\infty} (\ln a_n)} = e^1 = e.$$

In Exercises 63–66, find the limit of the sequence using L'Hôpital's Rule.

63. $a_n = \dfrac{(\ln n)^2}{n}$

SOLUTION

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln x)^2}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{\frac{2\ln x}{x}}{1} = \lim_{x\to\infty} \frac{2\ln x}{x}$$

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}2\ln x}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{\frac{2}{x}}{1} = \lim_{x\to\infty} \frac{2}{x} = 0$$

65. $c_n = n\left(\sqrt{n^2+1} - n\right)$

SOLUTION

$$\lim_{n\to\infty} n\left(\sqrt{n^2+1} - n\right) = \lim_{x\to\infty} x\left(\sqrt{x^2+1} - x\right) = \lim_{x\to\infty} \frac{x\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)}{\sqrt{x^2+1} + x}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1} + x} = \lim_{x\to\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}\sqrt{x^2+1} + x} = \lim_{x\to\infty} \frac{1}{1 + \frac{x}{\sqrt{x^2+1}}}$$

$$= \lim_{x\to\infty} \frac{1}{1 + \sqrt{\frac{x^2}{x^2+1}}} = \lim_{x\to\infty} \frac{1}{1 + \sqrt{\frac{1}{1+(1/x^2)}}} = \frac{1}{2}$$

In Exercises 67–70, use the Squeeze Theorem to evaluate $\lim_{n\to\infty} a_n$ by verifying the given inequality.

67. $a_n = \dfrac{1}{\sqrt{n^4+n^8}}, \quad \dfrac{1}{\sqrt{2n^4}} \le a_n \le \dfrac{1}{\sqrt{2n^2}}$

SOLUTION For all $n > 1$ we have $n^4 < n^8$, so the quotient $\frac{1}{\sqrt{n^4+n^8}}$ is smaller than $\frac{1}{\sqrt{n^4+n^4}}$ and larger than $\frac{1}{\sqrt{n^8+n^8}}$. That is,

$$a_n < \frac{1}{\sqrt{n^4+n^4}} = \frac{1}{\sqrt{n^4 \cdot 2}} = \frac{1}{\sqrt{2}n^2}; \text{ and}$$

$$a_n > \frac{1}{\sqrt{n^8+n^8}} = \frac{1}{\sqrt{2n^8}} = \frac{1}{\sqrt{2}n^4}.$$

Now, since $\lim_{n\to\infty} \dfrac{1}{\sqrt{2}n^4} = \lim_{n\to\infty} \dfrac{1}{\sqrt{2}n^2} = 0$, the Squeeze Theorem for Sequences implies that $\lim_{n\to\infty} a_n = 0$.

69. $a_n = (2^n + 3^n)^{1/n}, \quad 3 \le a_n \le (2 \cdot 3^n)^{1/n} = 2^{1/n} \cdot 3$

SOLUTION Clearly $2^n + 3^n \ge 3^n$ for all $n \ge 1$. Therefore:

$$(2^n + 3^n)^{1/n} \ge (3^n)^{1/n} = 3.$$

Also $2^n + 3^n \le 3^n + 3^n = 2 \cdot 3^n$, so

$$(2^n + 3^n)^{1/n} \le (2 \cdot 3^n)^{1/n} = 2^{1/n} \cdot 3.$$

Thus,

$$3 \le (2^n + 3^n)^{1/n} \le 2^{1/n} \cdot 3.$$

Because

$$\lim_{n\to\infty} 2^{1/n} \cdot 3 = 3 \lim_{n\to\infty} 2^{1/n} = 3 \cdot 1 = 3$$

and $\lim_{n\to\infty} 3 = 3$, the Squeeze Theorem for Sequences guarantees

$$\lim_{n\to\infty} (2^n + 3^n)^{1/n} = 3.$$

71. Which of the following statements is equivalent to the assertion $\lim_{n\to\infty} a_n = L$? Explain.

(a) For every $\epsilon > 0$, the interval $(L - \epsilon, L + \epsilon)$ contains at least one element of the sequence $\{a_n\}$.
(b) For every $\epsilon > 0$, the interval $(L - \epsilon, L + \epsilon)$ contains all but at most finitely many elements of the sequence $\{a_n\}$.

SOLUTION Statement (b) is equivalent to Definition 1 of the limit, since the assertion "$|a_n - L| < \epsilon$ for all $n > M$" means that $L - \epsilon < a_n < L + \epsilon$ for all $n > M$; that is, the interval $(L - \epsilon, L + \epsilon)$ contains all the elements a_n except (maybe) the finite number of elements $a_1, a_2, \ldots, a_M$.

Statement (a) is not equivalent to the assertion $\lim_{n\to\infty} a_n = L$. We show this, by considering the following sequence:

$$a_n = \begin{cases} \dfrac{1}{n} & \text{for odd } n \\ 1 + \dfrac{1}{n} & \text{for even } n \end{cases}$$

Clearly for every $\epsilon > 0$, the interval $(-\epsilon, \epsilon) = (L - \epsilon, L + \epsilon)$ for $L = 0$ contains at least one element of $\{a_n\}$, but the sequence diverges (rather than converges to $L = 0$). Since the terms in the odd places converge to 0 and the terms in the even places converge to 1. Hence, a_n does not approach one limit.

73. Show that $a_n = \dfrac{3n^2}{n^2+2}$ is increasing. Find an upper bound.

SOLUTION Let $f(x) = \frac{3x^2}{x^2+2}$. Then

$$f'(x) = \frac{6x(x^2+2) - 3x^2 \cdot 2x}{(x^2+2)^2} = \frac{12x}{(x^2+2)^2}.$$

$f'(x) > 0$ for $x > 0$, hence f is increasing on this interval. It follows that $a_n = f(n)$ is also increasing. We now show that $M = 3$ is an upper bound for a_n, by writing:

$$a_n = \frac{3n^2}{n^2+2} \le \frac{3n^2+6}{n^2+2} = \frac{3(n^2+2)}{n^2+2} = 3.$$

That is, $a_n \le 3$ for all n.

75. Give an example of a divergent sequence $\{a_n\}$ such that $\lim_{n\to\infty} |a_n|$ converges.

SOLUTION Let $a_n = (-1)^n$. The sequence $\{a_n\}$ diverges because the terms alternate between $+1$ and -1; however, the sequence $\{|a_n|\}$ converges because it is a constant sequence, all of whose terms are equal to 1.

77. Using the limit definition, prove that if $\{a_n\}$ converges and $\{b_n\}$ diverges, then $\{a_n + b_n\}$ diverges.

SOLUTION We will prove this result by contradiction. Suppose $\lim_{n\to\infty} a_n = L_1$ and that $\{a_n + b_n\}$ converges to a limit L_2. Now, let $\epsilon > 0$. Because $\{a_n\}$ converges to L_1 and $\{a_n + b_n\}$ converges to L_2, it follows that there exist numbers M_1 and M_2 such that:

$$|a_n - L_1| < \frac{\epsilon}{2} \qquad \text{for all } n > M_1,$$
$$|(a_n + b_n) - L_2| < \frac{\epsilon}{2} \qquad \text{for all } n > M_2.$$

Thus, for $n > M = \max\{M_1, M_2\}$,

$$|a_n - L_1| < \frac{\epsilon}{2} \quad \text{and} \quad |(a_n + b_n) - L_2| < \frac{\epsilon}{2}.$$

By the triangle inequality,

$$|b_n - (L_2 - L_1)| = |a_n + b_n - a_n - (L_2 - L_1)| = |(-a_n + L_1) + (a_n + b_n - L_2)|$$
$$\le |L_1 - a_n| + |a_n + b_n - L_2|.$$

Thus, for $n > M$,

$$|b_n - (L_2 - L_1)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon;$$

that is, $\{b_n\}$ converges to $L_2 - L_1$, in contradiction to the given data. Thus, $\{a_n + b_n\}$ must diverge.

79. Theorem 1 states that if $\lim_{x\to\infty} f(x) = L$, then the sequence $a_n = f(n)$ converges and $\lim_{n\to\infty} a_n = L$. Show that the *converse* is false. In other words, find a function $f(x)$ such that $a_n = f(n)$ converges but $\lim_{x\to\infty} f(x)$ does not exist.

SOLUTION Let $f(x) = \sin \pi x$ and $a_n = \sin \pi n$. Then $a_n = f(n)$. Since $\sin \pi x$ is oscillating between -1 and 1 the limit $\lim_{x\to\infty} f(x)$ does not exist. However, the sequence $\{a_n\}$ is the constant sequence in which $a_n = \sin \pi n = 0$ for all n, hence it converges to zero.

81. Let $b_n = a_{n+1}$. Use the limit definition to prove that if $\{a_n\}$ converges, then $\{b_n\}$ also converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

SOLUTION Suppose $\{a_n\}$ converges to L. Let $b_n = a_{n+1}$, and let $\epsilon > 0$. Because $\{a_n\}$ converges to L, there exists an M' such that $|a_n - L| < \epsilon$ for $n > M'$. Now, let $M = M' - 1$. Then, whenever $n > M$, $n + 1 > M + 1 = M'$. Thus, for $n > M$,

$$|b_n - L| = |a_{n+1} - L| < \epsilon.$$

Hence, $\{b_n\}$ converges to L.

83. Proceed as in Example 12 to show that the sequence $\sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, \ldots$ is increasing and bounded above by $M = 3$. Then prove that the limit exists and find its value.

SOLUTION This sequence is defined recursively by the formula:

$$a_{n+1} = \sqrt{3a_n}, \qquad a_1 = \sqrt{3}.$$

Consider the following inequalities:

$$\begin{aligned} a_2 &= \sqrt{3a_1} = \sqrt{3\sqrt{3}} > \sqrt{3} = a_1 &&\Rightarrow \quad a_2 > a_1; \\ a_3 &= \sqrt{3a_2} > \sqrt{3a_1} = a_2 &&\Rightarrow \quad a_3 > a_2; \\ a_4 &= \sqrt{3a_3} > \sqrt{3a_2} = a_3 &&\Rightarrow \quad a_4 > a_3. \end{aligned}$$

In general, if we assume that $a_k > a_{k-1}$, then

$$a_{k+1} = \sqrt{3a_k} > \sqrt{3a_{k-1}} = a_k.$$

Hence, by mathematical induction, $a_{n+1} > a_n$ for all n; that is, the sequence $\{a_n\}$ is increasing.

Because $a_{n+1} = \sqrt{3a_n}$, it follows that $a_n \geq 0$ for all n. Now, $a_1 = \sqrt{3} < 3$. If $a_k \leq 3$, then

$$a_{k+1} = \sqrt{3a_k} \leq \sqrt{3 \cdot 3} = 3.$$

Thus, by mathematical induction, $a_n \leq 3$ for all n.

Since $\{a_n\}$ is increasing and bounded, it follows by the Theorem on Bounded Monotonic Sequences that this sequence is converging. Denote the limit by $L = \lim_{n\to\infty} a_n$. Using Exercise 81, it follows that

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{3a_n} = \sqrt{3 \lim_{n\to\infty} a_n} = \sqrt{3L}.$$

Thus, $L^2 = 3L$, so $L = 0$ or $L = 3$. Because the sequence is increasing, we have $a_n \geq a_1 = \sqrt{3}$ for all n. Hence, the limit also satisfies $L \geq \sqrt{3}$. We conclude that the appropriate solution is $L = 3$; that is, $\lim_{n\to\infty} a_n = 3$.

Further Insights and Challenges

85. Show that $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$. *Hint:* Verify that $n! \geq (n/2)^{n/2}$ by observing that half of the factors of $n!$ are greater than or equal to $n/2$.

SOLUTION We show that $n! \geq \left(\frac{n}{2}\right)^{n/2}$. For $n \geq 4$ even, we have:

$$n! = \underbrace{1 \cdot \cdots \cdot \frac{n}{2}}_{\frac{n}{2}\text{ factors}} \cdot \underbrace{\left(\frac{n}{2}+1\right) \cdot \cdots \cdot n}_{\frac{n}{2}\text{ factors}} \geq \underbrace{\left(\frac{n}{2}+1\right) \cdot \cdots \cdot n}_{\frac{n}{2}\text{ factors}}.$$

Since each one of the $\frac{n}{2}$ factors is greater than $\frac{n}{2}$, we have:

$$n! \geq \underbrace{\left(\frac{n}{2}+1\right) \cdot \cdots \cdot n}_{\frac{n}{2}\text{ factors}} \geq \underbrace{\frac{n}{2} \cdot \cdots \cdot \frac{n}{2}}_{\frac{n}{2}\text{ factors}} = \left(\frac{n}{2}\right)^{n/2}.$$

For $n \geq 3$ odd, we have:

$$n! = \underbrace{1 \cdot \cdots \cdot \frac{n-1}{2}}_{\frac{n-1}{2}\text{ factors}} \cdot \underbrace{\frac{n+1}{2} \cdot \cdots \cdot n}_{\frac{n+1}{2}\text{ factors}} \geq \underbrace{\frac{n+1}{2} \cdot \cdots \cdot n}_{\frac{n+1}{2}\text{ factors}}.$$

Since each one of the $\frac{n+1}{2}$ factors is greater than $\frac{n}{2}$, we have:

$$n! \geq \underbrace{\frac{n+1}{2} \cdot \cdots \cdot n}_{\frac{n+1}{2} \text{ factors}} \geq \underbrace{\frac{n}{2} \cdot \cdots \cdot \frac{n}{2}}_{\frac{n+1}{2} \text{ factors}} = \left(\frac{n}{2}\right)^{(n+1)/2} = \left(\frac{n}{2}\right)^{n/2}\sqrt{\frac{n}{2}} \geq \left(\frac{n}{2}\right)^{n/2}.$$

In either case we have $n! \geq \left(\frac{n}{2}\right)^{n/2}$. Thus,

$$\sqrt[n]{n!} \geq \sqrt{\frac{n}{2}}.$$

Since $\lim_{n\to\infty} \sqrt{\frac{n}{2}} = \infty$, it follows that $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$. Thus, the sequence $a_n = \sqrt[n]{n!}$ diverges.

87. Given positive numbers $a_1 < b_1$, define two sequences recursively by

$$a_{n+1} = \sqrt{a_n b_n}, \qquad b_{n+1} = \frac{a_n + b_n}{2}$$

(a) Show that $a_n \leq b_n$ for all n (Figure 13).
(b) Show that $\{a_n\}$ is increasing and $\{b_n\}$ is decreasing.
(c) Show that $b_{n+1} - a_{n+1} \leq \dfrac{b_n - a_n}{2}$.
(d) Prove that both $\{a_n\}$ and $\{b_n\}$ converge and have the same limit. This limit, denoted AGM(a_1, b_1), is called the **arithmetic-geometric mean** of a_1 and b_1.
(e) Estimate AGM$(1, \sqrt{2})$ to three decimal places.

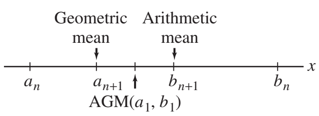


FIGURE 13

SOLUTION

(a) Examine the following:

$$b_{n+1} - a_{n+1} = \frac{a_n + b_n}{2} - \sqrt{a_n b_n} = \frac{a_n + b_n - 2\sqrt{a_n b_n}}{2} = \frac{\left(\sqrt{a_n}\right)^2 - 2\sqrt{a_n}\sqrt{b_n} + \left(\sqrt{b_n}\right)^2}{2}$$
$$= \frac{\left(\sqrt{a_n} - \sqrt{b_n}\right)^2}{2} \geq 0.$$

We conclude that $b_{n+1} \geq a_{n+1}$ for all $n > 1$. By the given information $b_1 > a_1$; hence, $b_n \geq a_n$ for all n.

(b) By part (a), $b_n \geq a_n$ for all n, so

$$a_{n+1} = \sqrt{a_n b_n} \geq \sqrt{a_n \cdot a_n} = \sqrt{a_n^2} = a_n$$

for all n. Hence, the sequence $\{a_n\}$ is increasing. Moreover, since $a_n \leq b_n$ for all n,

$$b_{n+1} = \frac{a_n + b_n}{2} \leq \frac{b_n + b_n}{2} = \frac{2b_n}{2} = b_n$$

for all n; that is, the sequence $\{b_n\}$ is decreasing.

(c) Since $\{a_n\}$ is increasing, $a_{n+1} \geq a_n$. Thus,

$$b_{n+1} - a_{n+1} \leq b_{n+1} - a_n = \frac{a_n + b_n}{2} - a_n = \frac{a_n + b_n - 2a_n}{2} = \frac{b_n - a_n}{2}.$$

Now, by part (a), $a_n \leq b_n$ for all n. By part (b), $\{b_n\}$ is decreasing. Hence $b_n \leq b_1$ for all n. Combining the two inequalities we conclude that $a_n \leq b_1$ for all n. That is, the sequence $\{a_n\}$ is increasing and bounded ($0 \leq a_n \leq b_1$). By the Theorem on Bounded Monotonic Sequences we conclude that $\{a_n\}$ converges. Similarly, since $\{a_n\}$ is increasing, $a_n \geq a_1$ for all n. We combine this inequality with $b_n \geq a_n$ to conclude that $b_n \geq a_1$ for all n. Thus, $\{b_n\}$ is decreasing and bounded ($a_1 \leq b_n \leq b_1$); hence this sequence converges.

To show that $\{a_n\}$ and $\{b_n\}$ converge to the same limit, note that

$$b_n - a_n \leq \frac{b_{n-1} - a_{n-1}}{2} \leq \frac{b_{n-2} - a_{n-2}}{2^2} \leq \cdots \leq \frac{b_1 - a_1}{2^{n-1}}.$$

Thus,

$$\lim_{n\to\infty}(b_n - a_n) = (b_1 - a_1)\lim_{n\to\infty}\frac{1}{2^{n-1}} = 0.$$

(d) We have

$$a_{n+1} = \sqrt{a_n b_n}, \quad a_1 = 1; \quad b_{n+1} = \frac{a_n + b_n}{2}, \quad b_1 = \sqrt{2}$$

Computing the values of a_n and b_n until the first three decimal digits are equal in successive terms, we obtain:

$$a_2 = \sqrt{a_1 b_1} = \sqrt{1 \cdot \sqrt{2}} = 1.1892$$
$$b_2 = \frac{a_1 + b_1}{2} = \frac{1 + \sqrt{2}}{2} = 1.2071$$
$$a_3 = \sqrt{a_2 b_2} = \sqrt{1.1892 \cdot 1.2071} = 1.1981$$
$$b_3 = \frac{a_2 + b_2}{2} = \frac{1.1892 \cdot 1.2071}{2} = 1.1981$$
$$a_4 = \sqrt{a_3 b_3} = 1.1981$$
$$b_4 = \frac{a_3 + b_3}{2} = 1.1981$$

Thus,

$$AGM\left(1, \sqrt{2}\right) \approx 1.198.$$

89. Let $a_n = H_n - \ln n$, where H_n is the nth harmonic number

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

(a) Show that $a_n \ge 0$ for $n \ge 1$. *Hint:* Show that $H_n \ge \int_1^{n+1} \frac{dx}{x}$.

(b) Show that $\{a_n\}$ is decreasing by interpreting $a_n - a_{n+1}$ as an area.

(c) Prove that $\lim_{n\to\infty} a_n$ exists.

This limit, denoted γ, is known as *Euler's Constant*. It appears in many areas of mathematics, including analysis and number theory, and has been calculated to more than 100 million decimal places, but it is still not known whether γ is an irrational number. The first 10 digits are $\gamma \approx 0.5772156649$.

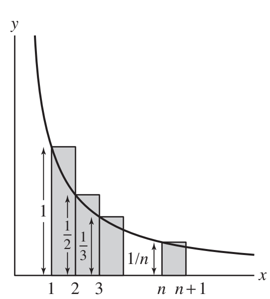
SOLUTION

(a) Since the function $y = \frac{1}{x}$ is decreasing, the left endpoint approximation to the integral $\int_1^{n+1} \frac{dx}{x}$ is greater than this integral; that is,

$$1 \cdot 1 + \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 + \cdots + \frac{1}{n} \cdot 1 \ge \int_1^{n+1} \frac{dx}{x}$$

or

$$H_n \ge \int_1^{n+1} \frac{dx}{x}.$$



Moreover, since the function $y = \frac{1}{x}$ is positive for $x > 0$, we have:

$$\int_1^{n+1} \frac{dx}{x} \ge \int_1^{n} \frac{dx}{x}.$$

Thus,

$$H_n \geq \int_1^n \frac{dx}{x} = \ln x \Big|_1^n = \ln n - \ln 1 = \ln n,$$

and

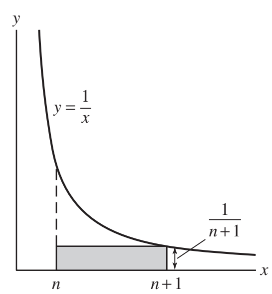
$$a_n = H_n - \ln n \geq 0 \qquad \text{for all } n \geq 1.$$

(b) To show that $\{a_n\}$ is decreasing, we consider the difference $a_n - a_{n+1}$:

$$\begin{aligned} a_n - a_{n+1} &= H_n - \ln n - \big(H_{n+1} - \ln(n+1)\big) = H_n - H_{n+1} + \ln(n+1) - \ln n \\ &= 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1}\right) + \ln(n+1) - \ln n \\ &= -\frac{1}{n+1} + \ln(n+1) - \ln n. \end{aligned}$$

Now, $\ln(n+1) - \ln n = \int_n^{n+1} \frac{dx}{x}$, whereas $\frac{1}{n+1}$ is the right endpoint approximation to the integral $\int_n^{n+1} \frac{dx}{x}$. Recalling $y = \frac{1}{x}$ is decreasing, it follows that

$$\int_n^{n+1} \frac{dx}{x} \geq \frac{1}{n+1}$$



so

$$a_n - a_{n+1} \geq 0.$$

(c) By parts (a) and (b), $\{a_n\}$ is decreasing and 0 is a lower bound for this sequence. Hence $0 \leq a_n \leq a_1$ for all n. A monotonic and bounded sequence is convergent, so $\lim_{n\to\infty} a_n$ exists.

10.2 Summing an Infinite Series

Preliminary Questions

1. What role do partial sums play in defining the sum of an infinite series?

SOLUTION The sum of an infinite series is defined as the limit of the sequence of partial sums. If the limit of this sequence does not exist, the series is said to diverge.

2. What is the sum of the following infinite series?

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$$

SOLUTION This is a geometric series with $c = \frac{1}{4}$ and $r = \frac{1}{2}$. The sum of the series is therefore

$$\frac{\frac{1}{4}}{1 - \frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}.$$

3. What happens if you apply the formula for the sum of a geometric series to the following series? Is the formula valid?

$$1 + 3 + 3^2 + 3^3 + 3^4 + \cdots$$

SOLUTION This is a geometric series with $c = 1$ and $r = 3$. Applying the formula for the sum of a geometric series then gives

$$\sum_{n=0}^{\infty} 3^n = \frac{1}{1-3} = -\frac{1}{2}.$$

Clearly, this is not valid: a series with all positive terms cannot have a negative sum. The formula is not valid in this case because a geometric series with $r = 3$ diverges.

4. Arvind asserts that $\sum_{n=1}^{\infty} \frac{1}{n^2} = 0$ because $\frac{1}{n^2}$ tends to zero. Is this valid reasoning?

SOLUTION Arvind's reasoning is not valid. Though the terms in the series do tend to zero, the general term in the sequence of partial sums,

$$S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2},$$

is clearly larger than 1. The sum of the series therefore cannot be zero.

5. Colleen claims that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges because

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Is this valid reasoning?

SOLUTION Colleen's reasoning is not valid. Although the general term of a convergent series must tend to zero, a series whose general term tends to zero need not converge. In the case of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, the series diverges even though its general term tends to zero.

6. Find an N such that $S_N > 25$ for the series $\sum_{n=1}^{\infty} 2$.

SOLUTION The Nth partial sum of the series is:

$$S_N = \sum_{n=1}^{N} 2 = \underbrace{2 + \cdots + 2}_{N} = 2N.$$

7. Does there exist an N such that $S_N > 25$ for the series $\sum_{n=1}^{\infty} 2^{-n}$? Explain.

SOLUTION The series $\sum_{n=1}^{\infty} 2^{-n}$ is a convergent geometric series with the common ratio $r = \frac{1}{2}$. The sum of the series is:

$$S = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

Notice that the sequence of partial sums $\{S_N\}$ is increasing and converges to 1; therefore $S_N \le 1$ for all N. Thus, there does not exist an N such that $S_N > 25$.

8. Give an example of a divergent infinite series whose general term tends to zero.

SOLUTION Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{9}{10}}}$. The general term tends to zero, since $\lim_{n\to\infty} \frac{1}{n^{\frac{9}{10}}} = 0$. However, the Nth partial sum satisfies the following inequality:

$$S_N = \frac{1}{1^{\frac{9}{10}}} + \frac{1}{2^{\frac{9}{10}}} + \cdots + \frac{1}{N^{\frac{9}{10}}} \ge \frac{N}{N^{\frac{9}{10}}} = N^{1-\frac{9}{10}} = N^{\frac{1}{10}}.$$

That is, $S_N \ge N^{\frac{1}{10}}$ for all N. Since $\lim_{N\to\infty} N^{\frac{1}{10}} = \infty$, the sequence of partial sums S_n diverges; hence, the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{9}{10}}}$ diverges.

Exercises

1. Find a formula for the general term a_n (not the partial sum) of the infinite series.

(a) $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots$

(b) $\frac{1}{1}+\frac{5}{2}+\frac{25}{4}+\frac{125}{8}+\cdots$

(c) $\frac{1}{1}-\frac{2^2}{2\cdot 1}+\frac{3^3}{3\cdot 2\cdot 1}-\frac{4^4}{4\cdot 3\cdot 2\cdot 1}+\cdots$

(d) $\frac{2}{1^2+1}+\frac{1}{2^2+1}+\frac{2}{3^2+1}+\frac{1}{4^2+1}+\cdots$

SOLUTION

(a) The denominators of the terms are powers of 3, starting with the first power. Hence, the general term is:

$$a_n = \frac{1}{3^n}.$$

(b) The numerators are powers of 5, and the denominators are the same powers of 2. The first term is $a_1 = 1$ so,

$$a_n = \left(\frac{5}{2}\right)^{n-1}.$$

(c) The general term of this series is,

$$a_n = (-1)^{n+1}\frac{n^n}{n!}.$$

(d) Notice that the numerators of a_n equal 2 for odd values of n and 1 for even values of n. Thus,

$$a_n = \begin{cases} \dfrac{2}{n^2+1} & \text{odd } n \\ \dfrac{1}{n^2+1} & \text{even } n \end{cases}$$

The formula can also be rewritten as follows:

$$a_n = \frac{1+\frac{(-1)^{n+1}+1}{2}}{n^2+1}.$$

In Exercises 3–6, compute the partial sums S_2, S_4, and S_6.

3. $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$

SOLUTION

$$S_2 = 1+\frac{1}{2^2} = \frac{5}{4};$$

$$S_4 = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2} = \frac{205}{144};$$

$$S_6 = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2} = \frac{5369}{3600}.$$

5. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$

SOLUTION

$$S_2 = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3} = \frac{1}{2}+\frac{1}{6} = \frac{4}{6} = \frac{2}{3};$$

$$S_4 = S_2+a_3+a_4 = \frac{2}{3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5} = \frac{2}{3}+\frac{1}{12}+\frac{1}{20} = \frac{4}{5};$$

$$S_6 = S_4+a_5+a_6 = \frac{4}{5}+\frac{1}{5\cdot 6}+\frac{1}{6\cdot 7} = \frac{4}{5}+\frac{1}{30}+\frac{1}{42} = \frac{6}{7}.$$

7. The series $S = 1 + \left(\frac{1}{5}\right) + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^3 + \cdots$ converges to $\frac{5}{4}$. Calculate S_N for $N = 1, 2, \ldots$ until you find an S_N that approximates $\frac{5}{4}$ with an error less than 0.0001.

SOLUTION

$$\begin{aligned}
S_1 &= 1\\
S_2 &= 1 + \frac{1}{5} = \frac{6}{5} = 1.2\\
S_3 &= 1 + \frac{1}{5} + \frac{1}{25} = \frac{31}{25} = 1.24\\
S_3 &= 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} = \frac{156}{125} = 1.248\\
S_4 &= 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} = \frac{781}{625} = 1.2496\\
S_5 &= 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \frac{1}{3125} = \frac{3906}{3125} = 1.24992
\end{aligned}$$

Note that

$$1.25 - S_5 = 1.25 - 1.24992 = 0.00008 < 0.0001$$

In Exercises 9 and 10, use a computer algebra system to compute S_{10}, S_{100}, S_{500}, and S_{1000} for the series. Do these values suggest convergence to the given value?

9. CAS

$$\frac{\pi - 3}{4} = \frac{1}{2 \cdot 3 \cdot 4} - \frac{1}{4 \cdot 5 \cdot 6} + \frac{1}{6 \cdot 7 \cdot 8} - \frac{1}{8 \cdot 9 \cdot 10} + \cdots$$

SOLUTION Write

$$a_n = \frac{(-1)^{n+1}}{2n \cdot (2n+1) \cdot (2n+2)}$$

Then

$$S_N = \sum_{i=1}^{N} a_n$$

Computing, we find

$$\begin{aligned}
\frac{\pi - 3}{4} &\approx 0.0353981635\\
S_{10} &\approx 0.03535167962\\
S_{100} &\approx 0.03539810274\\
S_{500} &\approx 0.03539816290\\
S_{1000} &\approx 0.03539816334
\end{aligned}$$

It appears that $S_N \to \frac{\pi - 3}{4}$.

11. Calculate S_3, S_4, and S_5 and then find the sum of the telescoping series

$$S = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

SOLUTION

$$\begin{aligned}
S_3 &= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) = \frac{1}{2} - \frac{1}{5} = \frac{3}{10};\\
S_4 &= S_3 + \left(\frac{1}{5} - \frac{1}{6}\right) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3};\\
S_5 &= S_4 + \left(\frac{1}{6} - \frac{1}{7}\right) = \frac{1}{2} - \frac{1}{7} = \frac{5}{14}.
\end{aligned}$$

The general term in the sequence of partial sums is

$$S_N = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{N+1} - \frac{1}{N+2}\right) = \frac{1}{2} - \frac{1}{N+2};$$

thus,

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(\frac{1}{2} - \frac{1}{N+2}\right) = \frac{1}{2}.$$

The sum of the telescoping series is therefore $\frac{1}{2}$.

13. Calculate S_3, S_4, and S_5 and then find the sum $S = \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$ using the identity

$$\frac{1}{4n^2-1} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

SOLUTION

$$S_3 = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) = \frac{1}{2}\left(1 - \frac{1}{7}\right) = \frac{3}{7};$$

$$S_4 = S_3 + \frac{1}{2}\left(\frac{1}{7} - \frac{1}{9}\right) = \frac{1}{2}\left(1 - \frac{1}{9}\right) = \frac{4}{9};$$

$$S_5 = S_4 + \frac{1}{2}\left(\frac{1}{9} - \frac{1}{11}\right) = \frac{1}{2}\left(1 - \frac{1}{11}\right) = \frac{5}{11}.$$

The general term in the sequence of partial sums is

$$S_N = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \frac{1}{2}\left(\frac{1}{2N-1} - \frac{1}{2N+1}\right) = \frac{1}{2}\left(1 - \frac{1}{2N+1}\right);$$

thus,

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1}{2}\left(1 - \frac{1}{2N+1}\right) = \frac{1}{2}.$$

15. Find the sum of $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$.

SOLUTION We may write this sum as

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right).$$

The general term in the sequence of partial sums is

$$S_N = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \frac{1}{2}\left(\frac{1}{2N-1} - \frac{1}{2N+1}\right) = \frac{1}{2}\left(1 - \frac{1}{2N+1}\right);$$

thus,

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1}{2}\left(1 - \frac{1}{2N+1}\right) = \frac{1}{2},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}.$$

In Exercises 17–22, use Theorem 3 to prove that the following series diverge.

17. $\sum_{n=1}^{\infty} \frac{n}{10n+12}$

SOLUTION The general term, $\frac{n}{10n+12}$, has limit

$$\lim_{n\to\infty} \frac{n}{10n+12} = \lim_{n\to\infty} \frac{1}{10 + (12/n)} = \frac{1}{10}$$

Since the general term does not tend to zero, the series diverges.

19. $\frac{0}{1} - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \cdots$

SOLUTION The general term $a_n = (-1)^{n-1}\frac{n-1}{n}$ does not tend to zero. In fact, because $\lim_{n\to\infty} \frac{n-1}{n} = 1$, $\lim_{n\to\infty} a_n$ does not exist. By Theorem 3, we conclude that the given series diverges.

21. $\cos\frac{1}{2} + \cos\frac{1}{3} + \cos\frac{1}{4} + \cdots$

SOLUTION The general term $a_n = \cos\frac{1}{n+1}$ tends to 1, not zero. By Theorem 3, we conclude that the given series diverges.

In Exercises 23–36, use the formula for the sum of a geometric series to find the sum or state that the series diverges.

23. $\frac{1}{1} + \frac{1}{8} + \frac{1}{8^2} + \cdots$

SOLUTION This is a geometric series with $c = 1$ and $r = \frac{1}{8}$, so its sum is

$$\frac{1}{1-\frac{1}{8}} = \frac{1}{7/8} = \frac{8}{7}$$

25. $\sum_{n=3}^{\infty} \left(\frac{3}{11}\right)^{-n}$

SOLUTION Rewrite this series as

$$\sum_{n=3}^{\infty} \left(\frac{11}{3}\right)^{n}$$

This is a geometric series with $r = \frac{11}{3} > 1$, so it is divergent.

27. $\sum_{n=-4}^{\infty} \left(-\frac{4}{9}\right)^{n}$

SOLUTION This is a geometric series with $c = 1$ and $r = -\frac{4}{9}$, starting at $n = -4$. Its sum is thus

$$\frac{cr^{-4}}{1-r} = \frac{c}{r^4 - r^5} = \frac{1}{\frac{4^4}{9^4} + \frac{4^5}{9^5}} = \frac{9^5}{9\cdot 4^4 + 4^5} = \frac{59{,}049}{3328}$$

29. $\sum_{n=1}^{\infty} e^{-n}$

SOLUTION Rewrite the series as

$$\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^{n}$$

to recognize it as a geometric series with $c = \frac{1}{e}$ and $r = \frac{1}{e}$. Thus,

$$\sum_{n=1}^{\infty} e^{-n} = \frac{\frac{1}{e}}{1-\frac{1}{e}} = \frac{1}{e-1}.$$

31. $\sum_{n=0}^{\infty} \frac{8+2^n}{5^n}$

SOLUTION Rewrite the series as

$$\sum_{n=0}^{\infty} \frac{8}{5^n} + \sum_{n=0}^{\infty} \frac{2^n}{5^n} = \sum_{n=0}^{\infty} 8\cdot\left(\frac{1}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n,$$

which is a sum of two geometric series. The first series has $c = 8\left(\frac{1}{5}\right)^0 = 8$ and $r = \frac{1}{5}$; the second has $c = \left(\frac{2}{5}\right)^0 = 1$ and $r = \frac{2}{5}$. Thus,

$$\sum_{n=0}^{\infty} 8\cdot\left(\frac{1}{5}\right)^n = \frac{8}{1-\frac{1}{5}} = \frac{8}{\frac{4}{5}} = 10,$$

$$\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = \frac{1}{1-\frac{2}{5}} = \frac{1}{\frac{3}{5}} = \frac{5}{3},$$

and

$$\sum_{n=0}^{\infty} \frac{8+2^n}{5^n} = 10 + \frac{5}{3} = \frac{35}{3}.$$

33. $5 - \frac{5}{4} + \frac{5}{4^2} - \frac{5}{4^3} + \cdots$

SOLUTION This is a geometric series with $c = 5$ and $r = -\frac{1}{4}$. Thus,

$$\sum_{n=0}^{\infty} 5 \cdot \left(-\frac{1}{4}\right)^n = \frac{5}{1 - \left(-\frac{1}{4}\right)} = \frac{5}{1 + \frac{1}{4}} = \frac{5}{\frac{5}{4}} = 4.$$

35. $\frac{7}{8} - \frac{49}{64} + \frac{343}{512} - \frac{2401}{4096} + \cdots$

SOLUTION This is a geometric series with $c = \frac{7}{8}$ and $r = -\frac{7}{8}$. Thus,

$$\sum_{n=0}^{\infty} \frac{7}{8} \cdot \left(-\frac{7}{8}\right)^n = \frac{\frac{7}{8}}{1 - \left(-\frac{7}{8}\right)} = \frac{\frac{7}{8}}{\frac{15}{8}} = \frac{7}{15}.$$

37. Which of the following are *not* geometric series?

(a) $\sum_{n=0}^{\infty} \frac{7^n}{29^n}$

(b) $\sum_{n=3}^{\infty} \frac{1}{n^4}$

(c) $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$

(d) $\sum_{n=5}^{\infty} \pi^{-n}$

SOLUTION

(a) $\sum_{n=0}^{\infty} \frac{7^n}{29^n} = \sum_{n=0}^{\infty} \left(\frac{7}{29}\right)^n$: this is a geometric series with common ratio $r = \frac{7}{29}$.

(b) The ratio between two successive terms is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^4}}{\frac{1}{n^4}} = \frac{n^4}{(n+1)^4} = \left(\frac{n}{n+1}\right)^4.$$

This ratio is not constant since it depends on n. Hence, the series $\sum_{n=3}^{\infty} \frac{1}{n^4}$ is not a geometric series.

(c) The ratio between two successive terms is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \frac{(n+1)^2}{n^2} \cdot \frac{2^n}{2^{n+1}} = \left(1 + \frac{1}{n}\right)^2 \cdot \frac{1}{2}.$$

This ratio is not constant since it depends on n. Hence, the series $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$ is not a geometric series.

(d) $\sum_{n=5}^{\infty} \pi^{-n} = \sum_{n=5}^{\infty} \left(\frac{1}{\pi}\right)^n$: this is a geometric series with common ratio $r = \frac{1}{\pi}$.

39. Prove that if $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} (a_n + b_n)$ diverges. *Hint:* If not, derive a contradiction by writing

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n + b_n) - \sum_{n=1}^{\infty} a_n$$

SOLUTION Suppose to the contrary that $\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=1}^{\infty} b_n$ diverges, but $\sum_{n=1}^{\infty} (a_n + b_n)$ converges. Then by the Linearity of Infinite Series, we have

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n + b_n) - \sum_{n=1}^{\infty} a_n$$

so that $\sum_{n=1}^{\infty} b_n$ converges, a contradiction.

41. Give a counterexample to show that each of the following statements is false.

(a) If the general term a_n tends to zero, then $\displaystyle\sum_{n=1}^{\infty} a_n = 0$.

(b) The Nth partial sum of the infinite series defined by $\{a_n\}$ is a_N.

(c) If a_n tends to zero, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

(d) If a_n tends to L, then $\displaystyle\sum_{n=1}^{\infty} a_n = L$.

SOLUTION

(a) Let $a_n = 2^{-n}$. Then $\lim_{n\to\infty} a_n = 0$, but a_n is a geometric series with $c = 2^0 = 1$ and $r = 1/2$, so its sum is $\dfrac{1}{1-(1/2)} = 2$.

(b) Let $a_n = 1$. Then the n^{th} partial sum is $a_1 + a_2 + \cdots + a_n = n$ while $a_n = 1$.

(c) Let $a_n = \dfrac{1}{\sqrt{n}}$. An example in the text shows that while a_n tends to zero, the sum $\sum_{n=1}^{\infty} a_n$ does not converge.

(d) Let $a_n = 1$. Then clearly a_n tends to $L = 1$, while the series $\sum_{n=1}^{\infty} a_n$ obviously diverges.

43. Compute the total area of the (infinitely many) triangles in Figure 4.

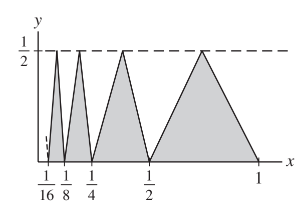


FIGURE 4

SOLUTION The area of a triangle with base B and height H is $A = \frac{1}{2}BH$. Because all of the triangles in Figure 4 have height $\frac{1}{2}$, the area of each triangle equals one-quarter of the base. Now, for $n \geq 0$, the nth triangle has a base which extends from $x = \frac{1}{2^{n+1}}$ to $x = \frac{1}{2^n}$. Thus,

$$B = \frac{1}{2^n} - \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}} \quad \text{and} \quad A = \frac{1}{4}B = \frac{1}{2^{n+3}}.$$

The total area of the triangles is then given by the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+3}} = \sum_{n=0}^{\infty} \frac{1}{8}\left(\frac{1}{2}\right)^n = \frac{\frac{1}{8}}{1-\frac{1}{2}} = \frac{1}{4}.$$

45. Find the total length of the infinite zigzag path in Figure 5 (each zag occurs at an angle of $\frac{\pi}{4}$).

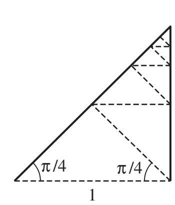


FIGURE 5

SOLUTION Because the angle at the lower left in Figure 5 has measure $\frac{\pi}{4}$ and each zag in the path occurs at an angle of $\frac{\pi}{4}$, every triangle in the figure is an isosceles right triangle. Accordingly, the length of each new segment in the path is $\frac{1}{\sqrt{2}}$ times the length of the previous segment. Since the first segment has length 1, the total length of the path is

$$\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n = \frac{1}{1-\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1} = 2+\sqrt{2}.$$

47. Show that if a is a positive integer, then

$$\sum_{n=1}^{\infty}\frac{1}{n(n+a)} = \frac{1}{a}\left(1+\frac{1}{2}+\cdots+\frac{1}{a}\right)$$

SOLUTION By partial fraction decomposition

$$\frac{1}{n\,(n+a)} = \frac{A}{n} + \frac{B}{n+a};$$

clearing the denominators gives

$$1 = A(n+a) + Bn.$$

Setting $n = 0$ then yields $A = \frac{1}{a}$, while setting $n = -a$ yields $B = -\frac{1}{a}$. Thus,

$$\frac{1}{n\,(n+a)} = \frac{\frac{1}{a}}{n} - \frac{\frac{1}{a}}{n+a} = \frac{1}{a}\left(\frac{1}{n} - \frac{1}{n+a}\right),$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n(n+a)} = \sum_{n=1}^{\infty}\frac{1}{a}\left(\frac{1}{n} - \frac{1}{n+a}\right).$$

For $N > a$, the Nth partial sum is

$$S_N = \frac{1}{a}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{a}\right) - \frac{1}{a}\left(\frac{1}{N+1}+\frac{1}{N+2}+\frac{1}{N+3}+\cdots+\frac{1}{N+a}\right).$$

Thus,

$$\sum_{n=1}^{\infty}\frac{1}{n(n+a)} = \lim_{N\to\infty} S_N = \frac{1}{a}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{a}\right).$$

49. Let $\{b_n\}$ be a sequence and let $a_n = b_n - b_{n-1}$. Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} b_n$ exists.

SOLUTION Let $a_n = b_n - b_{n-1}$. The general term in the sequence of partial sums for the series $\sum_{n=1}^{\infty} a_n$ is then

$$S_N = (b_1 - b_0) + (b_2 - b_1) + (b_3 - b_2) + \cdots + (b_N - b_{N-1}) = b_N - b_0.$$

Now, if $\lim_{N\to\infty} b_N$ exists, then so does $\lim_{N\to\infty} S_N$ and $\sum_{n=1}^{\infty} a_n$ converges. On the other hand, if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{N\to\infty} S_N$ exists, which implies that $\lim_{N\to\infty} b_N$ also exists. Thus, $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} b_n$ exists.

Further Insights and Challenges

Exercises 51–53 use the formula

$$1 + r + r^2 + \cdots + r^{N-1} = \frac{1-r^N}{1-r} \qquad \boxed{7}$$

51. Professor George Andrews of Pennsylvania State University observed that we can use Eq. (7) to calculate the derivative of $f(x) = x^N$ (for $N \ge 0$). Assume that $a \neq 0$ and let $x = ra$. Show that

$$f'(a) = \lim_{x\to a}\frac{x^N - a^N}{x-a} = a^{N-1}\lim_{r\to 1}\frac{r^N - 1}{r-1}$$

and evaluate the limit.

SOLUTION According to the definition of derivative of $f(x)$ at $x = a$

$$f'(a) = \lim_{x\to a} \frac{x^N - a^N}{x - a}.$$

Now, let $x = ra$. Then $x \to a$ if and only if $r \to 1$, and

$$f'(a) = \lim_{x\to a} \frac{x^N - a^N}{x - a} = \lim_{r\to 1} \frac{(ra)^N - a^N}{ra - a} = \lim_{r\to 1} \frac{a^N\left(r^N - 1\right)}{a(r-1)} = a^{N-1} \lim_{r\to 1} \frac{r^N - 1}{r - 1}.$$

By Eq. (7) for a geometric sum,

$$\frac{1 - r^N}{1 - r} = \frac{r^N - 1}{r - 1} = 1 + r + r^2 + \cdots + r^{N-1},$$

so

$$\lim_{r\to 1} \frac{r^N - 1}{r - 1} = \lim_{r\to 1}\left(1 + r + r^2 + \cdots + r^{N-1}\right) = 1 + 1 + 1^2 + \cdots + 1^{N-1} = N.$$

Therefore, $f'(a) = a^{N-1} \cdot N = Na^{N-1}$

53. Verify the Gregory–Leibniz formula as follows.

(a) Set $r = -x^2$ in Eq. (7) and rearrange to show that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^{N-1}x^{2N-2} + \frac{(-1)^N x^{2N}}{1+x^2}$$

(b) Show, by integrating over [0, 1], that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{N-1}}{2N-1} + (-1)^N \int_0^1 \frac{x^{2N}\,dx}{1+x^2}$$

(c) Use the Comparison Theorem for integrals to prove that

$$0 \le \int_0^1 \frac{x^{2N}\,dx}{1+x^2} \le \frac{1}{2N+1}$$

Hint: Observe that the integrand is $\le x^{2N}$.

(d) Prove that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Hint: Use (b) and (c) to show that the partial sums S_N of satisfy $\left|S_N - \frac{\pi}{4}\right| \le \frac{1}{2N+1}$, and thereby conclude that $\lim_{N\to\infty} S_N = \frac{\pi}{4}$.

SOLUTION

(a) Start with Eq. (7), and substitute $-x^2$ for r:

$$1 + r + r^2 + \cdots + r^{N-1} = \frac{1 - r^N}{1 - r}$$

$$1 - x^2 + x^4 + \cdots + (-1)^{N-1}x^{2N-2} = \frac{1 - (-1)^N x^{2N}}{1 - (-x^2)}$$

$$1 - x^2 + x^4 + \cdots + (-1)^{N-1}x^{2N-2} = \frac{1}{1+x^2} - \frac{(-1)^N x^{2N}}{1+x^2}$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 + \cdots + (-1)^{N-1}x^{2N-2} + \frac{(-1)^N x^{2N}}{1+x^2}$$

(b) The integrals of both sides must be equal. Now,

$$\int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1} x\Big|_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$$

while

$$\int_0^1 \left(1 - x^2 + x^4 + \cdots + (-1)^{N-1}x^{2N-2} + \frac{(-1)^N x^{2N}}{1+x^2}\right) dx$$

$$= \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots + (-1)^{N-1}\frac{1}{2N-1}x^{2N-1}\right) + (-1)^N \int_0^1 \frac{x^{2N}\,dx}{1+x^2}$$
$$= 1 - \frac{1}{3} + \frac{1}{5} + \cdots + (-1)^{N-1}\frac{1}{2N-1} + (-1)^N \int_0^1 \frac{x^{2N}\,dx}{1+x^2}$$

(c) Note that for $x \in [0, 1]$, we have $1 + x^2 \geq 1$, so that

$$0 \leq \frac{x^{2N}}{1+x^2} \leq x^{2N}$$

By the Comparison Theorem for integrals, we then see that

$$0 \leq \int_0^1 \frac{x^{2N}\,dx}{1+x^2} \leq \int_0^1 x^{2N}\,dx = \frac{1}{2N+1}x^{2N+1}\Big|_0^1 = \frac{1}{2N+1}$$

(d) Write

$$a_n = (-1)^n \frac{1}{2n-1}, \quad n \geq 1$$

and let S_N be the partial sums. Then

$$\left|S_N - \frac{\pi}{4}\right| = \left|(-1)^N \int_0^1 \frac{x^{2N}\,dx}{1+x^2}\right| = \int_0^1 \frac{x^{2N}\,dx}{1+x^2} \leq \frac{1}{2N+1}$$

Thus $\lim_{N\to\infty} S_N = \dfrac{\pi}{4}$ so that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

55. The **Koch snowflake** (described in 1904 by Swedish mathematician Helge von Koch) is an infinitely jagged "fractal" curve obtained as a limit of polygonal curves (it is continuous but has no tangent line at any point). Begin with an equilateral triangle (stage 0) and produce stage 1 by replacing each edge with four edges of one-third the length, arranged as in Figure 8. Continue the process: At the nth stage, replace each edge with four edges of one-third the length.
(a) Show that the perimeter P_n of the polygon at the nth stage satisfies $P_n = \frac{4}{3}P_{n-1}$. Prove that $\lim_{n\to\infty} P_n = \infty$. The snowflake has infinite length.
(b) Let A_0 be the area of the original equilateral triangle. Show that $(3)4^{n-1}$ new triangles are added at the nth stage, each with area $A_0/9^n$ (for $n \geq 1$). Show that the total area of the Koch snowflake is $\frac{8}{5}A_0$.

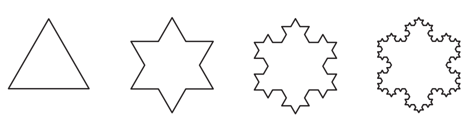


FIGURE 8

SOLUTION
(a) Each edge of the polygon at the $(n-1)$st stage is replaced by four edges of one-third the length; hence the perimeter of the polygon at the nth stage is $\frac{4}{3}$ times the perimeter of the polygon at the $(n-1)$th stage. That is, $P_n = \frac{4}{3}P_{n-1}$. Thus,

$$P_1 = \frac{4}{3}P_0; \quad P_2 = \frac{4}{3}P_1 = \left(\frac{4}{3}\right)^2 P_0, \quad P_3 = \frac{4}{3}P_2 = \left(\frac{4}{3}\right)^3 P_0,$$

and, in general, $P_n = \left(\frac{4}{3}\right)^n P_0$. As $n \to \infty$, it follows that

$$\lim_{n\to\infty} P_n = P_0 \lim_{n\to\infty} \left(\frac{4}{3}\right)^n = \infty.$$

(b) When each edge is replaced by four edges of one-third the length, one new triangle is created. At the $(n-1)$st stage, there are $3 \cdot 4^{n-1}$ edges in the snowflake, so $3 \cdot 4^{n-1}$ new triangles are generated at the nth stage. Because the area of an equilateral triangle is proportional to the square of its side length and the side length for each new triangle is one-third the side length of triangles from the previous stage, it follows that the area of the triangles added at each stage is reduced by a factor of $\frac{1}{9}$ from the area of the triangles added at the previous stage. Thus, each triangle added at the nth stage has an area of $A_0/9^n$. This means that the nth stage contributes

$$3 \cdot 4^{n-1} \cdot \frac{A_0}{9^n} = \frac{3}{4}A_0 \left(\frac{4}{9}\right)^n$$

to the area of the snowflake. The total area is therefore

$$A = A_0 + \frac{3}{4}A_0 \sum_{n=1}^{\infty} \left(\frac{4}{9}\right)^n = A_0 + \frac{3}{4}A_0 \frac{\frac{4}{9}}{1-\frac{4}{9}} = A_0 + \frac{3}{4}A_0 \cdot \frac{4}{5} = \frac{8}{5}A_0.$$

10.3 Convergence of Series with Positive Terms

Preliminary Questions

1. Let $S = \sum_{n=1}^{\infty} a_n$. If the partial sums S_N are increasing, then (choose the correct conclusion):

(a) $\{a_n\}$ is an increasing sequence.

(b) $\{a_n\}$ is a positive sequence.

SOLUTION The correct response is **(b)**. Recall that $S_N = a_1 + a_2 + a_3 + \cdots + a_N$; thus, $S_N - S_{N-1} = a_N$. If S_N is increasing, then $S_N - S_{N-1} \geq 0$. It then follows that $a_N \geq 0$; that is, $\{a_n\}$ is a positive sequence.

2. What are the hypotheses of the Integral Test?

SOLUTION The hypotheses for the Integral Test are: A function $f(x)$ such that $a_n = f(n)$ must be positive, decreasing, and continuous for $x \geq 1$.

3. Which test would you use to determine whether $\sum_{n=1}^{\infty} n^{-3.2}$ converges?

SOLUTION Because $n^{-3.2} = \frac{1}{n^{3.2}}$, we see that the indicated series is a p-series with $p = 3.2 > 1$. Therefore, the series converges.

4. Which test would you use to determine whether $\sum_{n=1}^{\infty} \frac{1}{2^n + \sqrt{n}}$ converges?

SOLUTION Because

$$\frac{1}{2^n + \sqrt{n}} < \frac{1}{2^n} = \left(\frac{1}{2}\right)^n,$$

and

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

is a convergent geometric series, the comparison test would be an appropriate choice to establish that the given series converges.

5. Ralph hopes to investigate the convergence of $\sum_{n=1}^{\infty} \frac{e^{-n}}{n}$ by comparing it with $\sum_{n=1}^{\infty} \frac{1}{n}$. Is Ralph on the right track?

SOLUTION No, Ralph is not on the right track. For $n \geq 1$,

$$\frac{e^{-n}}{n} < \frac{1}{n};$$

however, $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series. The Comparison Test therefore does not allow us to draw a conclusion about the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{e^{-n}}{n}$.

Exercises

In Exercises 1–14, use the Integral Test to determine whether the infinite series is convergent.

1. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

SOLUTION Let $f(x) = \frac{1}{x^4}$. This function is continuous, positive and decreasing on the interval $x \geq 1$, so the Integral Test applies. Moreover,

$$\int_1^\infty \frac{dx}{x^4} = \lim_{R\to\infty}\int_1^R x^{-4}\,dx = -\frac{1}{3}\lim_{R\to\infty}\left(\frac{1}{R^3}-1\right) = \frac{1}{3}.$$

The integral converges; hence, the series $\sum_{n=1}^{\infty}\frac{1}{n^4}$ also converges.

3. $\sum_{n=1}^{\infty} n^{-1/3}$

SOLUTION Let $f(x) = x^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{x}}$. This function is continuous, positive and decreasing on the interval $x \ge 1$, so the Integral Test applies. Moreover,

$$\int_1^\infty x^{-1/3}\,dx = \lim_{R\to\infty}\int_1^R x^{-1/3}\,dx = \frac{3}{2}\lim_{R\to\infty}\left(R^{2/3}-1\right) = \infty.$$

The integral diverges; hence, the series $\sum_{n=1}^{\infty} n^{-1/3}$ also diverges.

5. $\sum_{n=25}^{\infty}\frac{n^2}{(n^3+9)^{5/2}}$

SOLUTION Let $f(x) = \frac{x^2}{\left(x^3+9\right)^{5/2}}$. This function is positive and continuous for $x \ge 25$. Moreover, because

$$f'(x) = \frac{2x(x^3+9)^{5/2} - x^2\cdot\frac{5}{2}(x^3+9)^{3/2}\cdot 3x^2}{(x^3+9)^5} = \frac{x(36-11x^3)}{2(x^3+9)^{7/2}},$$

we see that $f'(x) < 0$ for $x \ge 25$, so f is decreasing on the interval $x \ge 25$. The Integral Test therefore applies. To evaluate the improper integral, we use the substitution $u = x^3+9$, $du = 3x^2dx$. We then find

$$\int_{25}^\infty \frac{x^2}{(x^3+9)^{5/2}}\,dx = \lim_{R\to\infty}\int_{25}^R \frac{x^2}{(x^3+9)^{5/2}}\,dx = \frac{1}{3}\lim_{R\to\infty}\int_{15634}^{R^3+9}\frac{du}{u^{5/2}}$$
$$= -\frac{2}{9}\lim_{R\to\infty}\left(\frac{1}{(R^3+9)^{3/2}} - \frac{1}{15634^{3/2}}\right) = \frac{2}{9\cdot 15634^{3/2}}.$$

The integral converges; hence, the series $\sum_{n=25}^{\infty}\frac{n^2}{\left(n^3+9\right)^{5/2}}$ also converges.

7. $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$

SOLUTION Let $f(x) = \frac{1}{x^2+1}$. This function is positive, decreasing and continuous on the interval $x \ge 1$, hence the Integral Test applies. Moreover,

$$\int_1^\infty \frac{dx}{x^2+1} = \lim_{R\to\infty}\int_1^R\frac{dx}{x^2+1} = \lim_{R\to\infty}\left(\tan^{-1}R - \frac{\pi}{4}\right) = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}.$$

The integral converges; hence, the series $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ also converges.

9. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

SOLUTION Let $f(x) = \frac{1}{x(x+1)}$. This function is positive, continuous and decreasing on the interval $x \ge 1$, so the Integral Test applies. We compute the improper integral using partial fractions:

$$\int_1^\infty \frac{dx}{x(x+1)} = \lim_{R\to\infty}\int_1^R\left(\frac{1}{x}-\frac{1}{x+1}\right)dx = \lim_{R\to\infty}\ln\frac{x}{x+1}\Bigg|_1^R = \lim_{R\to\infty}\left(\ln\frac{R}{R+1} - \ln\frac{1}{2}\right) = \ln 1 - \ln\frac{1}{2} = \ln 2.$$

The integral converges; hence, the series $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ converges.

11. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

SOLUTION Let $f(x) = \dfrac{1}{x(\ln x)^2}$. This function is positive and continuous for $x \geq 2$. Moreover,

$$f'(x) = -\frac{1}{x^2(\ln x)^4}\left(1 \cdot (\ln x)^2 + x \cdot 2(\ln x) \cdot \frac{1}{x}\right) = -\frac{1}{x^2(\ln x)^4}\left((\ln x)^2 + 2\ln x\right).$$

Since $\ln x > 0$ for $x > 1$, $f'(x)$ is negative for $x > 1$; hence, f is decreasing for $x \geq 2$. To compute the improper integral, we make the substitution $u = \ln x$, $du = \frac{1}{x}\,dx$. We obtain:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{R\to\infty}\int_2^R \frac{1}{x(\ln x)^2}\,dx = \lim_{R\to\infty}\int_{\ln 2}^{\ln R}\frac{du}{u^2}$$
$$= -\lim_{R\to\infty}\left(\frac{1}{\ln R} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

The integral converges; hence, the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ also converges.

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

SOLUTION Note that

$$2^{\ln n} = (e^{\ln 2})^{\ln n} = (e^{\ln n})^{\ln 2} = n^{\ln 2}.$$

Thus,

$$\sum_{n=1}^{\infty}\frac{1}{2^{\ln n}} = \sum_{n=1}^{\infty}\frac{1}{n^{\ln 2}}.$$

Now, let $f(x) = \dfrac{1}{x^{\ln 2}}$. This function is positive, continuous and decreasing on the interval $x \geq 1$; therefore, the Integral Test applies. Moreover,

$$\int_1^{\infty}\frac{dx}{x^{\ln 2}} = \lim_{R\to\infty}\int_1^R\frac{dx}{x^{\ln 2}} = \frac{1}{1-\ln 2}\lim_{R\to\infty}(R^{1-\ln 2} - 1) = \infty,$$

because $1 - \ln 2 > 0$. The integral diverges; hence, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^{\ln n}}$ also diverges.

15. Show that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3+8n}$ converges by using the Comparison Test with $\displaystyle\sum_{n=1}^{\infty} n^{-3}$.

SOLUTION We compare the series with the p-series $\displaystyle\sum_{n=1}^{\infty} n^{-3}$. For $n \geq 1$,

$$\frac{1}{n^3+8n} \leq \frac{1}{n^3}.$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges (it is a p-series with $p = 3 > 1$), the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3+8n}$ also converges by the Comparison Test.

17. Let $S = \displaystyle\sum_{n=1}^{\infty}\frac{1}{n+\sqrt{n}}$. Verify that for $n \geq 1$,

$$\frac{1}{n+\sqrt{n}} \leq \frac{1}{n}, \qquad \frac{1}{n+\sqrt{n}} \leq \frac{1}{\sqrt{n}}$$

Can either inequality be used to show that S diverges? Show that $\dfrac{1}{n+\sqrt{n}} \geq \dfrac{1}{2n}$ and conclude that S diverges.

SOLUTION For $n \geq 1$, $n + \sqrt{n} \geq n$ and $n + \sqrt{n} \geq \sqrt{n}$. Taking the reciprocal of each of these inequalities yields

$$\frac{1}{n+\sqrt{n}} \leq \frac{1}{n} \quad \text{and} \quad \frac{1}{n+\sqrt{n}} \leq \frac{1}{\sqrt{n}}.$$

These inequalities indicate that the series $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ is smaller than both $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$; however, $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ both diverge so neither inequality allows us to show that S diverges.

On the other hand, for $n \geq 1$, $n \geq \sqrt{n}$, so $2n \geq n + \sqrt{n}$ and

$$\frac{1}{n+\sqrt{n}} \geq \frac{1}{2n}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{2n} = 2\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, since the harmonic series diverges. The Comparison Test then lets us conclude that the larger series $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ also diverges.

In Exercises 19–30, use the Comparison Test to determine whether the infinite series is convergent.

19. $\sum_{n=1}^{\infty} \frac{1}{n2^n}$

SOLUTION We compare with the geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. For $n \geq 1$,

$$\frac{1}{n2^n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n.$$

Since $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges (it is a geometric series with $r = \frac{1}{2}$), we conclude by the Comparison Test that $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ also converges.

21. $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}+2^n}$

SOLUTION For $n \geq 1$,

$$\frac{1}{n^{1/3}+2^n} \leq \frac{1}{2^n}$$

The series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $r = \frac{1}{2}$, so it converges. By the Comparison test, so does $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}+2^n}$.

23. $\sum_{m=1}^{\infty} \frac{4}{m!+4^m}$

SOLUTION For $m \geq 1$,

$$\frac{4}{m!+4^m} \leq \frac{4}{4^m} = \left(\frac{1}{4}\right)^{m-1}.$$

The series $\sum_{m=1}^{\infty} \left(\frac{1}{4}\right)^{m-1}$ is a geometric series with $r = \frac{1}{4}$, so it converges. By the Comparison Test we can therefore conclude that the series $\sum_{m=1}^{\infty} \frac{4}{m!+4^m}$ also converges.

25. $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$

SOLUTION For $k \geq 1$, $0 \leq \sin^2 k \leq 1$, so

$$0 \leq \frac{\sin^2 k}{k^2} \leq \frac{1}{k^2}.$$

The series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a p-series with $p = 2 > 1$, so it converges. By the Comparison Test we can therefore conclude that the series $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$ also converges.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n + 3^{-n}}$

SOLUTION Since $3^{-n} > 0$ for all n,

$$\frac{2}{3^n + 3^{-n}} \le \frac{2}{3^n} = 2\left(\frac{1}{3}\right)^n.$$

The series $\displaystyle\sum_{n=1}^{\infty} 2\left(\frac{1}{3}\right)^n$ is a geometric series with $r = \frac{1}{3}$, so it converges. By the Comparison Theorem we can therefore conclude that the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n + 3^{-n}}$ also converges.

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$

SOLUTION Note that for $n \ge 2$,

$$(n+1)! = 1 \cdot \underbrace{2 \cdot 3 \cdots n \cdot (n+1)}_{n \text{ factors}} \le 2^n$$

so that

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)!} = 1 + \sum_{n=2}^{\infty} \frac{1}{(n+1)!} \le 1 + \sum_{n=2}^{\infty} \frac{1}{2^n}$$

But $\sum_{n=2}^{\infty} \frac{1}{2^n}$ is a geometric series with ratio $r = \frac{1}{2}$, so it converges. By the comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$ converges as well.

Exercise 31–36: For all $a > 0$ and $b > 1$, the inequalities

$$\ln n \le n^a, \qquad n^a < b^n$$

are true for n sufficiently large (this can be proved using L'Hopital's Rule). Use this, together with the Comparison Theorem, to determine whether the series converges or diverges.

31. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

SOLUTION For n sufficiently large (say $n = k$, although in this case $n = 1$ suffices), we have $\ln n \le n$, so that

$$\sum_{n=k}^{\infty} \frac{\ln n}{n^3} \le \sum_{n=k}^{\infty} \frac{n}{n^3} = \sum_{n=k}^{\infty} \frac{1}{n^2}$$

This is a p-series with $p = 2 > 1$, so it converges. Thus $\sum_{n=k}^{\infty} \frac{\ln n}{n^3}$ also converges; adding back in the finite number of terms for $1 \le n \le k$ does not affect this result.

33. $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^{100}}{n^{1.1}}$

SOLUTION Choose N so that $\ln n \le n^{0.0005}$ for $n \ge N$. Then also for $n > N$, $(\ln n)^{100} \le (n^{0.0005})^{100} = n^{0.05}$. Then

$$\sum_{n=N}^{\infty} \frac{(\ln n)^{100}}{n^{1.1}} \le \sum_{n=N}^{\infty} \frac{n^{0.05}}{n^{1.1}} = \sum_{n=N}^{\infty} \frac{1}{n^{1.05}}$$

But $\displaystyle\sum_{n=N}^{\infty} \frac{1}{n^{1.05}}$ is a p-series with $p = 1.05 > 1$, so is convergent. It follows that $\sum_{n=N}^{\infty} \frac{(\ln n)^100}{n^{1.1}}$ is also convergent; adding back in the finite number of terms for $n = 1, 2, \dots, N-1$ shows that $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^{100}}{n^{1.1}}$ converges as well.

35. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$

SOLUTION Choose N such that $n \le 2^n$ for $n \ge N$. Then

$$\sum_{n=N}^{\infty} \frac{n}{3^n} \le \sum_{n=N}^{\infty} \left(\frac{2}{3}\right)^n$$

The latter sum is a geometric series with $r = \frac{2}{3} < 1$, so it converges. Thus the series on the left converges as well. Adding back in the finite number of terms for $n < N$ shows that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$ converges.

37. Show that $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$ converges. *Hint:* Use $\sin x \le x$ for $x \ge 0$.

SOLUTION For $n \ge 1$,

$$0 \le \frac{1}{n^2} \le 1 < \pi;$$

therefore, $\sin \frac{1}{n^2} > 0$ for $n \ge 1$. Moreover, for $n \ge 1$,

$$\sin \frac{1}{n^2} \le \frac{1}{n^2}.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$, so it converges. By the Comparison Test we can therefore conclude that the series $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$ also converges.

In Exercises 39–48, use the Limit Comparison Test to prove convergence or divergence of the infinite series.

39. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^4 - 1}$

SOLUTION Let $a_n = \dfrac{n^2}{n^4 - 1}$. For large n, $\dfrac{n^2}{n^4 - 1} \approx \dfrac{n^2}{n^4} = \dfrac{1}{n^2}$, so we apply the Limit Comparison Test with $b_n = \dfrac{1}{n^2}$. We find

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2}{n^4-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4}{n^4 - 1} = 1.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$, so it converges; hence, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ also converges. Because L exists, by the Limit Comparison Test we can conclude that the series $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^4 - 1}$ converges.

41. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^3 + 1}}$

SOLUTION Let $a_n = \dfrac{n}{\sqrt{n^3 + 1}}$. For large n, $\dfrac{n}{\sqrt{n^3 + 1}} \approx \dfrac{n}{\sqrt{n^3}} = \dfrac{1}{\sqrt{n}}$, so we apply the Limit Comparison test with $b_n = \dfrac{1}{\sqrt{n}}$. We find

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^3+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n^3}}{\sqrt{n^3 + 1}} = 1.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} < 1$, so it diverges; hence, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ also diverges. Because $L > 0$, by the Limit Comparison Test we can conclude that the series $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^3 + 1}}$ diverges.

43. $\displaystyle\sum_{n=3}^{\infty} \frac{3n+5}{n(n-1)(n-2)}$

SOLUTION Let $a_n = \dfrac{3n+5}{n(n-1)(n-2)}$. For large n, $\dfrac{3n+5}{n(n-1)(n-2)} \approx \dfrac{3n}{n^3} = \dfrac{3}{n^2}$, so we apply the Limit Comparison Test with $b_n = \dfrac{1}{n^2}$. We find

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3n+5}{n(n+1)(n+2)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3n^3+5n^2}{n(n+1)(n+2)} = 3.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$, so it converges; hence, the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2}$ also converges. Because L exists, by the Limit Comparison Test we can conclude that the series $\displaystyle\sum_{n=3}^{\infty} \frac{3n+5}{n(n-1)(n-2)}$ converges.

45. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\ln n}$

SOLUTION Let

$$a_n = \frac{1}{\sqrt{n}+\ln n}$$

For large n, $\sqrt{n}+\ln n \approx \sqrt{n}$, so apply the Comparison Test with $b_n = \dfrac{1}{\sqrt{n}}$. We find

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}+\ln n} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{1}{1+\frac{\ln n}{\sqrt{n}}} = 1$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \dfrac{1}{2} < 1$, so it diverges. Because L exists, the Limit Comparison Test tells us the the original series also diverges.

47. $\displaystyle\sum_{n=1}^{\infty} \left(1 - \cos\frac{1}{n}\right)$ *Hint:* Compare with $\displaystyle\sum_{n=1}^{\infty} n^{-2}$.

SOLUTION Let $a_n = 1 - \cos\dfrac{1}{n}$, and apply the Limit Comparison Test with $b_n = \dfrac{1}{n^2}$. We find

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1-\cos\frac{1}{n}}{\frac{1}{n^2}} = \lim_{x\to\infty} \frac{1-\cos\frac{1}{x}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{-\frac{1}{x^2}\sin\frac{1}{x}}{-\frac{2}{x^3}} = \frac{1}{2}\lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}}.$$

As $x \to \infty$, $u = \frac{1}{x} \to 0$, so

$$L = \frac{1}{2}\lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = \frac{1}{2}\lim_{u\to 0} \frac{\sin u}{u} = \frac{1}{2}.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$, so it converges. Because L exists, by the Limit Comparison Test we can conclude that the series $\displaystyle\sum_{n=1}^{\infty} \left(1 - \cos\frac{1}{n}\right)$ also converges.

In Exercises 49–78, determine convergence or divergence using any method covered so far.

49. $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^2-9}$

SOLUTION Apply the Limit Comparison Test with $a_n = \dfrac{1}{n^2-9}$ and $b_n = \dfrac{1}{n^2}$:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2-9}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-9} = 1.$$

Since the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the series $\sum_{n=4}^{\infty} \frac{1}{n^2}$ also converges. Because L exists, by the Limit Comparison Test we can conclude that the series $\sum_{n=4}^{\infty} \frac{1}{n^2 - 9}$ converges.

51. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n+9}$

SOLUTION Apply the Limit Comparison Test with $a_n = \frac{\sqrt{n}}{4n+9}$ and $b_n = \frac{1}{\sqrt{n}}$:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{4n+9}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n}{4n+9} = \frac{1}{4}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series. Because $L > 0$, by the Limit Comparison Test we can conclude that the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n+9}$ also diverges.

53. $\sum_{n=1}^{\infty} \frac{n^2 - n}{n^5 + n}$

SOLUTION First rewrite $a_n = \frac{n^2 - n}{n^5 + n} = \frac{n\,(n-1)}{n\left(n^4+1\right)} = \frac{n-1}{n^4+1}$ and observe

$$\frac{n-1}{n^4+1} < \frac{n}{n^4} = \frac{1}{n^3}$$

for $n \geq 1$. The series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-series, so by the Comparison Test we can conclude that the series $\sum_{n=1}^{\infty} \frac{n^2 - n}{n^5 + n}$ also converges.

55. $\sum_{n=5}^{\infty} (4/5)^{-n}$

SOLUTION

$$\sum_{n=5}^{\infty} \left(\frac{4}{5}\right)^{-n} = \sum_{n=5}^{\infty} \left(\frac{5}{4}\right)^{n}$$

which is a geometric series starting at $n = 5$ with ratio $r = \frac{5}{4} > 1$. Thus the series diverges.

57. $\sum_{n=2}^{\infty} \frac{1}{n^{3/2} \ln n}$

SOLUTION For $n \geq 3$, $\ln n > 1$, so $n^{3/2} \ln n > n^{3/2}$ and

$$\frac{1}{n^{3/2} \ln n} < \frac{1}{n^{3/2}}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series, so the series $\sum_{n=3}^{\infty} \frac{1}{n^{3/2}}$ also converges. By the Comparison Test we can therefore conclude that the series $\sum_{n=3}^{\infty} \frac{1}{n^{3/2} \ln n}$ converges. Hence, the series $\sum_{n=2}^{\infty} \frac{1}{n^{3/2} \ln n}$ also converges.

59. $\sum_{k=1}^{\infty} 4^{1/k}$

SOLUTION

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} 4^{1/k} = 4^0 = 1 \neq 0;$$

therefore, the series $\sum_{k=1}^{\infty} 4^{1/k}$ diverges by the Divergence Test.

61. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^4}$

SOLUTION By the comment preceding Exercise 31, we can choose N so that for $n \geq N$, we have $\ln n < n^{1/8}$, so that $(\ln n)^4 < n^{1/2}$. Then

$$\sum_{n=N}^{\infty} \frac{1}{(\ln n)^4} > \sum_{n=N}^{\infty} \frac{1}{n^{1/2}}$$

which is a divergent p-series. Thus the series on the left diverges as well, and adding back in the finite number of terms for $n < N$ does not affect the result. Thus $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^4}$ diverges.

63. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \ln n - n}$

SOLUTION For $n \geq 2$, $n \ln n - n \leq n \ln n$; therefore,

$$\frac{1}{n \ln n - n} \geq \frac{1}{n \ln n}.$$

Now, let $f(x) = \dfrac{1}{x \ln x}$. For $x \geq 2$, this function is continuous, positive and decreasing, so the Integral Test applies. Using the substitution $u = \ln x$, $du = \frac{1}{x}\,dx$, we find

$$\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{R\to\infty} \int_2^{R} \frac{dx}{x \ln x} = \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{du}{u} = \lim_{R\to\infty} (\ln(\ln R) - \ln(\ln 2)) = \infty.$$

The integral diverges; hence, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ also diverges. By the Comparison Test we can therefore conclude that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n - n}$ diverges.

65. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

SOLUTION For $n \geq 2$, $n^n \geq 2^n$; therefore,

$$\frac{1}{n^n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n.$$

The series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a convergent geometric series, so $\displaystyle\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n$ also converges. By the Comparison Test we can therefore conclude that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^n}$ converges. Hence, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges.

67. $\displaystyle\sum_{n=1}^{\infty} \frac{1 + (-1)^n}{n}$

SOLUTION Let

$$a_n = \frac{1 + (-1)^n}{n}$$

Then

$$a_n = \begin{cases} 0 & n \text{ odd} \\ \frac{2}{2k} = \frac{1}{k} & n = 2k \text{ even} \end{cases}$$

Therefore, $\{a_n\}$ consists of 0s in the odd places and the harmonic series in the even places, so $\sum_{i=1}^{\infty} a_n$ is just the sum of the harmonic series, which diverges. Thus $\sum_{i=1}^{\infty} a_n$ diverges as well.

69. $\sum_{n=1}^{\infty} \sin \frac{1}{n}$

SOLUTION Apply the Limit Comparison Test with $a_n = \sin \frac{1}{n}$ and $b_n = \frac{1}{n}$:

$$L = \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{u\to 0} \frac{\sin u}{u} = 1,$$

where $u = \frac{1}{n}$. The harmonic series diverges. Because $L > 0$, by the Limit Comparison Test we can conclude that the series $\sum_{n=1}^{\infty} \sin \frac{1}{n}$ also diverges.

71. $\sum_{n=1}^{\infty} \frac{2n+1}{4^n}$

SOLUTION For $n \ge 3$, $2n + 1 < 2^n$, so

$$\frac{2n+1}{4^n} < \frac{2^n}{4^n} = \left(\frac{1}{2}\right)^n.$$

The series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a convergent geometric series, so $\sum_{n=3}^{\infty} \left(\frac{1}{2}\right)^n$ also converges. By the Comparison Test we can therefore conclude that the series $\sum_{n=3}^{\infty} \frac{2n+1}{4^n}$ converges. Finally, the series $\sum_{n=1}^{\infty} \frac{2n+1}{4^n}$ converges.

73. $\sum_{n=4}^{\infty} \frac{\ln n}{n^2 - 3n}$

SOLUTION By the comment preceding Exercise 31, we can choose $N \ge 4$ so that for $n \ge N$, $\ln n < n^{1/2}$. Then

$$\sum_{n=N}^{\infty} \frac{\ln n}{n^2 - 3n} \le \sum_{n=N}^{\infty} \frac{n^{1/2}}{n^2 - 3n} = \sum_{n=N}^{\infty} \frac{1}{n^{3/2} - 3n^{1/2}}$$

To evaluate convergence of the latter series, let $a_n = \frac{1}{n^{3/2} - 3n^{1/2}}$ and $b_n = \frac{1}{n^{3/2}}$, and apply the Limit Comparison Test:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^{3/2} - 3n^{1/2}} \cdot n^{3/2} = \lim_{n\to\infty} \frac{1}{1 - 3n^{-1}} = 0$$

Thus $\sum a_n$ converges if $\sum b_n$ does. But $\sum b_n$ is a convergent p-series. Thus $\sum a_n$ converges and, by the comparison test, so does the original series. Adding back in the finite number of terms for $n < N$ does not affect convergence.

75. $\sum_{n=2}^{\infty} \frac{1}{n^{1/2} \ln n}$

SOLUTION By the comment preceding Exercise 31, we can choose $N \ge 2$ so that for $n \ge N$, $\ln n < n^{1/4}$. Then

$$\sum_{n=N}^{\infty} \frac{1}{n^{1/2} \ln n} > \sum_{n=N}^{\infty} \frac{1}{n^{3/4}}$$

which is a divergent p-series. Thus the original series diverges as well - as usual, adding back in the finite number of terms for $n < N$ does not affect convergence.

77. $\sum_{n=1}^{\infty} \frac{4n^2 + 15n}{3n^4 - 5n^2 - 17}$

SOLUTION Apply the Limit Comparison Test with

$$a_n = \frac{4n^2 + 15n}{3n^4 - 5n^2 - 17}, \qquad b_n = \frac{4n^2}{3n^4} = \frac{4}{3n^2}$$

We have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4n^2+15n}{3n^4-5n^2-17} \cdot \frac{3n^2}{4} = \lim_{n\to\infty} \frac{12n^4+45n^3}{12n^4-20n^2-68} = \lim_{n\to\infty} \frac{12+45/n}{12-20/n^2-68/n^4} = 1$$

Now, $\sum_{n=1}^{\infty} b_n$ is a p-series with $p = 2 > 1$, so converges. Since $L = 1$, we see that $\displaystyle\sum_{n=1}^{\infty} \frac{4n^2+15n}{3n^4-5n^2-17}$ converges as well.

79. For which a does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^a}$ converge?

SOLUTION First consider the case $a > 0$ but $a \neq 1$. Let $f(x) = \dfrac{1}{x(\ln x)^a}$. This function is continuous, positive and decreasing for $x \geq 2$, so the Integral Test applies. Now,

$$\int_2^{\infty} \frac{dx}{x(\ln x)^a} = \lim_{R\to\infty} \int_2^R \frac{dx}{x(\ln x)^a} = \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{du}{u^a} = \frac{1}{1-a} \lim_{R\to\infty} \left(\frac{1}{(\ln R)^{a-1}} - \frac{1}{(\ln 2)^{a-1}} \right).$$

Because

$$\lim_{R\to\infty} \frac{1}{(\ln R)^{a-1}} = \begin{cases} \infty, & 0 < a < 1 \\ 0, & a > 1 \end{cases}$$

we conclude the integral diverges when $0 < a < 1$ and converges when $a > 1$. Therefore

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^a} \text{ converges for } a > 1 \text{ and diverges for } 0 < a < 1.$$

Next, consider the case $a = 1$. The series becomes $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$. Let $f(x) = \dfrac{1}{x \ln x}$. For $x \geq 2$, this function is continuous, positive and decreasing, so the Integral Test applies. Using the substitution $u = \ln x$, $du = \frac{1}{x}\,dx$, we find

$$\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{R\to\infty} \int_2^R \frac{dx}{x \ln x} = \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{du}{u} = \lim_{R\to\infty} (\ln(\ln R) - \ln(\ln 2)) = \infty.$$

The integral diverges; hence, the series also diverges.

Finally, consider the case $a < 0$. Let $b = -a > 0$ so the series becomes $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^b}{n}$. Since $\ln n > 1$ for all $n \geq 3$, it follows that

$$(\ln n)^b > 1 \quad \text{so} \quad \frac{(\ln n)^b}{n} > \frac{1}{n}.$$

The series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ diverges, so by the Comparison Test we can conclude that $\displaystyle\sum_{n=3}^{\infty} \frac{(\ln n)^b}{n}$ also diverges. Consequently, $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^b}{n}$ diverges. Thus,

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^a} \text{ diverges for } a < 0.$$

To summarize:

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^a} \text{ converges if } a > 1 \text{ and diverges if } a \leq 1.$$

Approximating Infinite Sums *In Exercises 81–83, let $a_n = f(n)$, where $f(x)$ is a continuous, decreasing function such that $f(x) \geq 0$ and $\int_1^{\infty} f(x)\,dx$ converges.*

81. Show that

$$\int_1^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^{\infty} f(x)\,dx$$

3

SOLUTION From the proof of the Integral Test, we know that

$$a_2 + a_3 + a_4 + \cdots + a_N \le \int_1^N f(x)\,dx \le \int_1^\infty f(x)\,dx;$$

that is,

$$S_N - a_1 \le \int_1^\infty f(x)\,dx \quad \text{or} \quad S_N \le a_1 + \int_1^\infty f(x)\,dx.$$

Also from the proof of the Integral test, we know that

$$\int_1^N f(x)\,dx \le a_1 + a_2 + a_3 + \cdots + a_{N-1} = S_N - a_N \le S_N.$$

Thus,

$$\int_1^N f(x)\,dx \le S_N \le a_1 + \int_1^\infty f(x)\,dx.$$

Taking the limit as $N \to \infty$ yields Eq. (3), as desired.

83. Let $S = \sum_{n=1}^{\infty} a_n$. Arguing as in Exercise 81, show that

$$\sum_{n=1}^{M} a_n + \int_{M+1}^\infty f(x)\,dx \le S \le \sum_{n=1}^{M+1} a_n + \int_{M+1}^\infty f(x)\,dx \qquad \boxed{4}$$

Conclude that

$$0 \le S - \left(\sum_{n=1}^{M} a_n + \int_{M+1}^\infty f(x)\,dx\right) \le a_{M+1} \qquad \boxed{5}$$

This provides a method for approximating S with an error of at most a_{M+1}.

SOLUTION Following the proof of the Integral Test and the argument in Exercise 81, but starting with $n = M + 1$ rather than $n = 1$, we obtain

$$\int_{M+1}^\infty f(x)\,dx \le \sum_{n=M+1}^{\infty} a_n \le a_{M+1} + \int_{M+1}^\infty f(x)\,dx.$$

Adding $\sum_{n=1}^{M} a_n$ to each part of this inequality yields

$$\sum_{n=1}^{M} a_n + \int_{M+1}^\infty f(x)\,dx \le \sum_{n=1}^{\infty} a_n = S \le \sum_{n=1}^{M+1} a_n + \int_{M+1}^\infty f(x)\,dx.$$

Subtracting $\sum_{n=1}^{M} a_n + \int_{M+1}^\infty f(x)\,dx$ from each part of this last inequality then gives us

$$0 \le S - \left(\sum_{n=1}^{M} a_n + \int_{M+1}^\infty f(x)\,dx\right) \le a_{M+1}.$$

85. CAS Apply Eq. (4) with $M = 40{,}000$ to show that

$$1.644934066 \le \sum_{n=1}^{\infty} \frac{1}{n^2} \le 1.644934068$$

Is this consistent with Euler's result, according to which this infinite series has sum $\pi^2/6$?

SOLUTION Using Eq. (4) with $f(x) = \frac{1}{x^2}$, $a_n = \frac{1}{n^2}$ and $M = 40{,}000$, we find

$$S_{40,000} + \int_{40,001}^\infty \frac{dx}{x^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2} \le S_{40,001} + \int_{40,001}^\infty \frac{dx}{x^2}.$$

Now,

$$S_{40,000} = 1.6449090672;$$
$$S_{40,001} = S_{40,000} + \frac{1}{40,001} = 1.6449090678;$$

and

$$\int_{40,001}^{\infty} \frac{dx}{x^2} = \lim_{R\to\infty} \int_{40,001}^{R} \frac{dx}{x^2} = -\lim_{R\to\infty} \left(\frac{1}{R} - \frac{1}{40,001} \right) = \frac{1}{40,001} = 0.0000249994.$$

Thus,

$$1.6449090672 + 0.0000249994 \le \sum_{n=1}^{\infty} \frac{1}{n^2} \le 1.6449090678 + 0.0000249994,$$

or

$$1.6449340665 \le \sum_{n=1}^{\infty} \frac{1}{n^2} \le 1.6449340672.$$

Since $\frac{\pi^2}{6} \approx 1.6449340668$, our approximation is consistent with Euler's result.

87. CAS Using a CAS and Eq. (5), determine the value of $\sum_{n=1}^{\infty} n^{-5}$ to within an error less than 10^{-4}.

SOLUTION Using Eq. (5) with $f(x) = x^{-5}$ and $a_n = n^{-5}$, we have

$$0 \le \sum_{n=1}^{\infty} n^{-5} - \left(\sum_{n=1}^{M+1} n^{-5} + \int_{M+1}^{\infty} x^{-5}\, dx \right) \le (M+1)^{-5}.$$

To guarantee an error less than 10^{-4}, we need $(M+1)^{-5} \le 10^{-4}$. This yields $M \ge 10^{4/5} - 1 \approx 5.3$, so we choose $M = 6$. Now,

$$\sum_{n=1}^{7} n^{-5} = 1.0368498887,$$

and

$$\int_{7}^{\infty} x^{-5}\, dx = \lim_{R\to\infty} \int_{7}^{R} x^{-5}\, dx = -\frac{1}{4} \lim_{R\to\infty} \left(R^{-4} - 7^{-4} \right) = \frac{1}{4 \cdot 7^4} = 0.0001041233.$$

Thus,

$$\sum_{n=1}^{\infty} n^{-5} \approx \sum_{n=1}^{7} n^{-5} + \int_{7}^{\infty} x^{-5}\, dx = 1.0368498887 + 0.0001041233 = 1.0369540120.$$

89. The following argument proves the divergence of the harmonic series $S = \sum_{n=1}^{\infty} 1/n$ without using the Integral Test. Let

$$S_1 = 1 + \frac{1}{3} + \frac{1}{5} + \cdots, \qquad S_2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$$

Show that if S converges, then

(a) S_1 and S_2 also converge and $S = S_1 + S_2$.

(b) $S_1 > S_2$ and $S_2 = \frac{1}{2} S$.

Observe that (b) contradicts (a), and conclude that S diverges.

SOLUTION Assume throughout that S converges; we will derive a contradiction. Write

$$a_n = \frac{1}{n}, \quad b_n = \frac{1}{2n-1}, \quad c_n = \frac{1}{2n}$$

for the n^{th} terms in the series S, S_1, and S_2. Since $2n - 1 \geq n$ for $n \geq 1$, we have $b_n < a_n$. Since $S = \sum a_n$ converges, so does $S_1 = \sum b_n$ by the Comparison Test. Also, $c_n = \frac{1}{2}a_n$, so again by the Comparison Test, the convergence of S implies the convergence of $S_2 = \sum c_n$. Now, define two sequences

$$b'_n = \begin{cases} b_{(n+1)/2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$c'_n = \begin{cases} 0 & n \text{ odd} \\ c_{n/2} & n \text{ even} \end{cases}$$

That is, b'_n and c'_n look like b_n and c_n, but have zeros inserted in the "missing" places compared to a_n. Then $a_n = b'_n + c'_n$; also $S_1 = \sum b_n = \sum b'_n$ and $S_2 = \sum c_n = \sum c'_n$. Finally, since S, S_1, and S_2 all converge, we have

$$S = \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(b'_n + c'_n) = \sum_{n=1}^{\infty} b'_n + \sum_{n=1}^{\infty} c'_n = \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} c_n = S_1 + S_2$$

Now, $b_n > c_n$ for every n, so that $S_1 > S_2$. Also, we showed above that $c_n = \frac{1}{2}a_n$, so that $2S_2 = S$. Putting all this together gives

$$S = S_1 + S_2 > S_2 + S_2 = 2S_2 = S$$

so that $S > S$, a contradiction. Thus S must diverge.

Further Insights and Challenges

91. Kummer's Acceleration Method Suppose we wish to approximate $S = \sum_{n=1}^{\infty} 1/n^2$. There is a similar telescoping series whose value can be computed exactly (Example 1 in Section 10.2):

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

(a) Verify that

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n(n+1)}\right)$$

Thus for M large,

$$S \approx 1 + \sum_{n=1}^{M} \frac{1}{n^2(n+1)} \qquad \boxed{6}$$

(b) Explain what has been gained. Why is Eq. (6) a better approximation to S than is $\sum_{n=1}^{M} 1/n^2$?

(c) CAS Compute

$$\sum_{n=1}^{1000} \frac{1}{n^2}, \qquad 1 + \sum_{n=1}^{100} \frac{1}{n^2(n+1)}$$

Which is a better approximation to S, whose exact value is $\pi^2/6$?

SOLUTION

(a) Because the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ both converge,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n(n+1)}\right) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2} = S.$$

Now,

$$\frac{1}{n^2} - \frac{1}{n(n+1)} = \frac{n+1}{n^2(n+1)} - \frac{n}{n^2(n+1)} = \frac{1}{n^2(n+1)},$$

so, for M large,

$$S \approx 1 + \sum_{n=1}^{M} \frac{1}{n^2(n+1)}.$$

(b) The series $\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$ converges more rapidly than $\sum_{n=1}^{\infty} \frac{1}{n^2}$ since the degree of n in the denominator is larger.

(c) Using a computer algebra system, we find

$$\sum_{n=1}^{1000} \frac{1}{n^2} = 1.6439345667 \quad \text{and} \quad 1 + \sum_{n=1}^{100} \frac{1}{n^2(n+1)} = 1.6448848903.$$

The second sum is more accurate because it is closer to the exact solution $\frac{\pi^2}{6} \approx 1.6449340668$.

10.4 Absolute and Conditional Convergence

Preliminary Questions

1. Give an example of a series such that $\sum a_n$ converges but $\sum |a_n|$ diverges.

SOLUTION The series $\sum \frac{(-1)^n}{\sqrt[3]{n}}$ converges by the Leibniz Test, but the positive series $\sum \frac{1}{\sqrt[3]{n}}$ is a divergent p-series.

2. Which of the following statements is equivalent to Theorem 1?

(a) If $\sum_{n=0}^{\infty} |a_n|$ diverges, then $\sum_{n=0}^{\infty} a_n$ also diverges.

(b) If $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=0}^{\infty} |a_n|$ also diverges.

(c) If $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} |a_n|$ also converges.

SOLUTION The correct answer is **(b)**: If $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=0}^{\infty} |a_n|$ also diverges. Take $a_n = (-1)^n \frac{1}{n}$ to see that statements **(a)** and **(c)** are not true in general.

3. Lathika argues that $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$ is an alternating series and therefore converges. Is Lathika right?

SOLUTION No. Although $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$ is an alternating series, the terms $a_n = \sqrt{n}$ do not form a decreasing sequence that tends to zero. In fact, $a_n = \sqrt{n}$ is an increasing sequence that tends to ∞, so $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$ diverges by the Divergence Test.

4. Suppose that a_n is positive, decreasing, and tends to 0, and let $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$. What can we say about $|S - S_{100}|$ if $a_{101} = 10^{-3}$? Is S larger or smaller than S_{100}?

SOLUTION From the text, we know that $|S - S_{100}| < a_{101} = 10^{-3}$. Also, the Leibniz test tells us that $S_{2N} < S < S_{2N+1}$ for any $N \geq 1$, so that $S_{100} < S$.

Exercises

1. Show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$$

converges absolutely.

SOLUTION The positive series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is a geometric series with $r = \frac{1}{2}$. Thus, the positive series converges, and the given series converges absolutely.

In Exercises 3–10, determine whether the series converges absolutely, conditionally, or not at all.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1/3}}$

SOLUTION The sequence $a_n = \frac{1}{n^{1/3}}$ is positive, decreasing, and tends to zero; hence, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1/3}}$ converges by the Leibniz Test. However, the positive series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ is a divergent p-series, so the original series converges conditionally.

5. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(1.1)^n}$

SOLUTION The positive series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{1.1}\right)^n$ is a convergent geometric series; thus, the original series converges absolutely.

7. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

SOLUTION Let $a_n = \frac{1}{n \ln n}$. Then a_n forms a decreasing sequence (note that n and $\ln n$ are both increasing functions of n) that tends to zero; hence, the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges by the Leibniz Test. However, the positive series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges, so the original series converges conditionally.

9. $\displaystyle\sum_{n=2}^{\infty} \frac{\cos n\pi}{(\ln n)^2}$

SOLUTION Since $\cos n\pi$ alternates between $+1$ and -1,

$$\sum_{n=2}^{\infty} \frac{\cos n\pi}{(lnn)^2} = \sum_{n=2}^{\infty} \frac{(-1)^n}{(lnn)^2}$$

This is an alternating series whose general term decreases to zero, so it converges. The associated positive series,

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$$

is a divergent series, so the original series converges conditionally.

11. Let $S = \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^3}$.

(a) Calculate S_n for $1 \le n \le 10$.

(b) Use Eq. (2) to show that $0.9 \le S \le 0.902$.

SOLUTION

(a)

$$S_1 = 1 \qquad S_6 = S_5 - \frac{1}{6^3} = 0.899782407$$

$$S_2 = 1 - \frac{1}{2^3} = \frac{7}{8} = 0.875 \qquad S_7 = S_6 + \frac{1}{7^3} = 0.902697859$$

$$S_3 = S_2 + \frac{1}{3^3} = 0.912037037 \qquad S_8 = S_7 - \frac{1}{8^3} = 0.900744734$$

$$S_4 = S_3 - \frac{1}{4^3} = 0.896412037 \qquad S_9 = S_8 + \frac{1}{9^3} = 0.902116476$$

$$S_5 = S_4 + \frac{1}{5^3} = 0.904412037 \qquad S_{10} = S_9 - \frac{1}{10^3} = 0.901116476$$

(b) By Eq. (2),

$$|S_{10} - S| \le a_{11} = \frac{1}{11^3},$$

so

$$S_{10} - \frac{1}{11^3} \le S \le S_{10} + \frac{1}{11^3},$$

or

$$0.900365161 \le S \le 0.901867791.$$

13. Approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ to three decimal places.

SOLUTION Let $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$, so that $a_n = \frac{1}{n^4}$. By Eq. (2),

$$|S_N - S| \le a_{N+1} = \frac{1}{(N+1)^4}.$$

To guarantee accuracy to three decimal places, we must choose N so that

$$\frac{1}{(N+1)^4} < 5 \times 10^{-4} \quad \text{or} \quad N > \sqrt[4]{2000} - 1 \approx 5.7.$$

The smallest value that satisfies the required inequality is then $N = 6$. Thus,

$$S \approx S_6 = 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} = 0.946767824.$$

In Exercises 15 and 16, find a value of N such that S_N approximates the series with an error of at most 10^{-5}. If you have a CAS, compute this value of S_N.

15. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+2)(n+3)}$

SOLUTION Let $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,(n+2)\,(n+3)}$, so that $a_n = \frac{1}{n\,(n+2)\,(n+3)}$. By Eq. (2),

$$|S_N - S| \le a_{N+1} = \frac{1}{(N+1)(N+3)(N+4)}.$$

We must choose N so that

$$\frac{1}{(N+1)(N+3)(N+4)} \le 10^{-5} \quad \text{or} \quad (N+1)(N+3)(N+4) \ge 10^5.$$

For $N = 43$, the product on the left hand side is 95,128, while for $N = 44$ the product is 101,520; hence, the smallest value of N which satisfies the required inequality is $N = 44$. Thus,

$$S \approx S_{44} = \sum_{n=1}^{44} \frac{(-1)^{n+1}}{n(n+2)(n+3)} = 0.0656746.$$

In Exercises 17–32, determine convergence or divergence by any method.

17. $\sum_{n=0}^{\infty} 7^{-n}$

SOLUTION This is a (positive) geometric series with $r = \frac{1}{7} < 1$, so it converges.

19. $\sum_{n=1}^{\infty} \frac{1}{5^n - 3^n}$

SOLUTION Use the Limit Comparison Test with $\frac{1}{5^n}$:

$$L = \lim_{n\to\infty} \frac{1/(5^n - 3^n)}{1/5^n} = \lim_{n\to\infty} \frac{5^n}{5^n - 3^n} = \lim_{n\to\infty} \frac{1}{1-(3/5)^n} = 1$$

But $\sum_{n=1}^{\infty} \frac{1}{5^n}$ is a convergent geometric series. Since $L = 1$, the Limit Comparison Test tells us that the original series converges as well.

21. $\sum_{n=1}^{\infty} \frac{1}{3n^4 + 12n}$

SOLUTION Use the Limit Comparison Test with $\frac{1}{3n^4}$:

$$L = \lim_{n\to\infty} \frac{(1/(3n^4 + 12n)}{1/3n^4} = \lim_{n\to\infty} \frac{3n^4}{3n^4 + 12n} = \lim_{n\to\infty} \frac{1}{1 + 4n^{-3}} = 1$$

But $\sum_{n=1}^{\infty} \frac{1}{3n^4} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a convergent p-series. Since $L = 1$, the Limit Comparison Test tells us that the original series converges as well.

23. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$

SOLUTION Apply the Limit Comparison Test and compare the series with the divergent harmonic series:

$$L = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1.$$

Because $L > 0$, we conclude that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges.

25. $\sum_{n=1}^{\infty} \frac{3^n + (-2)^n}{5^n}$

SOLUTION The series

$$\sum_{n=1}^{\infty} \frac{3^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$$

is a convergent geometric series, as is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, 2^n}{5^n} = \sum_{n=1}^{\infty} \left(-\frac{2}{5}\right)^n.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{3^n + (-1)^n 2^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n + \sum_{n=1}^{\infty} \left(-\frac{2}{5}\right)^n$$

also converges.

27. $\sum_{n=1}^{\infty} (-1)^n n^2 e^{-n^3/3}$

SOLUTION Consider the associated positive series $\sum_{n=1}^{\infty} n^2 e^{-n^3/3}$. This series can be seen to converge by the Integral Test:

$$\int_1^{\infty} x^2 e^{-x^3/3}\, dx = \lim_{R\to\infty} \int_1^R x^2 e^{-x^3/3}\, dx = -\lim_{R\to\infty} e^{-x^3/3}\Big|_1^R = e^{-1/3} + \lim_{R\to\infty} e^{-R^3/3} = e^{-1/3}.$$

The integral converges, so the original series converges absolutely.

29. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1/2}(\ln n)^2}$

SOLUTION This is an alternating series with $a_n = \dfrac{1}{n^{1/2}(\ln n)^2}$. Because a_n is a decreasing sequence which converges to zero, the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1/2}(\ln n)^2}$ converges by the Leibniz Test. (Note that the series converges only conditionally, not absolutely; the associated positive series is eventually greater than $\dfrac{1}{n^{3/4}}$, which is a divergent p-series).

31. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{1.05}}$

SOLUTION Choose N so that for $n \geq N$ we have $\ln n \leq n^{0.01}$. Then

$$\sum_{n=N}^{\infty} \frac{\ln n}{n^{1.05}} \leq \sum_{n=N}^{\infty} \frac{n^{0.01}}{n^{1.05}} = \sum_{n=N}^{\infty} \frac{1}{n^{1.04}}$$

This is a convergent p-series, so by the Comparison Test, the original series converges as well.

33. Show that

$$S = \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \cdots$$

converges by computing the partial sums. Does it converge absolutely?

SOLUTION The sequence of partial sums is

$$S_1 = \frac{1}{2}$$

$$S_2 = S_1 - \frac{1}{2} = 0$$

$$S_3 = S_2 + \frac{1}{3} = \frac{1}{3}$$

$$S_4 = S_3 - \frac{1}{3} = 0$$

and, in general,

$$S_N = \begin{cases} \dfrac{1}{N}, & \text{for odd } N \\ 0, & \text{for even } N \end{cases}$$

Thus, $\displaystyle\lim_{N\to\infty} S_N = 0$, and the series converges to 0. The positive series is

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \cdots = 2\sum_{n=2}^{\infty} \frac{1}{n};$$

which diverges. Therefore, the original series converges conditionally, not absolutely.

35. **Assumptions Matter** Show by counterexample that the Leibniz Test does not remain true if the sequence a_n tends to zero but is not assumed nonincreasing. *Hint:* Consider

$$R = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{8} + \frac{1}{4} - \frac{1}{16} + \cdots + \left(\frac{1}{n} - \frac{1}{2^n}\right) + \cdots$$

SOLUTION Let

$$R = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{8} + \frac{1}{4} - \frac{1}{16} + \cdots + \left(\frac{1}{n+1} - \frac{1}{2^{n+1}}\right) + \cdots$$

This is an alternating series with

$$a_n = \begin{cases} \dfrac{1}{k+1}, & n = 2k-1 \\ \dfrac{1}{2^{k+1}}, & n = 2k \end{cases}$$

Note that $a_n \to 0$ as $n \to \infty$, but the sequence $\{a_n\}$ is not decreasing. We will now establish that R diverges.

For sake of contradiction, suppose that R converges. The geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^{n+1}}$$

converges, so the sum of R and this geometric series must also converge; however,

$$R + \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \sum_{n=2}^{\infty} \frac{1}{n},$$

which diverges because the harmonic series diverges. Thus, the series R must diverge.

37. Prove that if $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges. Then give an example where $\sum a_n$ is only conditionally convergent and $\sum a_n^2$ diverges.

SOLUTION Suppose the series $\sum a_n$ converges absolutely. Because $\sum |a_n|$ converges, we know that

$$\lim_{n\to\infty} |a_n| = 0.$$

Therefore, there exists a positive integer N such that $|a_n| < 1$ for all $n \geq N$. It then follows that for $n \geq N$,

$$0 \leq a_n^2 = |a_n|^2 = |a_n| \cdot |a_n| < |a_n| \cdot 1 = |a_n|.$$

By the Comparison Test we can then conclude that $\sum a_n^2$ also converges.

Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. This series converges by the Leibniz Test, but the corresponding positive series is a divergent p-series; that is, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is conditionally convergent. Now, $\sum_{n=1}^{\infty} a_n^2$ is the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. Thus, $\sum a_n^2$ need not converge if $\sum a_n$ is only conditionally convergent.

Further Insights and Challenges

39. Use Exercise 38 to show that the following series converges:

$$S = \frac{1}{\ln 2} + \frac{1}{\ln 3} - \frac{2}{\ln 4} + \frac{1}{\ln 5} + \frac{1}{\ln 6} - \frac{2}{\ln 7} + \cdots$$

SOLUTION The given series has the structure of the generic series from Exercise 38 with $a_n = \frac{1}{\ln(n+1)}$. Because a_n is a positive, decreasing sequence with $\lim_{n\to\infty} a_n = 0$, we can conclude from Exercise 38 that the given series converges.

41. Show that the following series diverges:

$$S = 1 + \frac{1}{2} + \frac{1}{3} - \frac{2}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{2}{8} + \cdots$$

Hint: Use the result of Exercise 40 to write S as the sum of a convergent series and a divergent series.

SOLUTION Let

$$R = 1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{3}{8} + \cdots$$

and

$$S = 1 + \frac{1}{2} + \frac{1}{3} - \frac{2}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{2}{8} + \cdots$$

For sake of contradiction, suppose the series S converges. From Exercise 40, we know that the series R converges. Thus, the series $S - R$ must converge; however,

$$S - R = \frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \cdots = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k},$$

which diverges because the harmonic series diverges. Thus, the series S must diverge.

43. We say that $\{b_n\}$ is a rearrangement of $\{a_n\}$ if $\{b_n\}$ has the same terms as $\{a_n\}$ but occurring in a different order. Show that if $\{b_n\}$ is a rearrangement of $\{a_n\}$ and $S = \sum_{n=1}^{\infty} a_n$ converges absolutely, then $T = \sum_{n=1}^{\infty} b_n$ also converges absolutely. (This result does not hold if S is only conditionally convergent.) *Hint:* Prove that the partial sums $\sum_{n=1}^{N} |b_n|$ are bounded. It can be shown further that $S = T$.

SOLUTION Suppose the series $S = \sum_{n=1}^{\infty} a_n$ converges absolutely and denote the corresponding positive series by

$$S^+ = \sum_{n=1}^{\infty} |a_n|.$$

Further, let $T_N = \sum_{n=1}^{N} |b_n|$ denote the Nth partial sum of the series $\sum_{n=1}^{\infty} |b_n|$. Because $\{b_n\}$ is a rearrangement of $\{a_n\}$, we know that

$$0 \le T_N \le \sum_{n=1}^{\infty} |a_n| = S^+;$$

that is, the sequence $\{T_N\}$ is bounded. Moreover,

$$T_{N+1} = \sum_{n=1}^{N+1} |b_n| = T_N + |b_{N+1}| \ge T_N;$$

that is, $\{T_N\}$ is increasing. It follows that $\{T_N\}$ converges, so the series $\sum_{n=1}^{\infty} |b_n|$ converges, which means the series $\sum_{n=1}^{\infty} b_n$ converges absolutely.

10.5 The Ratio and Root Tests

Preliminary Questions

1. In the Ratio Test, is ρ equal to $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ or $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$?

SOLUTION In the Ratio Test ρ is the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

2. Is the Ratio Test conclusive for $\sum_{n=1}^{\infty} \frac{1}{2^n}$? Is it conclusive for $\sum_{n=1}^{\infty} \frac{1}{n}$?

SOLUTION The general term of $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is $a_n = \frac{1}{2^n}$; thus,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2^{n+1}} \cdot \frac{2^n}{1} = \frac{1}{2},$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} < 1.$$

Consequently, the Ratio Test guarantees that the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.

The general term of $\sum_{n=1}^{\infty} \frac{1}{n}$ is $a_n = \frac{1}{n}$; thus,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{n+1} \cdot \frac{n}{1} = \frac{n}{n+1},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n}{n+1} = 1.$$

The Ratio Test is therefore inconclusive for the series $\sum_{n=1}^{\infty}\frac{1}{n}$.

3. Can the Ratio Test be used to show convergence if the series is only conditionally convergent?

SOLUTION No. The Ratio Test can only establish absolute convergence and divergence, not conditional convergence.

Exercises

In Exercises 1–20, apply the Ratio Test to determine convergence or divergence, or state that the Ratio Test is inconclusive.

1. $\sum_{n=1}^{\infty}\frac{1}{5^n}$

SOLUTION With $a_n = \frac{1}{5^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5^{n+1}}\cdot\frac{5^n}{1} = \frac{1}{5} \quad\text{and}\quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty}\frac{1}{5^n}$ converges by the Ratio Test.

3. $\sum_{n=1}^{\infty}\frac{1}{n^n}$

SOLUTION With $a_n = \frac{1}{n^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(n+1)^{n+1}}\cdot\frac{n^n}{1} = \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{1}{n+1}\left(1+\frac{1}{n}\right)^{-n},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0\cdot\frac{1}{e} = 0 < 1.$$

Therefore, the series $\sum_{n=1}^{\infty}\frac{1}{n^n}$ converges by the Ratio Test.

5. $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$

SOLUTION With $a_n = \frac{n}{n^2+1}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{(n+1)^2+1}\cdot\frac{n^2+1}{n} = \frac{n+1}{n}\cdot\frac{n^2+1}{n^2+2n+2},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1\cdot 1 = 1.$$

Therefore, for the series $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$, the Ratio Test is inconclusive.

We can show that this series diverges by using the Limit Comparison Test and comparing with the divergent harmonic series.

7. $\sum_{n=1}^{\infty}\frac{2^n}{n^{100}}$

SOLUTION With $a_n = \frac{2^n}{n^{100}}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)^{100}}\cdot\frac{n^{100}}{2^n} = 2\left(\frac{n}{n+1}\right)^{100} \quad\text{and}\quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2\cdot 1^{100} = 2 > 1.$$

Therefore, the series $\sum_{n=1}^{\infty}\frac{2^n}{n^{100}}$ diverges by the Ratio Test.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{2^{n^2}}$

SOLUTION With $a_n = \frac{10^n}{2^{n^2}}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{10^{n+1}}{2^{(n+1)^2}} \cdot \frac{2^{n^2}}{10^n} = 10 \cdot \frac{1}{2^{2n+1}} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 10 \cdot 0 = 0 < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{2^{n^2}}$ converges by the Ratio Test.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^n}$

SOLUTION With $a_n = \frac{e^n}{n^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{e^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{e^n} = \frac{e}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{e}{n+1}\left(1+\frac{1}{n}\right)^{-n},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 \cdot \frac{1}{e} = 0 < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^n}$ converges by the Ratio Test.

13. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{6^n}$

SOLUTION With $a_n = \frac{n!}{6^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{6^{n+1}} \cdot \frac{6^n}{n!} = \frac{n+1}{6} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty > 1.$$

Therefore, the series $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{6^n}$ diverges by the Ratio Test.

15. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

SOLUTION With $a_n = \frac{1}{n \ln n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(n+1)\ln(n+1)} \cdot \frac{n \ln n}{1} = \frac{n}{n+1}\,\frac{\ln n}{\ln(n+1)},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \cdot \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}.$$

Now,

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} = \lim_{x\to\infty} \frac{1/(x+1)}{1/x} = \lim_{x\to\infty} \frac{x}{x+1} = 1.$$

Thus, $\rho = 1$, and the Ratio Test is inconclusive for the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

Using the Integral Test, we can show that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

17. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(2n+1)!}$

SOLUTION With $a_n = \frac{n^2}{(2n+1)!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2}{(2n+3)!} \cdot \frac{(2n+1)!}{n^2} = \left(\frac{n+1}{n}\right)^2 \frac{1}{(2n+3)(2n+2)},$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1^2 \cdot 0 = 0 < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(2n+1)!}$ converges by the Ratio Test.

19. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2^n+1}$

SOLUTION With $a_n = \dfrac{1}{2^n+1}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2^{n+1}+1} \cdot \frac{2^n+1}{1} = \frac{1+2^{-n}}{2+2^{-n}}$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} < 1$$

Therefore, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2^n+1}$ converges by the Ratio Test.

21. Show that $\displaystyle\sum_{n=1}^{\infty} n^k\, 3^{-n}$ converges for all exponents k.

SOLUTION With $a_n = n^k 3^{-n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^k 3^{-(n+1)}}{n^k 3^{-n}} = \frac{1}{3}\left(1+\frac{1}{n}\right)^k,$$

and, for all k,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3} \cdot 1 = \frac{1}{3} < 1.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} n^k\, 3^{-n}$ converges for all exponents k by the Ratio Test.

23. Show that $\displaystyle\sum_{n=1}^{\infty} 2^n x^n$ converges if $|x| < \frac{1}{2}$.

SOLUTION With $a_n = 2^n x^n$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}|x|^{n+1}}{2^n|x|^n} = 2|x| \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 2|x|.$$

Therefore, $\rho < 1$ and the series $\displaystyle\sum_{n=1}^{\infty} 2^n x^n$ converges by the Ratio Test provided $|x| < \frac{1}{2}$.

25. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{r^n}{n}$ converges if $|r| < 1$.

SOLUTION With $a_n = \frac{r^n}{n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|r|^{n+1}}{n+1} \cdot \frac{n}{|r|^n} = |r|\frac{n}{n+1} \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \cdot |r| = |r|.$$

Therefore, by the Ratio Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{r^n}{n}$ converges provided $|r| < 1$.

27. Show that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges. *Hint:* Use $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

SOLUTION With $a_n = \frac{n!}{n^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n = \left(1+\frac{1}{n}\right)^{-n},$$

and

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{e} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by the Ratio Test.

In Exercises 28–33, assume that $|a_{n+1}/a_n|$ converges to $\rho = \frac{1}{3}$. What can you say about the convergence of the given series?

29. $\sum_{n=1}^{\infty} n^3 a_n$

SOLUTION Let $b_n = n^3 a_n$. Then

$$\rho = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3 \left|\frac{a_{n+1}}{a_n}\right| = 1^3 \cdot \frac{1}{3} = \frac{1}{3} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} n^3 a_n$ converges by the Ratio Test.

31. $\sum_{n=1}^{\infty} 3^n a_n$

SOLUTION Let $b_n = 3^n a_n$. Then

$$\rho = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\frac{3^{n+1}}{3^n}\left|\frac{a_{n+1}}{a_n}\right| = 3 \cdot \frac{1}{3} = 1.$$

Therefore, the Ratio Test is inconclusive for the series $\sum_{n=1}^{\infty} 3^n a_n$.

33. $\sum_{n=1}^{\infty} a_n^2$

SOLUTION Let $b_n = a_n^2$. Then

$$\rho = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} a_n^2$ converges by the Ratio Test.

35. Is the Ratio Test conclusive for the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$?

SOLUTION With $a_n = \frac{1}{n^p}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(n+1)^p} \cdot \frac{n^p}{1} = \left(\frac{n}{n+1}\right)^p \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1^p = 1.$$

Therefore, the Ratio Test is inconclusive for the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

In Exercises 36–41, use the Root Test to determine convergence or divergence (or state that the test is inconclusive).

37. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

SOLUTION With $a_n = \frac{1}{n^n}$,

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{1}{n^n}} = \frac{1}{n} \quad \text{and} \quad \lim_{n\to\infty} \sqrt[n]{a_n} = 0 < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges by the Root Test.

39. $\sum_{k=0}^{\infty} \left(\frac{k}{3k+1}\right)^k$

SOLUTION With $a_k = \left(\frac{k}{3k+1}\right)^k$,

$$\sqrt[k]{a_k} = \sqrt[k]{\left(\frac{k}{3k+1}\right)^k} = \frac{k}{3k+1} \quad \text{and} \quad \lim_{k\to\infty} \sqrt[k]{a_k} = \frac{1}{3} < 1.$$

Therefore, the series $\sum_{k=0}^{\infty} \left(\frac{k}{3k+1}\right)^k$ converges by the Root Test.

41. $\sum_{n=4}^{\infty} \left(1 + \frac{1}{n}\right)^{-n^2}$

SOLUTION With $a_k = \left(1 + \frac{1}{n}\right)^{-n^2}$,

$$\sqrt[n]{a_n} = \sqrt[n]{\left(1 + \frac{1}{n}\right)^{-n^2}} = \left(1 + \frac{1}{n}\right)^{-n} \quad \text{and} \quad \lim_{n\to\infty} \sqrt[n]{a_n} = e^{-1} < 1.$$

Therefore, the series $\sum_{n=4}^{\infty} \left(1 + \frac{1}{n}\right)^{-n^2}$ converges by the Root Test.

In Exercises 43–56, determine convergence or divergence using any method covered in the text so far.

43. $\sum_{n=1}^{\infty} \frac{2^n + 4^n}{7^n}$

SOLUTION Because the series

$$\sum_{n=1}^{\infty} \frac{2^n}{7^n} = \sum_{n=1}^{\infty} \left(\frac{2}{7}\right)^n \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{4^n}{7^n} = \sum_{n=1}^{\infty} \left(\frac{4}{7}\right)^n$$

are both convergent geometric series, it follows that

$$\sum_{n=1}^{\infty} \frac{2^n + 4^n}{7^n} = \sum_{n=1}^{\infty} \left(\frac{2}{7}\right)^n + \sum_{n=1}^{\infty} \left(\frac{4}{7}\right)^n$$

also converges.

45. $\sum_{n=1}^{\infty} \frac{n^3}{5^n}$

SOLUTION The presence of the exponential term suggests applying the Ratio Test. With $a_n = \frac{n^3}{5^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{5^{n+1}} \cdot \frac{5^n}{n^3} = \frac{1}{5}\left(1 + \frac{1}{n}\right)^3 \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5} \cdot 1^3 = \frac{1}{5} < 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{n^3}{5^n}$ converges by the Ratio Test.

47. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - n^2}}$

SOLUTION This series is similar to a p-series; because

$$\frac{1}{\sqrt{n^3 - n^2}} \approx \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$$

for large n, we will apply the Limit Comparison Test comparing with the p-series with $p = \frac{3}{2}$. Now,

$$L = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^3-n^2}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \sqrt{\frac{n^3}{n^3 - n^2}} = 1.$$

The p-series with $p = \frac{3}{2}$ converges and L exists; therefore, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - n^2}}$ also converges.

49. $\displaystyle\sum_{n=1}^{\infty} n^{-0.8}$

SOLUTION

$$\sum_{n=1}^{\infty} n^{-0.8} = \sum_{n=1}^{\infty} \frac{1}{n^{0.8}}$$

so that this is a divergent p-series.

51. $\displaystyle\sum_{n=1}^{\infty} 4^{-2n+1}$

SOLUTION Observe

$$\sum_{n=1}^{\infty} 4^{-2n+1} = \sum_{n=1}^{\infty} 4 \cdot (4^{-2})^n = \sum_{n=1}^{\infty} 4\left(\frac{1}{16}\right)^n$$

is a geometric series with $r = \frac{1}{16}$; therefore, this series converges.

53. $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$

SOLUTION Here, we will apply the Limit Comparison Test, comparing with the p-series with $p = 2$. Now,

$$L = \lim_{n\to\infty} \frac{\sin \frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{u\to 0} \frac{\sin u}{u} = 1,$$

where $u = \frac{1}{n^2}$. The p-series with $p = 2$ converges and L exists; therefore, the series $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$ also converges.

55. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}$

SOLUTION Because

$$\lim_{n\to\infty} \frac{2^n}{\sqrt{n}} = \lim_{x\to\infty} \frac{2^x}{\sqrt{x}} = \lim_{x\to\infty} \frac{2^x \ln 2}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} 2^{x+1}\sqrt{x}\ln 2 = \infty \neq 0,$$

the general term in the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}$ does not tend toward zero; therefore, the series diverges by the Divergence Test.

Further Insights and Challenges

57. **Proof of the Root Test** Let $S = \displaystyle\sum_{n=0}^{\infty} a_n$ be a positive series, and assume that $L = \displaystyle\lim_{n\to\infty} \sqrt[n]{a_n}$ exists.

(a) Show that S converges if $L < 1$. *Hint:* Choose R with $L < R < 1$ and show that $a_n \leq R^n$ for n sufficiently large. Then compare with the geometric series $\sum R^n$.

(b) Show that S diverges if $L > 1$.

SOLUTION Suppose $\lim_{n\to\infty} \sqrt[n]{a_n} = L$ exists.

(a) If $L < 1$, let $\epsilon = \dfrac{1-L}{2}$. By the definition of a limit, there is a positive integer N such that

$$-\epsilon \le \sqrt[n]{a_n} - L \le \epsilon$$

for $n \ge N$. From this, we conclude that

$$0 \le \sqrt[n]{a_n} \le L + \epsilon$$

for $n \ge N$. Now, let $R = L + \epsilon$. Then

$$R = L + \frac{1-L}{2} = \frac{L+1}{2} < \frac{1+1}{2} = 1,$$

and

$$0 \le \sqrt[n]{a_n} \le R \quad \text{or} \quad 0 \le a_n \le R^n$$

for $n \ge N$. Because $0 \le R < 1$, the series $\displaystyle\sum_{n=N}^{\infty} R^n$ is a convergent geometric series, so the series $\displaystyle\sum_{n=N}^{\infty} a_n$ converges by the Comparison Test. Therefore, the series $\displaystyle\sum_{n=0}^{\infty} a_n$ also converges.

(b) If $L > 1$, let $\epsilon = \dfrac{L-1}{2}$. By the definition of a limit, there is a positive integer N such that

$$-\epsilon \le \sqrt[n]{a_n} - L \le \epsilon$$

for $n \ge N$. From this, we conclude that

$$L - \epsilon \le \sqrt[n]{a_n}$$

for $n \ge N$. Now, let $R = L - \epsilon$. Then

$$R = L - \frac{L-1}{2} = \frac{L+1}{2} > \frac{1+1}{2} = 1,$$

and

$$R \le \sqrt[n]{a_n} \quad \text{or} \quad R^n \le a_n$$

for $n \ge N$. Because $R > 1$, the series $\displaystyle\sum_{n=N}^{\infty} R^n$ is a divergent geometric series, so the series $\displaystyle\sum_{n=N}^{\infty} a_n$ diverges by the Comparison Test. Therefore, the series $\displaystyle\sum_{n=0}^{\infty} a_n$ also diverges.

59. Let $S = \displaystyle\sum_{n=1}^{\infty} \frac{c^n n!}{n^n}$, where c is a constant.

(a) Prove that S converges absolutely if $|c| < e$ and diverges if $|c| > e$.

(b) It is known that $\displaystyle\lim_{n\to\infty} \frac{e^n n!}{n^{n+1/2}} = \sqrt{2\pi}$. Verify this numerically.

(c) Use the Limit Comparison Test to prove that S diverges for $c = e$.

SOLUTION

(a) With $a_n = \frac{c^n n!}{n^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|c|^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{|c|^n n!} = |c| \left(\frac{n}{n+1}\right)^n = |c| \left(1 + \frac{1}{n}\right)^{-n},$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |c|e^{-1}.$$

Thus, by the Ratio Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{c^n n!}{n^n}$ converges when $|c|e^{-1} < 1$, or when $|c| < e$. The series diverges when $|c| > e$.

(b) The table below lists the value of $\frac{e^n n!}{n^{n+1/2}}$ for several increasing values of n. Since $\sqrt{2\pi} = 2.506628275$, the numerical evidence verifies that

$$\lim_{n\to\infty} \frac{e^n n!}{n^{n+1/2}} = \sqrt{2\pi}.$$

n	100	1000	10000	100000
$\frac{e^n n!}{n^{n+1/2}}$	2.508717995	2.506837169	2.506649163	2.506630363

(c) With $c = e$, the series S becomes $\sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$. Using the result from part (b),

$$L = \lim_{n\to\infty} \frac{\frac{e^n n!}{n^n}}{\sqrt{n}} = \lim_{n\to\infty} \frac{e^n n!}{n^{n+1/2}} = \sqrt{2\pi}.$$

Because the series $\sum_{n=1}^{\infty} \sqrt{n}$ diverges by the Divergence Test and $L > 0$, we conclude that $\sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$ diverges by the Limit Comparison Test.

10.6 Power Series

Preliminary Questions

1. Suppose that $\sum a_n x^n$ converges for $x = 5$. Must it also converge for $x = 4$? What about $x = -3$?

SOLUTION The power series $\sum a_n x^n$ is centered at $x = 0$. Because the series converges for $x = 5$, the radius of convergence must be at least 5 and the series converges absolutely at least for the interval $|x| < 5$. Both $x = 4$ and $x = -3$ are inside this interval, so the series converges for $x = 4$ and for $x = -3$.

2. Suppose that $\sum a_n (x-6)^n$ converges for $x = 10$. At which of the points (a)–(d) must it also converge?

(a) $x = 8$ **(b)** $x = 11$ **(c)** $x = 3$ **(d)** $x = 0$

SOLUTION The given power series is centered at $x = 6$. Because the series converges for $x = 10$, the radius of convergence must be at least $|10 - 6| = 4$ and the series converges absolutely at least for the interval $|x - 6| < 4$, or $2 < x < 10$.

(a) $x = 8$ is inside the interval $2 < x < 10$, so the series converges for $x = 8$.

(b) $x = 11$ is not inside the interval $2 < x < 10$, so the series may or may not converge for $x = 11$.

(c) $x = 3$ is inside the interval $2 < x < 10$, so the series converges for $x = 2$.

(d) $x = 0$ is not inside the interval $2 < x < 10$, so the series may or may not converge for $x = 0$.

3. What is the radius of convergence of $F(3x)$ if $F(x)$ is a power series with radius of convergence $R = 12$?

SOLUTION If the power series $F(x)$ has radius of convergence $R = 12$, then the power series $F(3x)$ has radius of convergence $R = \frac{12}{3} = 4$.

4. The power series $F(x) = \sum_{n=1}^{\infty} nx^n$ has radius of convergence $R = 1$. What is the power series expansion of $F'(x)$ and what is its radius of convergence?

SOLUTION We obtain the power series expansion for $F'(x)$ by differentiating the power series expansion for $F(x)$ term-by-term. Thus,

$$F'(x) = \sum_{n=1}^{\infty} n^2 x^{n-1}.$$

The radius of convergence for this series is $R = 1$, the same as the radius of convergence for the series expansion for $F(x)$.

Exercises

1. Use the Ratio Test to determine the radius of convergence R of $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$. Does it converge at the endpoints $x = \pm R$?

SOLUTION With $a_n = \frac{x^n}{2^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{2^{n+1}} \cdot \frac{2^n}{|x|^n} = \frac{|x|}{2} \quad \text{and} \quad \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{2}.$$

By the Ratio Test, the series converges when $\rho = \frac{|x|}{2} < 1$, or $|x| < 2$, and diverges when $\rho = \frac{|x|}{2} > 1$, or $|x| > 2$. The radius of convergence is therefore $R = 2$. For $x = -2$, the left endpoint, the series becomes $\sum_{n=0}^{\infty}(-1)^n$, which is divergent. For $x = 2$, the right endpoint, the series becomes $\sum_{n=0}^{\infty} 1$, which is also divergent. Thus the series diverges at both endpoints.

3. Show that the power series (a)–(c) have the same radius of convergence. Then show that (a) diverges at both endpoints, (b) converges at one endpoint but diverges at the other, and (c) converges at both endpoints.

(a) $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$ **(b)** $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$ **(c)** $\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$

SOLUTION

(a) With $a_n = \frac{x^n}{3^n}$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{3}\right| = \left|\frac{x}{3}\right|$$

Then $\rho < 1$ if $|x| < 3$, so that the radius of convergence is $R = 3$. For the endpoint $x = 3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{3^n}{3^n} = \sum_{n=1}^{\infty} 1,$$

which diverges by the Divergence Test. For the endpoint $x = -3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n,$$

which also diverges by the Divergence Test.

(b) With $a_n = \frac{x^n}{n3^n}$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{3}\left(\frac{n}{n+1}\right)\right| = \left|\frac{x}{3}\right|.$$

Then $\rho < 1$ when $|x| < 3$, so that the radius of convergence is $R = 3$. For the endpoint $x = 3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is the divergent harmonic series. For the endpoint $x = -3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the Leibniz Test.

(c) With $a_n = \frac{x^n}{n^2 3^n}$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{3}\left(\frac{n}{n+1}\right)^2\right| = \left|\frac{x}{3}\right|$$

Then $\rho < 1$ when $|x| < 3$, so that the radius of convergence is $R = 3$. For the endpoint $x = 3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is a convergent p-series. For the endpoint $x = -3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

which converges by the Leibniz Test.

5. Show that $\sum_{n=0}^{\infty} n^n x^n$ diverges for all $x \neq 0$.

SOLUTION With $a_n = n^n x^n$, and assuming $x \neq 0$,

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1} x^{n+1}}{n^n x^n} \right| = \lim_{n\to\infty} \left| x \left(1 + \frac{1}{n}\right)^n (n+1) \right| = \infty$$

$\rho < 1$ only if $x = 0$, so that the radius of convergence is therefore $R = 0$. In other words, the power series converges only for $x = 0$.

7. Use the Ratio Test to show that $\sum_{n=0}^{\infty} \frac{x^{2n}}{3^n}$ has radius of convergence $R = \sqrt{3}$.

SOLUTION With $a_n = \frac{x^{2n}}{3^n}$,

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{3^{n+1}} \cdot \frac{3^n}{x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{3} \right| = \left| \frac{x^2}{3} \right|$$

Then $\rho < 1$ when $|x^2| < 3$, or $x = \sqrt{3}$, so the radius of convergence is $R = \sqrt{3}$.

In Exercises 9–34, find the interval of convergence.

9. $\sum_{n=0}^{\infty} n x^n$

SOLUTION With $a_n = n x^n$,

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{n x^n} \right| = \lim_{n\to\infty} \left| x \frac{n+1}{n} \right| = |x|$$

Then $\rho < 1$ when $|x| < 1$, so that the radius of convergence is $R = 1$, and the series converges absolutely on the interval $|x| < 1$, or $-1 < x < 1$. For the endpoint $x = 1$, the series becomes $\sum_{n=0}^{\infty} n$, which diverges by the Divergence Test. For the endpoint $x = -1$, the series becomes $\sum_{n=1}^{\infty} (-1)^n n$, which also diverges by the Divergence Test. Thus, the series $\sum_{n=0}^{\infty} n x^n$ converges for $-1 < x < 1$ and diverges elsewhere.

11. $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2^n n}$

SOLUTION With $a_n = (-1)^n \frac{x^{2n+1}}{2^n n}$,

$$\rho = \lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{2^{n+1}(n+1)} \cdot \frac{2^n n}{x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{2} \cdot \frac{n}{n+1} \right| = \left| \frac{x^2}{2} \right|$$

Then $\rho < 1$ when $|x| < \sqrt{2}$, so the radius of convergence is $R = \sqrt{2}$, and the series converges absolutely on the interval $-\sqrt{2} < x < \sqrt{2}$. For the endpoint $x = -\sqrt{2}$, the series becomes $\sum_{n=1}^{\infty} (-1)^n \frac{-\sqrt{2}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{2}}{n}$, which converges by the Leibniz test. For the endpoint $x = \sqrt{2}$, the series becomes $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{2}}{n}$ which also converges by the Leibniz test. Thus the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2^n n}$ converges for $-\sqrt{2} \leq x \leq \sqrt{2}$ and diverges elsewhere.

13. $\displaystyle\sum_{n=4}^{\infty} \frac{x^n}{n^5}$

SOLUTION With $a_n = \frac{x^n}{n^5}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^5} \cdot \frac{n^5}{x^n}\right| = \lim_{n\to\infty} \left|x\left(\frac{n}{n+1}\right)^5\right| = |x|$$

Then $\rho < 1$ when $|x| < 1$, so the radius of convergence is $R = 1$, and the series converges absolutely on the interval $|x| < 1$, or $-1 < x < 1$. For the endpoint $x = 1$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$, which is a convergent p-series. For the endpoint $x = -1$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$, which converges by the Leibniz Test. Thus, the series $\displaystyle\sum_{n=4}^{\infty} \frac{x^n}{n^5}$ converges for $-1 \le x \le 1$ and diverges elsewhere.

15. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$

SOLUTION With $a_n = \frac{x^n}{(n!)^2}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{((n+1)!)^2} \cdot \frac{(n!)^2}{x^n}\right| = \lim_{n\to\infty} \left|x\left(\frac{1}{n+1}\right)^2\right| = 0$$

$\rho < 1$ for all x, so the radius of convergence is $R = \infty$, and the series converges absolutely for all x.

17. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^3} x^n$

SOLUTION With $a_n = \frac{(2n)!x^n}{(n!)^3}$, and assuming $x \neq 0$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(2(n+1))!x^{n+1}}{((n+1)!)^3} \cdot \frac{(n!)^3}{(2n)!x^n}\right| = \lim_{n\to\infty} \left|x\frac{(2n+2)(2n+1)}{(n+1)^3}\right|$$

$$= \lim_{n\to\infty} \left|x\frac{4n^2+6n+2}{n^3+3n^2+3n+1}\right| = \lim_{n\to\infty} \left|x\frac{4n^{-1}+6n^{-1}+2n^{-3}}{1+3n^{-1}+3n^{-2}+n^{-3}}\right| = 0$$

Then $\rho < 1$ for all x, so the radius of convergence is $R = \infty$, and the series converges absolutely for all x.

19. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+1}}$

SOLUTION With $a_n = \frac{(-1)^n x^n}{\sqrt{n^2+1}}$,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{n+1}}{\sqrt{n^2+2n+2}} \cdot \frac{\sqrt{n^2+1}}{(-1)^n x^n}\right|$$

$$= \lim_{n\to\infty} \left|x\frac{\sqrt{n^2+1}}{\sqrt{n^2+2n+2}}\right| = \lim_{n\to\infty} \left|x\sqrt{\frac{n^2+1}{n^2+2n+2}}\right| = \lim_{n\to\infty} \left|x\sqrt{\frac{1+1/n^2}{1+2/n+2/n^2}}\right|$$

$$= |x|$$

Then $\rho < 1$ when $|x| < 1$, so the radius of convergence is $R = 1$, and the series converges absolutely on the interval $-1 < x < 1$. For the endpoint $x = 1$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$, which converges by the Leibniz Test. For the endpoint $x = -1$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$, which diverges by the Limit Comparison Test comparing with the divergent harmonic series. Thus, the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+1}}$ converges for $-1 < x \le 1$ and diverges elsewhere.

21. $\displaystyle\sum_{n=15}^{\infty} \frac{x^{2n+1}}{3n+1}$

SOLUTION With $a_n = \dfrac{x^{2n+1}}{3n+1}$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{3n+4}\cdot\frac{3n+1}{x^{2n+1}}\right| = \lim_{n\to\infty}\left|x^2\frac{3n+1}{3n+4}\right| = |x^2|$$

Then $\rho < 1$ when $|x^2| < 1$, so the radius of convergence is $R = 1$, and the series converges absolutely for $-1 < x < 1$. For the endpoint $x = 1$, the series becomes $\displaystyle\sum_{n=15}^{\infty}\frac{1}{3n+1}$, which diverges by the Limit Comparison Test comparing with the divergent harmonic series. For the endpoint $x = -1$, the series becomes $\displaystyle\sum_{n=15}^{\infty}\frac{-1}{3n+1}$, which also diverges by the Limit Comparison Test comparing with the divergent harmonic series. Thus, the series $\displaystyle\sum_{n=15}^{\infty}\frac{x^{2n+1}}{3n+1}$ converges for $-1 < x < 1$ and diverges elsewhere.

23. $\displaystyle\sum_{n=2}^{\infty}\frac{x^n}{\ln n}$

SOLUTION With $a_n = \frac{x^n}{\ln n}$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{\ln(n+1)}\cdot\frac{\ln n}{x^n}\right| = \lim_{n\to\infty}\left|x\frac{\ln(n+1)}{\ln n}\right| = \lim_{n\to\infty}\left|x\frac{1/(n+1)}{1/n}\right| = \lim_{n\to\infty}\left|x\frac{n}{n+1}\right| = |x|$$

using L'Hôpital's rule. Then $\rho < 1$ when $|x| < 1$, so the radius of convergence is 1, and the series converges absolutely on the interval $|x| < 1$, or $-1 < x < 1$. For the endpoint $x = 1$, the series becomes $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\ln n}$. Because $\frac{1}{\ln n} > \frac{1}{n}$ and $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n}$ is the divergent harmonic series, the endpoint series diverges by the Comparison Test. For the endpoint $x = -1$, the series becomes $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$, which converges by the Leibniz Test. Thus, the series $\displaystyle\sum_{n=2}^{\infty}\frac{x^n}{\ln n}$ converges for $-1 \le x < 1$ and diverges elsewhere.

25. $\displaystyle\sum_{n=1}^{\infty} n(x-3)^n$

SOLUTION With $a_n = n(x-3)^n$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-3)^{n+1}}{n(x-3)^n}\right| = \lim_{n\to\infty}\left|(x-3)\cdot\frac{n+1}{n}\right| = |x-3|$$

Then $\rho < 1$ when $|x-3| < 1$, so the radius of convergence is 1, and the series converges absolutely on the interval $|x-3| < 1$, or $2 < x < 4$. For the endpoint $x = 4$, the series becomes $\displaystyle\sum_{n=1}^{\infty} n$, which diverges by the Divergence Test. For the endpoint $x = 2$, the series becomes $\displaystyle\sum_{n=1}^{\infty}(-1)^n n$, which also diverges by the Divergence Test. Thus, the series $\displaystyle\sum_{n=1}^{\infty} n(x-3)^n$ converges for $2 < x < 4$ and diverges elsewhere.

27. $\displaystyle\sum_{n=1}^{\infty}(-1)^n n^5(x-7)^n$

SOLUTION With $a_n = (-1)^n n^5(x-7)^n$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^5(x-7)^{n+1}}{(-1)^n n^5(x-7)^n}\right| = \lim_{n\to\infty}\left|(x-7)\cdot\frac{(n+1)^5}{n^5}\right|$$

$$= \lim_{n\to\infty}\left|(x-7)\cdot\frac{n^5+\ldots}{n^5}\right| = |x-7|$$

Then $\rho < 1$ when $|x-7| < 1$, so the radius of convergence is 1, and the series converges absolutely on the interval $|x-7| < 1$, or $6 < x < 8$. For the endpoint $x = 6$, the series becomes $\sum_{n=1}^{\infty}(-1)^{2n}n^5 = \sum_{n=1}^{\infty} n^5$, which diverges by the Divergence Test. For the endpoint $x = 8$, the series becomes $\sum_{n=1}^{\infty}(-1)^n n^5$, which also diverges by the Divergence Test. Thus, the series $\sum_{n=1}^{\infty}(-1)^n n^5 (x-7)^n$ converges for $6 < x < 8$ and diverges elsewhere.

29. $\sum_{n=1}^{\infty} \frac{2^n}{3n}(x+3)^n$

SOLUTION With $a_n = \frac{2^n(x+3)^n}{3n}$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x+3)^{n+1}}{3(n+1)} \cdot \frac{3n}{2^n(x+3)^n}\right| = \lim_{n\to\infty}\left|2(x+3)\cdot\frac{3n}{3n+3}\right|$$
$$= \lim_{n\to\infty}\left|2(x+3)\cdot\frac{1}{1+1/n}\right| = |2(x+3)|$$

Then $\rho < 1$ when $|2(x+3)| < 1$, so when $|x+3| < \frac{1}{2}$. Thus the radius of convergence is $\frac{1}{2}$, and the series converges absolutely on the interval $|x+3| < \frac{1}{2}$, or $-\frac{7}{2} < x < -\frac{5}{2}$. For the endpoint $x = -\frac{5}{2}$, the series becomes $\sum_{n=1}^{\infty}\frac{1}{3n}$, which diverges because it is a multiple of the divergent harmonic series. For the endpoint $x = -\frac{7}{2}$, the series becomes $\sum_{n=1}^{\infty}\frac{(-1)^n}{3n}$, which converges by the Leibniz Test. Thus, the series $\sum_{n=1}^{\infty}\frac{2^n}{3n}(x+3)^n$ converges for $-\frac{7}{2} \le x < -\frac{5}{2}$ and diverges elsewhere.

31. $\sum_{n=0}^{\infty} \frac{(-5)^n}{n!}(x+10)^n$

SOLUTION With $a_n = \frac{(-5)^n}{n!}(x+10)^n$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-5)^{n+1}(x+10)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-5)^n(x+10)^n}\right| = \lim_{n\to\infty}\left|5(x+10)\frac{1}{n}\right| = 0$$

Thus $\rho < 1$ for all x, so the radius of convergence is infinite, and $\sum_{n=0}^{\infty}\frac{(-5)^n}{n!}(x+10)^n$ converges for all x.

33. $\sum_{n=12}^{\infty} e^n(x-2)^n$

SOLUTION With $a_n = e^n(x-2)^n$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^{n+1}(x-2)^{n+1}}{e^n(x-2)^n}\right| = \lim_{n\to\infty}|e(x-2)| = |e(x-2)|$$

Thus $\rho < 1$ when $|e(x-2)| < 1$, so when $|x-2| < e^{-1}$. Thus the radius of convergence is e^{-1}, and the series converges absolutely on the interval $|x-2| < e^{-1}$, or $2 - e^{-1} < x < 2 + e^{-1}$. For the endpoint $x = 2 + e^{-1}$, the series becomes $\sum_{n=1}^{\infty} 1$, which diverges by the Divergence Test. For the endpoint $x = 2 - e^{-1}$, the series becomes $\sum_{n=1}^{\infty}(-1)^n$, which also diverges by the Divergence Test. Thus, the series $\sum_{n=12}^{\infty} e^n(x-2)^n$ converges for $2 - e^{-1} < x < 2 + e^{-1}$ and diverges elsewhere.

In Exercises 35–40, use Eq. (2) to expand the function in a power series with center $c = 0$ and determine the interval of convergence.

35. $f(x) = \dfrac{1}{1-3x}$

SOLUTION Substituting $3x$ for x in Eq. (2), we obtain

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} 3^n x^n.$$

This series is valid for $|3x| < 1$, or $|x| < \frac{1}{3}$.

37. $f(x) = \dfrac{1}{3-x}$

SOLUTION First write

$$\frac{1}{3-x} = \frac{1}{3} \cdot \frac{1}{1-\frac{x}{3}}.$$

Substituting $\frac{x}{3}$ for x in Eq. (2), we obtain

$$\frac{1}{1-\frac{x}{3}} = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{3^n};$$

Thus,

$$\frac{1}{3-x} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{x^n}{3^n} = \sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}}.$$

This series is valid for $|x/3| < 1$, or $|x| < 3$.

39. $f(x) = \dfrac{1}{1+x^2}$

SOLUTION Substituting $-x^2$ for x in Eq. (2), we obtain

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

This series is valid for $|x| < 1$.

41. Use the equalities

$$\frac{1}{1-x} = \frac{1}{-3-(x-4)} = \frac{-\frac{1}{3}}{1+\left(\frac{x-4}{3}\right)}$$

to show that for $|x-4| < 3$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-4)^n}{3^{n+1}}$$

SOLUTION Substituting $-\frac{x-4}{3}$ for x in Eq. (2), we obtain

$$\frac{1}{1+\left(\frac{x-4}{3}\right)} = \sum_{n=0}^{\infty} \left(-\frac{x-4}{3}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{(x-4)^n}{3^n}.$$

Thus,

$$\frac{1}{1-x} = -\frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \frac{(x-4)^n}{3^n} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-4)^n}{3^{n+1}}.$$

This series is valid for $|-\frac{x-4}{3}| < 1$, or $|x-4| < 3$.

43. Use the method of Exercise 41 to expand $1/(4-x)$ in a power series with center $c = 5$. Determine the interval of convergence.

SOLUTION First write

$$\frac{1}{4-x} = \frac{1}{-1-(x-5)} = -\frac{1}{1+(x-5)}.$$

Substituting $-(x-5)$ for x in Eq. (2), we obtain

$$\frac{1}{1+(x-5)} = \sum_{n=0}^{\infty}(-(x-5))^n = \sum_{n=0}^{\infty}(-1)^n(x-5)^n.$$

Thus,

$$\frac{1}{4-x} = -\sum_{n=0}^{\infty}(-1)^n(x-5)^n = \sum_{n=0}^{\infty}(-1)^{n+1}(x-5)^n.$$

This series is valid for $|-(x-5)| < 1$, or $|x-5| < 1$.

45. Apply integration to the expansion

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

to prove that for $-1 < x < 1$,

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

SOLUTION To obtain the first expansion, substitute $-x$ for x in Eq. (2):

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n.$$

This expansion is valid for $|-x| < 1$, or $-1 < x < 1$.

Upon integrating both sides of the above equation, we find

$$\ln(1+x) = \int \frac{dx}{1+x} = \int \left(\sum_{n=0}^{\infty}(-1)^n x^n\right) dx.$$

Integrating the series term-by-term then yields

$$\ln(1+x) = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}.$$

To determine the constant C, set $x = 0$. Then $0 = \ln(1+0) = C$. Finally,

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}.$$

47. Let $F(x) = (x+1)\ln(1+x) - x$.

(a) Apply integration to the result of Exercise 45 to prove that for $-1 < x < 1$,

$$F(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{n+1}}{n(n+1)}$$

(b) Evaluate at $x = \frac{1}{2}$ to prove

$$\frac{3}{2}\ln\frac{3}{2} - \frac{1}{2} = \frac{1}{1\cdot 2\cdot 2^2} - \frac{1}{2\cdot 3\cdot 2^3} + \frac{1}{3\cdot 4\cdot 2^4} - \frac{1}{4\cdot 5\cdot 2^5} + \cdots$$

(c) Use a calculator to verify that the partial sum S_4 approximates the left-hand side with an error no greater than the term a_5 of the series.

SOLUTION

(a) Note that

$$\int \ln(x+1)\,dx = (x+1)\ln(x+1) - x + C$$

Then integrating both sides of the result of Exercise 45 gives

$$(x+1)\ln(x+1) - x = \int \ln(x+1)\,dx = \int \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}\,dx$$

For $-1 < x < 1$, which is the interval of convergence of the series in Exercise 45, therefore, we can integrate term by term to get

$$(x+1)\ln(x+1) - x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int x^n\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \frac{x^{n+1}}{n+1} + C = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+1}}{n(n+1)} + C$$

(noting that $(-1)^{n-1} = (-1)^{n+1}$). To determine C, evaluate both sides at $x = 0$ to get

$$0 = \ln 1 - 0 = 0 + C$$

so that $C = 0$ and we get finally

$$(x+1)\ln(x+1) - x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+1}}{n(n+1)}$$

(b) Evaluating the result of part(a) at $x = \frac{1}{2}$ gives

$$\frac{3}{2}\ln\frac{3}{2} - \frac{1}{2} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n(n+1)2^{n+1}}$$
$$= \frac{1}{1\cdot 2\cdot 2^2} - \frac{1}{2\cdot 3\cdot 2^3} + \frac{1}{3\cdot 4\cdot 2^4} - \frac{1}{4\cdot 5\cdot 2^5} + \dots$$

(c)

$$S_4 = \frac{1}{1\cdot 2\cdot 2^2} - \frac{1}{2\cdot 3\cdot 2^3} + \frac{1}{3\cdot 4\cdot 2^4} - \frac{1}{4\cdot 5\cdot 2^5} = 0.1078125$$
$$a_5 = \frac{1}{5\cdot 6\cdot 2^6} \approx 0.0005208$$
$$\frac{3}{2}\ln\frac{3}{2} - \frac{1}{2} \approx 0.10819766$$

and

$$\left| S_4 - \frac{3}{2}\ln\frac{3}{2} - \frac{1}{2} \right| \approx 0.0003852 < a_5$$

49. Use the result of Example 7 to show that

$$F(x) = \frac{x^2}{1\cdot 2} - \frac{x^4}{3\cdot 4} + \frac{x^6}{5\cdot 6} - \frac{x^8}{7\cdot 8} + \cdots$$

is an antiderivative of $f(x) = \tan^{-1} x$ satisfying $F(0) = 0$. What is the radius of convergence of this power series?

SOLUTION For $-1 < x < 1$, which is the interval of convergence for the power series for arctangent, we can integrate term-by-term, so integrate that power series to get

$$F(x) = \int \tan^{-1} x\,dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{2n+1}}{2n+1}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)}$$
$$= \frac{x^2}{1\cdot 2} - \frac{x^4}{3\cdot 4} + \frac{x^6}{5\cdot 6} - \frac{x^8}{7\cdot 8} + \cdots + C$$

If we assume $F(0) = 0$, then we have $C = 0$. The radius of convergence of this power series is the same as that of the original power series, which is 1.

51. Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$. *Hint:* Use differentiation to show that

$$(1-x)^{-2} = \sum_{n=1}^{\infty} nx^{n-1} \quad (\text{for } |x| < 1)$$

SOLUTION Differentiate both sides of Eq. (2) to obtain

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}.$$

Setting $x = \frac{1}{2}$ then yields

$$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4.$$

Divide this equation by 2 to obtain

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2.$$

53. Show that the following series converges absolutely for $|x| < 1$ and compute its sum:

$$F(x) = 1 - x - x^2 + x^3 - x^4 - x^5 + x^6 - x^7 - x^8 + \cdots$$

Hint: Write $F(x)$ as a sum of three geometric series with common ratio x^3.

SOLUTION Because the coefficients in the power series are all ± 1, we find

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1.$$

The radius of convergence is therefore $R = r^{-1} = 1$, and the series converges absolutely for $|x| < 1$.

By Exercise 43 of Section 10.4, any rearrangement of the terms of an absolutely convergent series yields another absolutely convergent series with the same sum as the original series. Following the hint, we now rearrange the terms of $F(x)$ as the sum of three geometric series:

$$F(x) = \left(1 + x^3 + x^6 + \cdots\right) - \left(x + x^4 + x^7 + \cdots\right) - \left(x^2 + x^5 + x^8 + \cdots\right)$$

$$= \sum_{n=0}^{\infty} (x^3)^n - \sum_{n=0}^{\infty} x(x^3)^n - \sum_{n=0}^{\infty} x^2(x^3)^n = \frac{1}{1-x^3} - \frac{x}{1-x^3} - \frac{x^2}{1-x^3} = \frac{1-x-x^2}{1-x^3}.$$

55. Find all values of x such that $\displaystyle\sum_{n=1}^{\infty} \frac{x^{n^2}}{n!}$ converges.

SOLUTION With $a_n = \frac{x^{n^2}}{n!}$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|^{(n+1)^2}}{(n+1)!} \cdot \frac{n!}{|x|^{n^2}} = \frac{|x|^{2n+1}}{n+1}.$$

if $|x| \le 1$, then

$$\lim_{n\to\infty} \frac{|x|^{2n+1}}{n+1} = 0,$$

and the series converges absolutely. On the other hand, if $|x| > 1$, then

$$\lim_{n\to\infty} \frac{|x|^{2n+1}}{n+1} = \infty,$$

and the series diverges. Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{x^{n^2}}{n!}$ converges for $-1 \le x \le 1$ and diverges elsewhere.

57. Find a power series $P(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ satisfying the differential equation $y' = -y$ with initial condition $y(0) = 1$. Then use Theorem 1 of Section 5.8 to conclude that $P(x) = e^{-x}$.

SOLUTION Let $P(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ and note that $P(0) = a_0$; thus, to satisfy the initial condition $P(0) = 1$, we must take $a_0 = 1$. Now,

$$P'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

so

$$P'(x) + P(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} + a_n\right] x^n.$$

In order for this series to be equal to zero, the coefficient of x^n must be equal to zero for each n; thus

$$(n+1)a_{n+1} + a_n = 0 \quad \text{or} \quad a_{n+1} = -\frac{a_n}{n+1}.$$

Starting from $a_0 = 1$, we then calculate

$$a_1 = -\frac{a_0}{1} = -1;$$

$$a_2 = -\frac{a_1}{2} = \frac{1}{2};$$

$$a_3 = -\frac{a_2}{3} = -\frac{1}{6} = -\frac{1}{3!};$$

and, in general,

$$a_n = (-1)^n \frac{1}{n!}.$$

Hence,

$$P(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}.$$

The solution to the initial value problem $y' = -y$, $y(0) = 1$ is $y = e^{-x}$. Because this solution is unique, it follows that

$$P(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = e^{-x}.$$

59. Use the power series for $y = e^x$ to show that

$$\frac{1}{e} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$$

Use your knowledge of alternating series to find an N such that the partial sum S_N approximates e^{-1} to within an error of at most 10^{-3}. Confirm this using a calculator to compute both S_N and e^{-1}.

SOLUTION Recall that the series for e^x is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Setting $x = -1$ yields

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - + \cdots = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - + \cdots.$$

This is an alternating series with $a_n = \frac{1}{(n+1)!}$. The error in approximating e^{-1} with the partial sum S_N is therefore bounded by

$$|S_N - e^{-1}| \le a_{N+1} = \frac{1}{(N+2)!}.$$

To make the error at most 10^{-3}, we must choose N such that

$$\frac{1}{(N+2)!} \le 10^{-3} \quad \text{or} \quad (N+2)! \ge 1000.$$

For $N = 4$, $(N+2)! = 6! = 720 < 1000$, but for $N = 5$, $(N+2)! = 7! = 5040$; hence, $N = 5$ is the smallest value that satisfies the error bound. The corresponding approximation is

$$S_5 = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} = 0.368055555$$

Now, $e^{-1} = 0.367879441$, so

$$|S_5 - e^{-1}| = 1.761 \times 10^{-4} < 10^{-3}.$$

61. Find a power series $P(x)$ satisfying the differential equation

$$y'' - xy' + y = 0$$ 9

with initial condition $y(0) = 1$, $y'(0) = 0$. What is the radius of convergence of the power series?

SOLUTION Let $P(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$P'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad P''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Note that $P(0) = a_0$ and $P'(0) = a_1$; in order to satisfy the initial conditions $P(0) = 1$, $P'(0) = 0$, we must have $a_0 = 1$ and $a_1 = 0$. Now,

$$\begin{aligned} P''(x) - xP'(x) + P(x) &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= 2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - n a_n + a_n\right] x^n. \end{aligned}$$

In order for this series to be equal to zero, the coefficient of x^n must be equal to zero for each n; thus, $2a_2 + a_0 = 0$ and $(n+2)(n+1)a_{n+2} - (n-1)a_n = 0$, or

$$a_2 = -\frac{1}{2}a_0 \quad \text{and} \quad a_{n+2} = \frac{n-1}{(n+2)(n+1)} a_n.$$

Starting from $a_1 = 0$, we calculate

$$a_3 = \frac{1-1}{(3)(2)} a_1 = 0;$$

$$a_5 = \frac{2}{(5)(4)} a_3 = 0;$$

$$a_7 = \frac{4}{(7)(6)} a_5 = 0;$$

and, in general, all of the odd coefficients are zero. As for the even coefficients, we have $a_0 = 1$, $a_2 = -\frac{1}{2}$,

$$a_4 = \frac{1}{(4)(3)} a_2 = -\frac{1}{4!};$$

$$a_6 = \frac{3}{(6)(5)} a_4 = -\frac{3}{6!};$$

$$a_8 = \frac{5}{(8)(7)} a_6 = -\frac{15}{8!}$$

and so on. Thus,

$$P(x) = 1 - \frac{1}{2}x^2 - \frac{1}{4!}x^4 - \frac{3}{6!}x^6 - \frac{15}{8!}x^8 - \cdots$$

To determine the radius of convergence, treat this as a series in the variable x^2, and observe that

$$r = \lim_{k \to \infty} \left| \frac{a_{2k+2}}{a_{2k}} \right| = \lim_{k \to \infty} \frac{2k-1}{(2k+2)(2k+1)} = 0.$$

Thus, the radius of convergence is $R = r^{-1} = \infty$.

63. Prove that

$$J_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+2}\, k!\, (k+3)!} x^{2k+2}$$

is a solution of the Bessel differential equation of order 2:

$$x^2 y'' + xy' + (x^2 - 4)y = 0$$

SOLUTION Let $J_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+2}\, k!\,(k+2)!} x^{2k+2}$. Then

$$J_2'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (k+1)}{2^{2k+1}\, k!\,(k+2)!} x^{2k+1}$$

$$J_2''(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (k+1)(2k+1)}{2^{2k+1}\, k!\,(k+2)!} x^{2k}$$

and

$$\begin{aligned} x^2 J_2''(x) + x J_2'(x) + (x^2-4) J_2(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k (k+1)(2k+1)}{2^{2k+1}\, k!\,(k+2)!} x^{2k+2} + \sum_{k=0}^{\infty} \frac{(-1)^k (k+1)}{2^{2k+1}\, k!\,(k+2)!} x^{2k+2} \\ &\quad - \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+2}\, k!\,(k+2)!} x^{2k+4} - \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}\, k!\,(k+2)!} x^{2k+2} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k k(k+2)}{2^{2k} k!(k+2)!} x^{2k+2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^{2k}\,(k-1)!\,(k+1)!} x^{2k+2} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k}(k-1)!(k+1)!} x^{2k+2} - \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k}(k-1)!(k+1)!} x^{2k+2} = 0. \end{aligned}$$

Further Insights and Challenges

65. Suppose that the coefficients of $F(x) = \sum_{n=0}^{\infty} a_n x^n$ are *periodic*; that is, for some whole number $M > 0$, we have $a_{M+n} = a_n$. Prove that $F(x)$ converges absolutely for $|x| < 1$ and that

$$F(x) = \frac{a_0 + a_1 x + \cdots + a_{M-1} x^{M-1}}{1 - x^M}$$

Hint: Use the hint for Exercise 53.

SOLUTION Suppose the coefficients of $F(x)$ are periodic, with $a_{M+n} = a_n$ for some whole number M and all n. The $F(x)$ can be written as the sum of M geometric series:

$$\begin{aligned} F(x) &= a_0\left(1 + x^M + x^{2M} + \cdots\right) + a_1\left(x + x^{M+1} + x^{2M+1} + \cdots\right) + \\ &= a_2\left(x^2 + x^{M+2} + x^{2M+2} + \cdots\right) + \cdots + a_{M-1}\left(x^{M-1} + x^{2M-1} + x^{3M-1} + \cdots\right) \\ &= \frac{a_0}{1-x^M} + \frac{a_1 x}{1-x^M} + \frac{a_2 x^2}{1-x^M} + \cdots + \frac{a_{M-1}x^{M-1}}{1-x^M} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_{M-1}x^{M-1}}{1-x^M}. \end{aligned}$$

As each geometric series converges absolutely for $|x| < 1$, it follows that $F(x)$ also converges absolutely for $|x| < 1$.

10.7 Taylor Series

Preliminary Questions

1. Determine $f(0)$ and $f'''(0)$ for a function $f(x)$ with Maclaurin series

$$T(x) = 3 + 2x + 12x^2 + 5x^3 + \cdots$$

SOLUTION The Maclaurin series for a function f has the form

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Matching this general expression with the given series, we find $f(0) = 3$ and $\frac{f'''(0)}{3!} = 5$. From this latter equation, it follows that $f'''(0) = 30$.

2. Determine $f(-2)$ and $f^{(4)}(-2)$ for a function with Taylor series

$$T(x) = 3(x+2) + (x+2)^2 - 4(x+2)^3 + 2(x+2)^4 + \cdots$$

SOLUTION The Taylor series for a function f centered at $x = -2$ has the form

$$f(-2) + \frac{f'(-2)}{1!}(x+2) + \frac{f''(-2)}{2!}(x+2)^2 + \frac{f'''(-2)}{3!}(x+2)^3 + \frac{f^{(4)}(-2)}{4!}(x+2)^4 + \cdots$$

Matching this general expression with the given series, we find $f(-2) = 0$ and $\frac{f^{(4)}(-2)}{4!} = 2$. From this latter equation, it follows that $f^{(4)}(-2) = 48$.

3. What is the easiest way to find the Maclaurin series for the function $f(x) = \sin(x^2)$?

SOLUTION The easiest way to find the Maclaurin series for $\sin\left(x^2\right)$ is to substitute x^2 for x in the Maclaurin series for $\sin x$.

4. Find the Taylor series for $f(x)$ centered at $c = 3$ if $f(3) = 4$ and $f'(x)$ has a Taylor expansion

$$f'(x) = \sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$

SOLUTION Integrating the series for $f'(x)$ term-by-term gives

$$f(x) = C + \sum_{n=1}^{\infty} \frac{(x-3)^{n+1}}{n(n+1)}.$$

Substituting $x = 3$ then yields

$$f(3) = C = 4;$$

so

$$f(x) = 4 + \sum_{n=1}^{\infty} \frac{(x-3)^{n+1}}{n(n+1)}.$$

5. Let $T(x)$ be the Maclaurin series of $f(x)$. Which of the following guarantees that $f(2) = T(2)$?

(a) $T(x)$ converges for $x = 2$.
(b) The remainder $R_k(2)$ approaches a limit as $k \to \infty$.
(c) The remainder $R_k(2)$ approaches zero as $k \to \infty$.

SOLUTION The correct response is **(c)**: $f(2) = T(2)$ if and only if the remainder $R_k(2)$ approaches zero as $k \to \infty$.

Exercises

1. Write out the first four terms of the Maclaurin series of $f(x)$ if

$$f(0) = 2, \quad f'(0) = 3, \quad f''(0) = 4, \quad f'''(0) = 12$$

SOLUTION The first four terms of the Maclaurin series of $f(x)$ are

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 2 + 3x + \frac{4}{2}x^2 + \frac{12}{6}x^3 = 2 + 3x + 2x^2 + 2x^3.$$

In Exercises 3–18, find the Maclaurin series and find the interval on which the expansion is valid.

3. $f(x) = \dfrac{1}{1-2x}$

SOLUTION Substituting $2x$ for x in the Maclaurin series for $\frac{1}{1-x}$ gives

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n.$$

This series is valid for $|2x| < 1$, or $|x| < \frac{1}{2}$.

5. $f(x) = \cos 3x$

SOLUTION Substituting $3x$ for x in the Maclaurin series for $\cos x$ gives

$$\cos 3x = \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{9^n x^{2n}}{(2n)!}.$$

This series is valid for all x.

7. $f(x) = \sin(x^2)$

SOLUTION Substituting x^2 for x in the Maclaurin series for $\sin x$ gives

$$\sin x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}.$$

This series is valid for all x.

9. $f(x) = \ln(1 - x^2)$

SOLUTION Substituting $-x^2$ for x in the Maclaurin series for $\ln(1 + x)$ gives

$$\ln(1 - x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-x^2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1} x^{2n}}{n} = -\sum_{n=1}^{\infty} \frac{x^{2n}}{n}.$$

This series is valid for $|x| < 1$.

11. $f(x) = \tan^{-1}(x^2)$

SOLUTION Substituting x^2 for x in the Maclaurin series for $\tan^{-1} x$ gives

$$\tan^{-1}(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}.$$

This series is valid for $|x| \le 1$.

13. $f(x) = e^{x-2}$

SOLUTION $e^{x-2} = e^{-2}e^x$; thus,

$$e^{x-2} = e^{-2} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{e^2 n!}.$$

This series is valid for all x.

15. $f(x) = \ln(1 - 5x)$

SOLUTION Substituting $-5x$ for x in the Maclaurin series for $\ln(1 + x)$ gives

$$\ln(1 - 5x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-5x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1} 5^n x^n}{n} = -\sum_{n=1}^{\infty} \frac{5^n x^n}{n}.$$

This series is valid for $|5x| < 1$, or $|x| < \frac{1}{5}$, and for $x = -\frac{1}{5}$.

17. $f(x) = \sinh x$

SOLUTION Recall that

$$\sinh x = \frac{1}{2}(e^x - e^{-x}).$$

Therefore,

$$\sinh x = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{x^n}{2(n!)} \left(1 - (-1)^n\right).$$

Now,

$$1 - (-1)^n = \begin{cases} 0, & n \text{ even} \\ 2, & n \text{ odd} \end{cases}$$

so

$$\sinh x = \sum_{k=0}^{\infty} 2 \frac{x^{2k+1}}{2(2k+1)!} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.$$

This series is valid for all x.

In Exercises 19–28, find the terms through degree four of the Maclaurin series of $f(x)$. Use multiplication and substitution as necessary.

19. $f(x) = e^x \sin x$

SOLUTION Multiply the fourth-order Taylor Polynomials for e^x and $\sin x$:

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\right)\left(x - \frac{x^3}{6}\right)$$
$$= x + x^2 - \frac{x^3}{6} + \frac{x^3}{2} - \frac{x^4}{6} + \frac{x^4}{6} + \text{ higher-order terms}$$
$$= x + x^2 + \frac{x^3}{3} + \text{ higher-order terms.}$$

The terms through degree four in the Maclaurin series for $f(x) = e^x \sin x$ are therefore

$$x + x^2 + \frac{x^3}{3}.$$

21. $f(x) = \dfrac{\sin x}{1 - x}$

SOLUTION Multiply the fourth order Taylor Polynomials for $\sin x$ and $\dfrac{1}{1-x}$:

$$\left(x - \frac{x^3}{6}\right)\left(1 + x + x^2 + x^3 + x^4\right)$$
$$= x + x^2 - \frac{x^3}{6} + x^3 + x^4 - \frac{x^4}{6} + \text{ higher-order terms}$$
$$= x + x^2 + \frac{5x^3}{6} + \frac{5x^4}{6} + \text{ higher-order terms.}$$

The terms through order four of the Maclaurin series for $f(x) = \dfrac{\sin x}{1-x}$ are therefore

$$x + x^2 + \frac{5x^3}{6} + \frac{5x^4}{6}.$$

23. $f(x) = (1 + x)^{1/4}$

SOLUTION The first five generalized binomial coefficients for $a = \frac{1}{4}$ are

$$1, \quad \frac{1}{4}, \quad \frac{\frac{1}{4}\left(\frac{-3}{4}\right)}{2!} = -\frac{3}{32}, \quad \frac{\frac{1}{4}\left(\frac{-3}{4}\right)\left(\frac{-7}{4}\right)}{3!} = \frac{7}{128}, \quad \frac{\frac{1}{4}\left(\frac{-3}{4}\right)\left(\frac{-7}{4}\right)\left(\frac{-11}{4}\right)}{4!} = \frac{-77}{2048}$$

Therefore, the first four terms in the binomial series for $(1 + x)^{1/4}$ are

$$1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3 - \frac{77}{2048}x^4$$

25. $f(x) = e^x \tan^{-1} x$

SOLUTION Using the Maclaurin series for e^x and $\tan^{-1} x$, we find

$$e^x \tan^{-1} x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(x - \frac{x^3}{3} + \cdots\right) = x + x^2 - \frac{x^3}{3} + \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^4}{3} + \cdots$$
$$= x + x^2 + \frac{1}{6}x^3 - \frac{1}{6}x^4 + \cdots.$$

27. $f(x) = e^{\sin x}$

SOLUTION Substituting $\sin x$ for x in the Maclaurin series for e^x and then using the Maclaurin series for $\sin x$, we find

$$\begin{aligned} e^{\sin x} &= 1 + \sin x + \frac{\sin^2 x}{2} + \frac{\sin^3 x}{6} + \frac{\sin^4 x}{24} + \cdots \\ &= 1 + \left(x - \frac{x^3}{6} + \cdots\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + \cdots\right)^2 + \frac{1}{6}(x - \cdots)^3 + \frac{1}{24}(x - \cdots)^4 \\ &= 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{6}x^3 - \frac{1}{6}x^4 + \frac{1}{24}x^4 + \cdots \\ &= 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots . \end{aligned}$$

In Exercises 29–38, find the Taylor series centered at c and find the interval on which the expansion is valid.

29. $f(x) = \frac{1}{x}$, $c = 1$

SOLUTION Write

$$\frac{1}{x} = \frac{1}{1 + (x-1)},$$

and then substitute $-(x-1)$ for x in the Maclaurin series for $\frac{1}{1-x}$ to obtain

$$\frac{1}{x} = \sum_{n=0}^{\infty} [-(x-1)]^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n.$$

This series is valid for $|x - 1| < 1$.

31. $f(x) = \frac{1}{1-x}$, $c = 5$

SOLUTION Write

$$\frac{1}{1-x} = \frac{1}{-4-(x-5)} = -\frac{1}{4} \cdot \frac{1}{1 + \frac{x-5}{4}}.$$

Substituting $-\frac{x-5}{4}$ for x in the Maclaurin series for $\frac{1}{1-x}$ yields

$$\frac{1}{1 + \frac{x-5}{4}} = \sum_{n=0}^{\infty} \left(-\frac{x-5}{4}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{(x-5)^n}{4^n}.$$

Thus,

$$\frac{1}{1-x} = -\frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \frac{(x-5)^n}{4^n} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-5)^n}{4^{n+1}}.$$

This series is valid for $\left|\frac{x-5}{4}\right| < 1$, or $|x - 5| < 4$.

33. $f(x) = x^4 + 3x - 1$, $c = 2$

SOLUTION To determine the Taylor series with center $c = 2$, we compute

$$f'(x) = 4x^3 + 3, \quad f''(x) = 12x^2, \quad f'''(x) = 24x,$$

and $f^{(4)}(x) = 24$. All derivatives of order five and higher are zero. Now,

$$f(2) = 21, \quad f'(2) = 35, \quad f''(2) = 48, \quad f'''(2) = 48,$$

and $f^{(4)}(2) = 24$. Therefore, the Taylor series is

$$21 + 35(x-2) + \frac{48}{2}(x-2)^2 + \frac{48}{6}(x-2)^3 + \frac{24}{24}(x-2)^4,$$

or

$$21 + 35(x-2) + 24(x-2)^2 + 8(x-2)^3 + (x-2)^4.$$

35. $f(x) = \dfrac{1}{x^2}, \quad c = 4$

SOLUTION We will first find the Taylor series for $\frac{1}{x}$ and then differentiate to obtain the series for $\frac{1}{x^2}$. Write

$$\frac{1}{x} = \frac{1}{4 + (x-4)} = \frac{1}{4} \cdot \frac{1}{1 + \frac{x-4}{4}}.$$

Now substitute $-\frac{x-4}{4}$ for x in the Maclaurin series for $\frac{1}{1-x}$ to obtain

$$\frac{1}{x} = \frac{1}{4} \sum_{n=}^{\infty} \left(-\frac{x-4}{4} \right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{(x-4)^n}{4^{n+1}}.$$

Differentiating term-by-term yields

$$-\frac{1}{x^2} = \sum_{n=1}^{\infty} (-1)^n n \frac{(x-4)^{n-1}}{4^{n+1}},$$

so that

$$\frac{1}{x^2} = \sum_{n=1}^{\infty} (-1)^{n-1} n \frac{(x-4)^{n-1}}{4^{n+1}} = \sum_{n=0}^{\infty} (-1)^n (n+1) \frac{(x-4)^n}{4^{n+2}}.$$

This series is valid for $\left| \frac{x-4}{4} \right| < 1$, or $|x - 4| < 4$.

37. $f(x) = \dfrac{1}{1-x^2}, \quad c = 3$

SOLUTION By partial fraction decomposition

$$\frac{1}{1-x^2} = \frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x},$$

so

$$\frac{1}{1-x^2} = \frac{\frac{1}{2}}{-2-(x-3)} + \frac{\frac{1}{2}}{4+(x-3)} = -\frac{1}{4} \cdot \frac{1}{1 + \frac{x-3}{2}} + \frac{1}{8} \cdot \frac{1}{1 + \frac{x-3}{4}}.$$

Substituting $-\frac{x-3}{2}$ for x in the Maclaurin series for $\frac{1}{1-x}$ gives

$$\frac{1}{1 + \frac{x-3}{2}} = \sum_{n=0}^{\infty} \left(-\frac{x-3}{2} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} (x-3)^n,$$

while substituting $-\frac{x-3}{4}$ for x in the same series gives

$$\frac{1}{1 + \frac{x-3}{4}} = \sum_{n=0}^{\infty} \left(-\frac{x-3}{4} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} (x-3)^n.$$

Thus,

$$\frac{1}{1-x^2} = -\frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} (x-3)^n + \frac{1}{8} \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} (x-3)^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+2}} (x-3)^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+3}} (x-3)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{(-1)^{n+1}}{2^{n+2}} + \frac{(-1)^n}{2^{2n+3}} \right) (x-3)^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2^{n+1} - 1)}{2^{2n+3}} (x-3)^n.$$

This series is valid for $|x - 3| < 2$.

39. Use the identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ to find the Maclaurin series for $\cos^2 x$.

SOLUTION The Maclaurin series for $\cos 2x$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}$$

so the Maclaurin series for $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ is

$$\frac{1 + \left(1 + \sum_{n=1}^{\infty}(-1)^n \frac{2^{2n}x^{2n}}{(2n)!}\right)}{2} = 1 + \sum_{n=1}^{\infty}(-1)^n \frac{2^{2n-1}x^{2n}}{(2n)!}$$

41. Use the Maclaurin series for $\ln(1+x)$ and $\ln(1-x)$ to show that

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

for $|x| < 1$. What can you conclude by comparing this result with that of Exercise 40?

SOLUTION Using the Maclaurin series for $\ln(1+x)$ and $\ln(1-x)$, we have for $|x| < 1$

$$\ln(1+x) - \ln(1-x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(-x)^n$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n + \sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{1 + (-1)^{n-1}}{n}x^n.$$

Since $1 + (-1)^{n-1} = 0$ for even n and $1 + (-1)^{n-1} = 2$ for odd n,

$$\ln(1+x) - \ln(1-x) = \sum_{k=0}^{\infty} \frac{2}{2k+1}x^{2k+1}.$$

Thus,

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\left(\ln(1+x) - \ln(1-x)\right) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{2}{2k+1}x^{2k+1} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}.$$

Observe that this is the same series we found in Exercise 40; therefore,

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \tanh^{-1} x.$$

43. Show, by integrating the Maclaurin series for $f(x) = \dfrac{1}{\sqrt{1-x^2}}$, that for $|x| < 1$,

$$\sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{x^{2n+1}}{2n+1}$$

SOLUTION From Example 10, we know that for $|x| < 1$

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}x^{2n} = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}x^{2n},$$

so, for $|x| < 1$,

$$\sin^{-1} x = \int \frac{dx}{\sqrt{1-x^2}} = C + x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{x^{2n+1}}{2n+1}.$$

Since $\sin^{-1} 0 = 0$, we find that $C = 0$. Thus,

$$\sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{x^{2n+1}}{2n+1}.$$

45. How many terms of the Maclaurin series of $f(x) = \ln(1+x)$ are needed to compute $\ln 1.2$ to within an error of at most 0.0001? Make the computation and compare the result with the calculator value.

SOLUTION Substitute $x = 0.2$ into the Maclaurin series for $\ln(1+x)$ to obtain:

$$\ln 1.2 = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(0.2)^n}{n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{5^n n}.$$

This is an alternating series with $a_n = \dfrac{1}{n \cdot 5^n}$. Using the error bound for alternating series

$$|\ln 1.2 - S_N| \le a_{N+1} = \frac{1}{(N+1)5^{N+1}},$$

so we must choose N so that

$$\frac{1}{(N+1)5^{N+1}} < 0.0001 \quad \text{or} \quad (N+1)5^{N+1} > 10{,}000.$$

For $N = 3$, $(N+1)5^{N+1} = 4 \cdot 5^4 = 2500 < 10,000$, and for $N = 4$, $(N+1)5^{N+1} = 5 \cdot 5^5 = 15,625 > 10,000$; thus, the smallest acceptable value for N is $N = 4$. The corresponding approximation is:

$$S_4 = \sum_{n=1}^{4} \frac{(-1)^{n-1}}{5^n \cdot n} = \frac{1}{5} - \frac{1}{5^2 \cdot 2} + \frac{1}{5^3 \cdot 3} - \frac{1}{5^4 \cdot 4} = 0.182266666.$$

Now, $\ln 1.2 = 0.182321556$, so

$$|\ln 1.2 - S_4| = 5.489 \times 10^{-5} < 0.0001.$$

47. Use the Maclaurin expansion for e^{-t^2} to express the function $F(x) = \int_0^x e^{-t^2}\,dt$ as an alternating power series in x (Figure 4).

(a) How many terms of the Maclaurin series are needed to approximate the integral for $x = 1$ to within an error of at most 0.001?

(b) CAS Carry out the computation and check your answer using a computer algebra system.

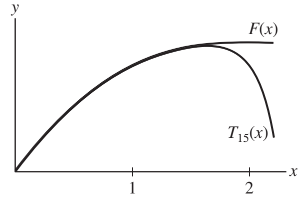


FIGURE 4 The Maclaurin polynomial $T_{15}(x)$ for $F(t) = \int_0^x e^{-t^2}\,dt$.

SOLUTION Substituting $-t^2$ for t in the Maclaurin series for e^t yields

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!};$$

thus,

$$\int_0^x e^{-t^2}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}.$$

(a) For $x = 1$,

$$\int_0^1 e^{-t^2}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(2n+1)}.$$

This is an alternating series with $a_n = \frac{1}{n!(2n+1)}$; therefore, the error incurred by using S_N to approximate the value of the definite integral is bounded by

$$\left| \int_0^1 e^{-t^2}\,dt - S_N \right| \le a_{N+1} = \frac{1}{(N+1)!(2N+3)}.$$

To guarantee the error is at most 0.001, we must choose N so that

$$\frac{1}{(N+1)!(2N+3)} < 0.001 \quad \text{or} \quad (N+1)!(2N+3) > 1000.$$

For $N = 3$, $(N+1)!(2N+3) = 4! \cdot 9 = 216 < 1000$ and for $N = 4$, $(N+1)!(2N+3) = 5! \cdot 11 = 1320 > 1000$; thus, the smallest acceptable value for N is $N = 4$. The corresponding approximation is

$$S_4 = \sum_{n=0}^{4} \frac{(-1)^n}{n!(2n+1)} = 1 - \frac{1}{3} + \frac{1}{2! \cdot 5} - \frac{1}{3! \cdot 7} + \frac{1}{4! \cdot 9} = 0.747486772.$$

(b) Using a computer algebra system, we find

$$\int_0^1 e^{-t^2}\,dt = 0.746824133;$$

therefore

$$\left|\int_0^1 e^{-t^2}\,dt - S_4\right| = 6.626 \times 10^{-4} < 10^{-3}.$$

In Exercises 49–52, express the definite integral as an infinite series and find its value to within an error of at most 10^{-4}.

49. $\displaystyle\int_0^1 \cos(x^2)\,dx$

SOLUTION Substituting x^2 for x in the Maclaurin series for $\cos x$ yields

$$\cos(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!};$$

therefore,

$$\int_0^1 \cos(x^2)\,dx = \sum_{n=0}^{\infty} (-1)^n \left. \frac{x^{4n+1}}{(2n)!(4n+1)} \right|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(4n+1)}.$$

This is an alternating series with $a_n = \frac{1}{(2n)!(4n+1)}$; therefore, the error incurred by using S_N to approximate the value of the definite integral is bounded by

$$\left|\int_0^1 \cos(x^2)\,dx - S_N\right| \le a_{N+1} = \frac{1}{(2N+2)!(4N+5)}.$$

To guarantee the error is at most 0.0001, we must choose N so that

$$\frac{1}{(2N+2)!(4N+5)} < 0.0001 \quad \text{or} \quad (2N+2)!(4N+5) > 10{,}000.$$

For $N = 2$, $(2N+2)!(4N+5) = 6! \cdot 13 = 9360 < 10{,}000$ and for $N = 3$, $(2N+2)!(4N+5) = 8! \cdot 17 = 685{,}440 > 10{,}000$; thus, the smallest acceptable value for N is $N = 3$. The corresponding approximation is

$$S_3 = \sum_{n=0}^{3} \frac{(-1)^n}{(2n)!(4n+1)} = 1 - \frac{1}{5 \cdot 2!} + \frac{1}{9 \cdot 4!} - \frac{1}{13 \cdot 6!} = 0.904522792.$$

51. $\displaystyle\int_0^1 e^{-x^3}\,dx$

SOLUTION Substituting $-x^3$ for x in the Maclaurin series for e^x yields

$$e^{-x^3} = \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{n!};$$

therefore,

$$\int_0^1 e^{-x^3}\,dx = \sum_{n=0}^{\infty} (-1)^n \left. \frac{x^{3n+1}}{n!(3n+1)} \right|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(3n+1)}.$$

This is an alternating series with $a_n = \frac{1}{n!(3n+1)}$; therefore, the error incurred by using S_N to approximate the value of the definite integral is bounded by

$$\left|\int_0^1 e^{-x^3}\,dx - S_N\right| \le a_{N+1} = \frac{1}{(N+1)!(3N+4)}.$$

To guarantee the error is at most 0.0001, we must choose N so that

$$\frac{1}{(N+1)!(3N+4)} < 0.0001 \quad \text{or} \quad (N+1)!(3N+4) > 10{,}000.$$

For $N = 4$, $(N+1)!(3N+4) = 5! \cdot 16 = 1920 < 10{,}000$ and for $N = 5$, $(N+1)!(3N+4) = 6! \cdot 19 = 13{,}680 > 10{,}000$; thus, the smallest acceptable value for N is $N = 5$. The corresponding approximation is

$$S_5 = \sum_{n=0}^{5} \frac{(-1)^n}{n!(3n+1)} = 0.807446200.$$

In Exercises 53–56, express the integral as an infinite series.

53. $\displaystyle\int_0^x \frac{1-\cos(t)}{t}\,dt$, for all x

SOLUTION The Maclaurin series for $\cos t$ is

$$\cos t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!},$$

so

$$1 - \cos t = -\sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!},$$

and

$$\frac{1-\cos t}{t} = \frac{1}{t}\sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n-1}}{(2n)!}.$$

Thus,

$$\int_0^x \frac{1-\cos(t)}{t}\,dt = \sum_{n=1}^{\infty} (-1)^{n+1} \left.\frac{t^{2n}}{(2n)!2n}\right|_0^x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{(2n)!2n}.$$

55. $\displaystyle\int_0^x \ln(1+t^2)\,dt$, for $|x| < 1$

SOLUTION Substituting t^2 for t in the Maclaurin series for $\ln(1+t)$ yields

$$\ln(1+t^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(t^2)^n}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{n}.$$

Thus,

$$\int_0^x \ln(1+t^2)\,dt = \sum_{n=1}^{\infty} (-1)^n \left.\frac{t^{2n+1}}{n(2n+1)}\right|_0^x = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{n(2n+1)}.$$

57. Which function has Maclaurin series $\displaystyle\sum_{n=0}^{\infty} (-1)^n 2^n x^n$?

SOLUTION We recognize that

$$\sum_{n=0}^{\infty} (-1)^n 2^n x^n = \sum_{n=0}^{\infty} (-2x)^n$$

is the Maclaurin series for $\frac{1}{1-x}$ with x replaced by $-2x$. Therefore,

$$\sum_{n=0}^{\infty} (-1)^n 2^n x^n = \frac{1}{1-(-2x)} = \frac{1}{1+2x}.$$

In Exercises 59–62, use Theorem 2 to prove that the $f(x)$ is represented by its Maclaurin series for all x.

59. $f(x) = \sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{3}\right)$,

SOLUTION All derivatives of $f(x)$ consist of sin or cos applied to each of $x/2$ and $x/3$ and added together, so each summand is bounded by 1. Thus $\left|f^{(n)}(x)\right| \le 2$ for all n and x. By Theorem 2, $f(x)$ is represented by its Taylor series for every x.

61. $f(x) = \sinh x$

SOLUTION By definition, $\sinh x = \frac{1}{2}(e^x - e^{-x})$, so if both e^x and e^{-x} are represented by their Taylor series centered at c, then so is $\sinh x$. But the previous exercise shows that e^{-x} is so represented, and the text shows that e^x is.

In Exercises 63–66, find the functions with the following Maclaurin series (refer to Table 1 on page 599).

63. $1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \cdots$

SOLUTION We recognize

$$1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} = \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!}$$

as the Maclaurin series for e^x with x replaced by x^3. Therefore,

$$1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \cdots = e^{x^3}.$$

65. $1 - \frac{5^3x^3}{3!} + \frac{5^5x^5}{5!} - \frac{5^7x^7}{7!} + \cdots$

SOLUTION Note

$$1 - \frac{5^3x^3}{3!} + \frac{5^5x^5}{5!} - \frac{5^7x^7}{7!} + \cdots = 1 - 5x + \left(5x - \frac{5^3x^3}{3!} + \frac{5^5x^5}{5!} - \frac{5^7x^7}{7!} + \cdots\right)$$

$$= 1 - 5x + \sum_{n=0}^{\infty} (-1)^n \frac{(5x)^{2n+1}}{(2n+1)!}.$$

The series is the Maclaurin series for $\sin x$ with x replaced by $5x$, so

$$1 - \frac{5^3x^3}{3!} + \frac{5^5x^5}{5!} - \frac{5^7x^7}{7!} + \cdots = 1 - 5x + \sin(5x).$$

In Exercises 67 and 68, let

$$f(x) = \frac{1}{(1-x)(1-2x)}$$

67. Find the Maclaurin series of $f(x)$ using the identity

$$f(x) = \frac{2}{1-2x} - \frac{1}{1-x}$$

SOLUTION Substituting $2x$ for x in the Maclaurin series for $\frac{1}{1-x}$ gives

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$$

which is valid for $|2x| < 1$, or $|x| < \frac{1}{2}$. Because the Maclaurin series for $\frac{1}{1-x}$ is valid for $|x| < 1$, the two series together are valid for $|x| < \frac{1}{2}$. Thus, for $|x| < \frac{1}{2}$,

$$\frac{1}{(1-2x)(1-x)} = \frac{2}{1-2x} - \frac{1}{1-x} = 2\sum_{n=0}^{\infty} 2^n x^n - \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty} 2^{n+1} x^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left(2^{n+1} - 1\right) x^n.$$

69. When a voltage V is applied to a series circuit consisting of a resistor R and an inductor L, the current at time t is

$$I(t) = \left(\frac{V}{R}\right)\left(1 - e^{-Rt/L}\right)$$

Expand $I(t)$ in a Maclaurin series. Show that $I(t) \approx \frac{Vt}{L}$ for small t.

SOLUTION Substituting $-\frac{Rt}{L}$ for t in the Maclaurin series for e^t gives

$$e^{-Rt/L} = \sum_{n=0}^{\infty} \frac{\left(-\frac{Rt}{L}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\frac{R}{L}\right)^n t^n = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\left(\frac{R}{L}\right)^n t^n$$

Thus,

$$1 - e^{-Rt/L} = 1 - \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\left(\frac{R}{L}\right)^n t^n\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}\left(\frac{Rt}{L}\right)^n,$$

and

$$I(t) = \frac{V}{R}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}\left(\frac{Rt}{L}\right)^n = \frac{Vt}{L} + \frac{V}{R}\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n!}\left(\frac{Rt}{L}\right)^n.$$

If t is small, then we can approximate $I(t)$ by the first (linear) term, and ignore terms with higher powers of t; then we find

$$V(t) \approx \frac{Vt}{L}.$$

71. Find the Maclaurin series for $f(x) = \cos(x^3)$ and use it to determine $f^{(6)}(0)$.

SOLUTION The Maclaurin series for $\cos x$ is

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Substituting x^3 for x gives

$$\cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!}$$

Now, the coefficient of x^6 in this series is

$$-\frac{1}{2!} = -\frac{1}{2} = \frac{f^{(6)}(0)}{6!}$$

so

$$f^{(6)}(0) = -\frac{6!}{2} = -360$$

73. Use substitution to find the first three terms of the Maclaurin series for $f(x) = e^{x^{20}}$. How does the result show that $f^{(k)}(0) = 0$ for $1 \le k \le 19$?

SOLUTION Substituting x^{20} for x in the Maclaurin series for e^x yields

$$e^{x^{20}} = \sum_{n=0}^{\infty} \frac{(x^{20})^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{20n}}{n!};$$

the first three terms in the series are then

$$1 + x^{20} + \frac{1}{2}x^{40}.$$

Recall that the coefficient of x^k in the Maclaurin series for f is $\frac{f^{(k)}(0)}{k!}$. For $1 \le k \le 19$, the coefficient of x^k in the Maclaurin series for $f(x) = e^{x^{20}}$ is zero; it therefore follows that

$$\frac{f^{(k)}(0)}{k!} = 0 \quad \text{or} \quad f^{(k)}(0) = 0$$

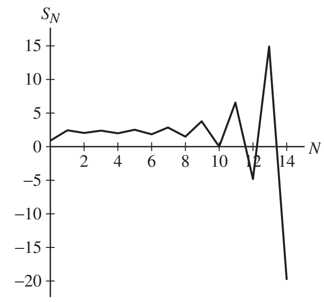
for $1 \le k \le 19$.

75. Does the Maclaurin series for $f(x) = (1+x)^{3/4}$ converge to $f(x)$ at $x = 2$? Give numerical evidence to support your answer.

SOLUTION The Taylor series for $f(x) = (1+x)^{3/4}$ converges to $f(x)$ for $|x| < 1$; because $x = 2$ is not contained on this interval, the series does not converge to $f(x)$ at $x = 2$. The graph below displays

$$S_N = \sum_{n=0}^{N} \binom{\frac{3}{4}}{n} 2^n$$

for $0 \le N \le 14$. The divergent nature of the sequence of partial sums is clear.



77. GU Let $f(x) = \sqrt{1+x}$.

(a) Use a graphing calculator to compare the graph of f with the graphs of the first five Taylor polynomials for f. What do they suggest about the interval of convergence of the Taylor series?

(b) Investigate numerically whether or not the Taylor expansion for f is valid for $x = 1$ and $x = -1$.

SOLUTION

(a) The five first terms of the Binomial series with $a = \frac{1}{2}$ are

$$\begin{aligned}\sqrt{1+x} &= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}x^4 + \cdots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{9}{4}x^3 - \frac{45}{2}x^4 + \cdots\end{aligned}$$

Therefore, the first five Taylor polynomials are

$$T_0(x) = 1;$$

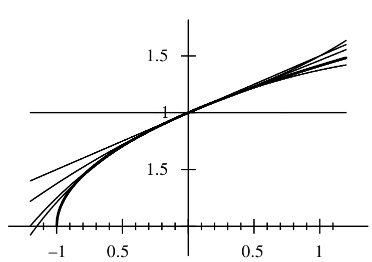
$$T_1(x) = 1 + \frac{1}{2}x;$$

$$T_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2;$$

$$T_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{8}x^3;$$

$$T_4(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{8}x^3 - \frac{5}{128}x^4.$$

The figure displays the graphs of these Taylor polynomials, along with the graph of the function $f(x) = \sqrt{1+x}$, which is shown in red.



The graphs suggest that the interval of convergence for the Taylor series is $-1 < x < 1$.

(b) Using a computer algebra system to calculate $S_N = \sum_{n=0}^{N} \binom{\frac{1}{2}}{n} x^n$ for $x = 1$ we find

$$S_{10} = 1.409931183, \quad S_{100} = 1.414073048, \quad S_{1000} = 1.414209104,$$

which appears to be converging to $\sqrt{2}$ as expected. At $x = -1$ we calculate $S_N = \sum_{n=0}^{N} \binom{\frac{1}{2}}{n} \cdot (-1)^n$, and find

$$S_{10} = 0.176197052, \quad S_{100} = 0.056348479, \quad S_{1000} = 0.017839011,$$

which appears to be converging to zero, though slowly.

79. Use Example 11 and the approximation $\sin x \approx x$ to show that the period T of a pendulum released at an angle θ has the following second-order approximation:

$$T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{\theta^2}{16}\right)$$

SOLUTION The period T of a pendulum of length L released from an angle θ is

$$T = 4\sqrt{\frac{L}{g}}E(k),$$

where $g \approx 9.8\ \text{m/s}^2$ is the acceleration due to gravity, $E(k)$ is the elliptic function of the first kind and $k = \sin\frac{\theta}{2}$. From Example 11, we know that

$$E(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right)^2 k^{2n}.$$

Using the approximation $\sin x \approx x$, we have

$$k = \sin\frac{\theta}{2} \approx \frac{\theta}{2};$$

moreover, using the first two terms of the series for $E(k)$, we find

$$E(k) \approx \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2\left(\frac{\theta}{2}\right)^2\right] = \frac{\pi}{2}\left(1 + \frac{\theta^2}{16}\right).$$

Therefore,

$$T = 4\sqrt{\frac{L}{g}}E(k) \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{\theta^2}{16}\right).$$

In Exercises 80–83, find the Maclaurin series of the function and use it to calculate the limit.

81. $\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$

SOLUTION Using the Maclaurin series for $\sin x$, we find

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} + \sum_{n=3}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

Thus,

$$\sin x - x + \frac{x^3}{6} = \frac{x^5}{120} + \sum_{n=3}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

and

$$\frac{\sin x - x + \frac{x^3}{6}}{x^5} = \frac{1}{120} + \sum_{n=3}^{\infty}(-1)^n\frac{x^{2n-4}}{(2n+1)!}$$

Note that the radius of convergence for this series is infinite, and recall from the previous section that a convergent power series is continuous within its radius of convergence. Thus to calculate the limit of this power series as $x \to 0$ it suffices to evaluate it at $x = 0$:

$$\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x\to 0}\left(\frac{1}{120} + \sum_{n=3}^{\infty}(-1)^n\frac{x^{2n-4}}{(2n+1)!}\right) = \frac{1}{120} + 0 = \frac{1}{120}$$

83. $\lim_{x\to 0}\left(\frac{\sin(x^2)}{x^4}-\frac{\cos x}{x^2}\right)$

SOLUTION We start with

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\qquad \cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

so that

$$\frac{\sin(x^2)}{x^4}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{(2n+1)!x^4}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n-2}}{(2n+1)!}$$

$$\frac{\cos x}{x^2}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n-2}}{(2n)!}$$

Expanding the first few terms gives

$$\frac{\sin(x^2)}{x^4}=\frac{1}{x^2}-\sum_{n=1}^{\infty}(-1)^n\frac{x^{4n-2}}{(2n+1)!}$$

$$\frac{\cos x}{x^2}=\frac{1}{x^2}-\frac{1}{2}+\sum_{n=2}^{\infty}(-1)^n\frac{x^{2n-2}}{(2n)!}$$

so that

$$\frac{\sin(x^2)}{x^4}-\frac{\cos x}{x^2}=\frac{1}{2}-\sum_{n=1}^{\infty}(-1)^n\frac{x^{4n-2}}{(2n+1)!}-\sum_{n=2}^{\infty}(-1)^n\frac{x^{2n-2}}{(2n)!}$$

Note that all terms under the summation signs have positive powers of x. Now, the radius of convergence of the series for both sin and cos is infinite, so the radius of convergence of this series is infinite. Recall from the previous section that a convergent power series is continuous within its radius of convergence. Thus to calculate the limit of this power series as $x\to 0$ it suffices to evaluate it at $x=0$:

$$\lim_{x\to 0}\left(\frac{\sin(x^2)}{x^4}-\frac{\cos x}{x^2}\right)=\lim_{x\to 0}\left(\frac{1}{2}-\sum_{n=1}^{\infty}(-1)^n\frac{x^{4n-2}}{(2n+1)!}-\sum_{n=2}^{\infty}(-1)^n\frac{x^{2n-2}}{(2n)!}\right)=\frac{1}{2}+0=\frac{1}{2}$$

Further Insights and Challenges

85. Let $g(t)=\frac{1}{1+t^2}-\frac{t}{1+t^2}$.

(a) Show that $\int_0^1 g(t)\,dt=\frac{\pi}{4}-\frac{1}{2}\ln 2$.

(b) Show that $g(t)=1-t-t^2+t^3-t^4-t^5-t^6+\dots$

(c) Evaluate $S=1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\dots$

SOLUTION

(a)

$$\int_0^1 g(t)\,dt=\left(\tan^{-1}t-\frac{1}{2}\ln(t^2+1)\right)\Big|_0^1=\tan^{-1}1-\frac{1}{2}\ln 2=\frac{\pi}{4}-\frac{1}{2}\ln 2$$

(b) Start with the Taylor series for $\frac{1}{1+t}$:

$$\frac{1}{1+t}=\sum_{n=0}^{\infty}(-1)^n t^n$$

and substitute t^2 for t to get

$$\frac{1}{1+t^2}=\sum_{n=0}^{\infty}(-1)^n t^{2n}=1-t^2+t^4-t^6+\dots$$

so that

$$\frac{t}{1+t^2} = \sum_{n=0}^{\infty}(-1)^n t^{2n+1} = t - t^3 + t^5 - t^7 + \dots$$

Finally,

$$g(t) = \frac{1}{1+t^2} - \frac{t}{1+t^2} = 1 - t - t^2 + t^3 + t^4 - t^5 - t^6 + t^7 + \dots$$

(c) We have

$$\int g(t)\,dt = \int (1 - t - t^2 + t^3 + t^4 - t^5 - \dots)\,dt = t - \frac{1}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{4}t^4 + \frac{1}{5}t^5 - \frac{1}{6}t^6 - \dots + C$$

The radius of convergence of the series for $g(t)$ is 1, so the radius of convergence of this series is also 1. However, this series converges at the right endpoint, $t = 1$, since

$$\left(1 - \frac{1}{2}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) - \dots$$

is an alternating series with general term decreasing to zero. Thus by part (a),

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \dots = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

In Exercises 86 and 87, we investigate the convergence of the binomial series

$$T_a(x) = \sum_{n=0}^{\infty} \binom{a}{n} x^n$$

87. By Exercise 86, $T_a(x)$ converges for $|x| < 1$, but we do not yet know whether $T_a(x) = (1+x)^a$.

(a) Verify the identity

$$a\binom{a}{n} = n\binom{a}{n} + (n+1)\binom{a}{n+1}$$

(b) Use (a) to show that $y = T_a(x)$ satisfies the differential equation $(1+x)y' = ay$ with initial condition $y(0) = 1$.

(c) Prove that $T_a(x) = (1+x)^a$ for $|x| < 1$ by showing that the derivative of the ratio $\dfrac{T_a(x)}{(1+x)^a}$ is zero.

SOLUTION

(a)

$$\begin{aligned} n\binom{a}{n} + (n+1)\binom{a}{n+1} &= n \cdot \frac{a(a-1)\cdots(a-n+1)}{n!} + (n+1)\cdot\frac{a(a-1)\cdots(a-n+1)(a-n)}{(n+1)!} \\ &= \frac{a(a-1)\cdots(a-n+1)}{(n-1)!} + \frac{a(a-1)\cdots(a-n+1)(a-n)}{n!} \\ &= \frac{a(a-1)\cdots(a-n+1)(n+(a-n))}{n!} = a\cdot\binom{a}{n} \end{aligned}$$

(b) Differentiating $T_a(x)$ term-by-term yields

$$T_a'(x) = \sum_{n=1}^{\infty} n\binom{a}{n} x^{n-1}.$$

Thus,

$$\begin{aligned} (1+x)T_a'(x) &= \sum_{n=1}^{\infty} n\binom{a}{n}x^{n-1} + \sum_{n=1}^{\infty} n\binom{a}{n}x^n = \sum_{n=0}^{\infty}(n+1)\binom{a}{n+1}x^n + \sum_{n=0}^{\infty} n\binom{a}{n}x^n \\ &= \sum_{n=0}^{\infty}\left[(n+1)\binom{a}{n+1} + n\binom{a}{n}\right]x^n = a\sum_{n=0}^{\infty}\binom{a}{n}x^n = aT_a(x). \end{aligned}$$

Moreover,

$$T_a(0) = \binom{a}{0} = 1.$$

(c)

$$\frac{d}{dx}\left(\frac{T_a(x)}{(1+x)^a}\right) = \frac{(1+x)^a T_a'(x) - a(1+x)^{a-1}T_a(x)}{(1+x)^{2a}} = \frac{(1+x)T_a'(x) - aT_a(x)}{(1+x)^{a+1}} = 0.$$

Thus,

$$\frac{T_a(x)}{(1+x)^a} = C,$$

for some constant C. For $x = 0$,

$$\frac{T_a(0)}{(1+0)^a} = \frac{1}{1} = 1, \text{ so } C = 1.$$

Finally, $T_a(x) = (1+x)^a$.

89. Assume that $a < b$ and let L be the arc length (circumference) of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ shown in Figure 5. There is no explicit formula for L, but it is known that $L = 4bG(k)$, with $G(k)$ as in Exercise 88 and $k = \sqrt{1 - a^2/b^2}$. Use the first three terms of the expansion of Exercise 88 to estimate L when $a = 4$ and $b = 5$.

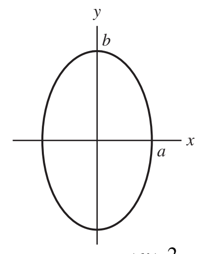


FIGURE 5 The ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

SOLUTION With $a = 4$ and $b = 5$,

$$k = \sqrt{1 - \frac{4^2}{5^2}} = \frac{3}{5},$$

and the arc length of the ellipse $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$ is

$$L = 20G\left(\frac{3}{5}\right) = 20\left(\frac{\pi}{2} - \frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right)^2 \frac{\left(\frac{3}{5}\right)^{2n}}{2n-1}\right).$$

Using the first three terms in the series for $G(k)$ gives

$$L \approx 10\pi - 10\pi\left(\left(\frac{1}{2}\right)^2\cdot\frac{(3/5)^2}{1} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\cdot\frac{(3/5)^4}{3}\right) = 10\pi\left(1 - \frac{9}{100} - \frac{243}{40{,}000}\right) = \frac{36{,}157\pi}{4000} \approx 28.398.$$

91. Irrationality of *e* Prove that e is an irrational number using the following argument by contradiction. Suppose that $e = M/N$, where M, N are nonzero integers.

(a) Show that $M!\,e^{-1}$ is a whole number.

(b) Use the power series for e^x at $x = -1$ to show that there is an integer B such that $M!\,e^{-1}$ equals

$$B + (-1)^{M+1}\left(\frac{1}{M+1} - \frac{1}{(M+1)(M+2)} + \cdots\right)$$

(c) Use your knowledge of alternating series with decreasing terms to conclude that $0 < |M!\,e^{-1} - B| < 1$ and observe that this contradicts (a). Hence, e is not equal to M/N.

SOLUTION Suppose that $e = M/N$, where M, N are nonzero integers.

(a) With $e = M/N$,

$$M!e^{-1} = M!\frac{N}{M} = (M-1)!N,$$

which is a whole number.

(b) Substituting $x = -1$ into the Maclaurin series for e^x and multiplying the resulting series by $M!$ yields

$$M!e^{-1} = M!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^k}{k!} + \cdots\right).$$

For all $k \le M$, $\frac{M!}{k!}$ is a whole number, so

$$M!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^k}{M!}\right)$$

is an integer. Denote this integer by B. Thus,

$$M!\,e^{-1} = B + M!\left(\frac{(-1)^{M+1}}{(M+1)!} + \frac{(-1)^{M+2}}{(M+2)!} + \cdots\right) = B + (-1)^{M+1}\left(\frac{1}{M+1} - \frac{1}{(M+1)(M+2)} + \cdots\right).$$

(c) The series for $M!\,e^{-1}$ obtained in part (b) is an alternating series with $a_n = \frac{M!}{n!}$. Using the error bound for an alternating series and noting that $B = S_M$, we have

$$\left|M!\,e^{-1} - B\right| \le a_{M+1} = \frac{1}{M+1} < 1.$$

This inequality implies that $M!\,e^{-1} - B$ is not a whole number; however, B is a whole number so $M!\,e^{-1}$ cannot be a whole number. We get a contradiction to the result in part (a), which proves that the original assumption that e is a rational number is false.

CHAPTER REVIEW EXERCISES

1. Let $a_n = \frac{n-3}{n!}$ and $b_n = a_{n+3}$. Calculate the first three terms in each sequence.

(a) a_n^2

(b) b_n

(c) $a_n b_n$

(d) $2a_{n+1} - 3a_n$

SOLUTION

(a)

$$a_1^2 = \left(\frac{1-3}{1!}\right)^2 = (-2)^2 = 4;$$

$$a_2^2 = \left(\frac{2-3}{2!}\right)^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4};$$

$$a_3^2 = \left(\frac{3-3}{3!}\right)^2 = 0.$$

(b)

$$b_1 = a_4 = \frac{4-3}{4!} = \frac{1}{24};$$

$$b_2 = a_5 = \frac{5-3}{5!} = \frac{1}{60};$$

$$b_3 = a_6 = \frac{6-3}{6!} = \frac{1}{240}.$$

(c) Using the formula for a_n and the values in (b) we obtain:

$$a_1 b_1 = \frac{1-3}{1!} \cdot \frac{1}{24} = -\frac{1}{12};$$

$$a_2 b_2 = \frac{2-3}{2!} \cdot \frac{1}{60} = -\frac{1}{120};$$

$$a_3 b_3 = \frac{3-3}{3!} \cdot \frac{1}{240} = 0.$$

(d)

$$2a_2 - 3a_1 = 2\left(-\frac{1}{2}\right) - 3(-2) = 5;$$

$$2a_3 - 3a_2 = 2 \cdot 0 - 3\left(-\frac{1}{2}\right) = \frac{3}{2};$$

$$2a_4 - 3a_3 = 2 \cdot \frac{1}{24} - 3 \cdot 0 = \frac{1}{12}.$$

In Exercises 3–8, compute the limit (or state that it does not exist) assuming that $\lim_{n\to\infty} a_n = 2$.

3. $\lim_{n\to\infty} (5a_n - 2a_n^2)$

SOLUTION

$$\lim_{n\to\infty} \left(5a_n - 2a_n^2\right) = 5 \lim_{n\to\infty} a_n - 2 \lim_{n\to\infty} a_n^2 = 5 \lim_{n\to\infty} a_n - 2\left(\lim_{n\to\infty} a_n\right)^2 = 5 \cdot 2 - 2 \cdot 2^2 = 2.$$

5. $\lim_{n\to\infty} e^{a_n}$

SOLUTION The function $f(x) = e^x$ is continuous, hence:

$$\lim_{n\to\infty} e^{a_n} = e^{\lim_{n\to\infty} a_n} = e^2.$$

7. $\lim_{n\to\infty} (-1)^n a_n$

SOLUTION Because $\lim_{n\to\infty} a_n \neq 0$, it follows that $\lim_{n\to\infty} (-1)^n a_n$ does not exist.

In Exercises 9–22, determine the limit of the sequence or show that the sequence diverges.

9. $a_n = \sqrt{n+5} - \sqrt{n+2}$

SOLUTION First rewrite a_n as follows:

$$a_n = \frac{\left(\sqrt{n+5} - \sqrt{n+2}\right)\left(\sqrt{n+5} + \sqrt{n+2}\right)}{\sqrt{n+5} + \sqrt{n+2}} = \frac{(n+5) - (n+2)}{\sqrt{n+5} + \sqrt{n+2}} = \frac{3}{\sqrt{n+5} + \sqrt{n+2}}.$$

Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3}{\sqrt{n+5} + \sqrt{n+2}} = 0.$$

11. $a_n = 2^{1/n^2}$

SOLUTION The function $f(x) = 2^x$ is continuous, so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{1/n^2} = 2^{\lim_{n\to\infty} (1/n^2)} = 2^0 = 1.$$

13. $b_m = 1 + (-1)^m$

SOLUTION Because $1 + (-1)^m$ is equal to 0 for m odd and is equal to 2 for m even, the sequence $\{b_m\}$ does not approach one limit; hence this sequence diverges.

15. $b_n = \tan^{-1}\left(\frac{n+2}{n+5}\right)$

SOLUTION The function $\tan^{-1} x$ is continuous, so

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \tan^{-1}\left(\frac{n+2}{n+5}\right) = \tan^{-1}\left(\lim_{n\to\infty} \frac{n+2}{n+5}\right) = \tan^{-1} 1 = \frac{\pi}{4}.$$

17. $b_n = \sqrt{n^2+n} - \sqrt{n^2+1}$

SOLUTION Rewrite b_n as

$$b_n = \frac{\left(\sqrt{n^2+n} - \sqrt{n^2+1}\right)\left(\sqrt{n^2+n} + \sqrt{n^2+1}\right)}{\sqrt{n^2+n} + \sqrt{n^2+1}} = \frac{\left(n^2+n\right) - \left(n^2+1\right)}{\sqrt{n^2+n} + \sqrt{n^2+1}} = \frac{n-1}{\sqrt{n^2+n} + \sqrt{n^2+1}}.$$

Then

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\frac{n}{n} - \frac{1}{n}}{\sqrt{\frac{n^2}{n^2} + \frac{n}{n^2}} + \sqrt{\frac{n^2}{n^2} + \frac{1}{n^2}}} = \lim_{n\to\infty} \frac{1 - \frac{1}{n}}{\sqrt{1 + \frac{1}{n}} + \sqrt{1 + \frac{1}{n^2}}} = \frac{1 - 0}{\sqrt{1+0} + \sqrt{1+0}} = \frac{1}{2}.$$

19. $b_m = \left(1 + \frac{1}{m}\right)^{3m}$

SOLUTION $\lim_{m\to\infty} b_m = \lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^m = e.$

21. $b_n = n\big(\ln(n+1) - \ln n\big)$

SOLUTION Write

$$b_n = n \ln\left(\frac{n+1}{n}\right) = \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}}.$$

Using L'Hôpital's Rule, we find

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\left(1 + \frac{1}{x}\right)^{-1} \cdot \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{-1} = 1.$$

23. Use the Squeeze Theorem to show that $\lim_{n\to\infty} \frac{\arctan(n^2)}{\sqrt{n}} = 0.$

SOLUTION For all x,

$$-\frac{\pi}{2} < \arctan x < \frac{\pi}{2},$$

so

$$-\frac{\pi/2}{\sqrt{n}} < \frac{\arctan(n^2)}{\sqrt{n}} < \frac{\pi/2}{\sqrt{n}},$$

for all n. Because

$$\lim_{n\to\infty} \left(-\frac{\pi/2}{\sqrt{n}}\right) = \lim_{n\to\infty} \frac{\pi/2}{\sqrt{n}} = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{n\to\infty} \frac{\arctan(n^2)}{\sqrt{n}} = 0.$$

25. Calculate $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, where $a_n = \frac{1}{2}3^n - \frac{1}{3}2^n$.

SOLUTION Because

$$\frac{1}{2}3^n - \frac{1}{3}2^n \geq \frac{1}{2}3^n - \frac{1}{3}3^n = \frac{3^n}{6}$$

and

$$\lim_{n\to\infty} \frac{3^n}{6} = \infty,$$

we conclude that $\lim_{n\to\infty} a_n = \infty$, so L'Hôpital's rule may be used:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{2}3^{n+1} - \frac{1}{3}2^{n+1}}{\frac{1}{2}3^n - \frac{1}{3}2^n} = \lim_{n\to\infty} \frac{3^{n+2} - 2^{n+2}}{3^{n+1} - 2^{n+1}} = \lim_{n\to\infty} \frac{3 - 2\left(\frac{2}{3}\right)^{n+1}}{1 - \left(\frac{2}{3}\right)^{n+1}} = \frac{3-0}{1-0} = 3.$$

27. Calculate the partial sums S_4 and S_7 of the series $\sum_{n=1}^{\infty} \frac{n-2}{n^2+2n}$.

SOLUTION

$$S_4 = -\frac{1}{3} + 0 + \frac{1}{15} + \frac{2}{24} = -\frac{11}{60} = -0.183333;$$

$$S_7 = -\frac{1}{3} + 0 + \frac{1}{15} + \frac{2}{24} + \frac{3}{35} + \frac{4}{48} + \frac{5}{63} = \frac{287}{4410} = 0.065079.$$

29. Find the sum $\frac{4}{9}+\frac{8}{27}+\frac{16}{81}+\frac{32}{243}+\cdots$.

SOLUTION This is a geometric series with common ratio $r=\frac{2}{3}$. Therefore,

$$\frac{4}{9}+\frac{8}{27}+\frac{16}{81}+\frac{32}{243}+\cdots=\frac{\frac{4}{9}}{1-\frac{2}{3}}=\frac{4}{3}.$$

31. Find the sum $\sum_{n=-1}^{\infty}\frac{2^{n+3}}{3^n}$.

SOLUTION Note

$$\sum_{n=-1}^{\infty}\frac{2^{n+3}}{3^n}=2^3\sum_{n=-1}^{\infty}\frac{2^n}{3^n}=8\sum_{n=-1}^{\infty}\left(\frac{2}{3}\right)^n;$$

therefore,

$$\sum_{n=-1}^{\infty}\frac{2^{n+3}}{3^n}=8\cdot\frac{3}{2}\cdot\frac{1}{1-\frac{2}{3}}=36.$$

33. Give an example of divergent series $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ such that $\sum_{n=1}^{\infty}(a_n+b_n)=1$.

SOLUTION Let $a_n=\left(\frac{1}{2}\right)^n+1$, $b_n=-1$. The corresponding series diverge by the Divergence Test; however,

$$\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1.$$

35. Evaluate $S=\sum_{n=3}^{\infty}\frac{1}{n(n+3)}$.

SOLUTION Note that

$$\frac{1}{n(n+3)}=\frac{1}{3}\left(\frac{1}{n}-\frac{1}{n+3}\right)$$

so that

$$\begin{aligned}\sum_{n=3}^{N}\frac{1}{n(n+3)}&=\frac{1}{3}\sum_{n=3}^{N}\left(\frac{1}{n}-\frac{1}{n+3}\right)\\&=\frac{1}{3}\left(\left(\frac{1}{3}-\frac{1}{6}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{5}-\frac{1}{8}\right)\right.\\&\qquad\left.\left(\frac{1}{6}-\frac{1}{9}\right)+\cdots+\left(\frac{1}{N-1}-\frac{1}{N+2}\right)+\left(\frac{1}{N}-\frac{1}{N+3}\right)\right)\\&=\frac{1}{3}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{N+1}-\frac{1}{N+2}-\frac{1}{N+3}\right)\end{aligned}$$

Thus

$$\begin{aligned}\sum_{n=3}^{\infty}\frac{1}{n(n+3)}&=\frac{1}{3}\lim_{N\to\infty}\sum_{n=3}^{N}\left(\frac{1}{n}-\frac{1}{n+3}\right)\\&=\frac{1}{3}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{N+1}-\frac{1}{N+2}-\frac{1}{N+3}\right)=\frac{1}{3}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)=\frac{47}{180}\end{aligned}$$

In Exercises 37–40, use the Integral Test to determine whether the infinite series converges.

37. $\sum_{n=1}^{\infty}\frac{n^2}{n^3+1}$

SOLUTION Let $f(x)=\frac{x^2}{x^3+1}$. This function is continuous and positive for $x\geq 1$. Because

$$f'(x)=\frac{(x^3+1)(2x)-x^2(3x^2)}{(x^3+1)^2}=\frac{x(2-x^3)}{(x^3+1)^2},$$

we see that $f'(x) < 0$ and f is decreasing on the interval $x \geq 2$. Therefore, the Integral Test applies on the interval $x \geq 2$. Now,

$$\int_2^{\infty} \frac{x^2}{x^3+1}\,dx = \lim_{R\to\infty} \int_2^R \frac{x^2}{x^3+1}\,dx = \frac{1}{3} \lim_{R\to\infty} \left(\ln(R^3+1) - \ln 9\right) = \infty.$$

The integral diverges; hence, the series $\sum_{n=2}^{\infty} \frac{n^2}{n^3+1}$ diverges, as does the series $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$.

39. $\sum_{n=1}^{\infty} \frac{1}{(n+2)(\ln(n+2))^3}$

SOLUTION Let $f(x) = \frac{1}{(x+2)\ln^3(x+2)}$. Using the substitution $u = \ln(x+2)$, so that $du = \frac{1}{x+2}\,dx$, we have

$$\int_0^{\infty} f(x)\,dx = \int_{\ln 2}^{\infty} \frac{1}{u^3}\,du = \lim_{R\to\infty} \int_{\ln 2}^{\infty} \frac{1}{u^3}\,du = \lim_{R\to\infty} \left(-\frac{1}{2u^2}\bigg|_{\ln 2}^{R} \right)$$

$$= \lim_{R\to\infty} \left(\frac{1}{2(\ln 2)^2} - \frac{1}{2(\ln R)^2} \right) = \frac{1}{2(\ln 2)^2}$$

Since the integral of $f(x)$ converges, so does the series.

In Exercises 41–48, use the Comparison or Limit Comparison Test to determine whether the infinite series converges.

41. $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$

SOLUTION For all $n \geq 1$,

$$0 < \frac{1}{n+1} < \frac{1}{n} \quad \text{so} \quad \frac{1}{(n+1)^2} < \frac{1}{n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, so the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges by the Comparison Test.

43. $\sum_{n=2}^{\infty} \frac{n^2+1}{n^{3.5}-2}$

SOLUTION Apply the Limit Comparison Test with $a_n = \frac{n^2+1}{n^{3.5}-2}$ and $b_n = \frac{1}{n^{1.5}}$. Now,

$$L = \lim_{n\to\infty} \frac{\frac{n^2+1}{n^{3.5}-2}}{\frac{1}{n^{1.5}}} = \lim_{n\to\infty} \frac{n^{3.5}+n^{1.5}}{n^{3.5}-2} = 1.$$

Because L exists and $\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ is a convergent p-series, we conclude by the Limit Comparison Test that the series $\sum_{n=2}^{\infty} \frac{n^2+1}{n^{3.5}-2}$ also converges.

45. $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^5+5}}$

SOLUTION For all $n \geq 2$,

$$\frac{n}{\sqrt{n^5+5}} < \frac{n}{n^{5/2}} = \frac{1}{n^{3/2}}.$$

The series $\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series, so the series $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^5+5}}$ converges by the Comparison Test.

47. $\sum_{n=1}^{\infty} \frac{n^{10}+10^n}{n^{11}+11^n}$

SOLUTION Apply the Limit Comparison Test with $a_n = \frac{n^{10}+10^n}{n^{11}+11^n}$ and $b_n = \left(\frac{10}{11}\right)^n$. Then,

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^{10}+10^n}{n^{11}+11^n}}{\left(\frac{10}{11}\right)^n} = \lim_{n\to\infty} \frac{\frac{n^{10}+10^n}{10^n}}{\frac{n^{11}+11^n}{11^n}} = \lim_{n\to\infty} \frac{\frac{n^{10}}{10^n}+1}{\frac{n^{11}}{11^n}+1} = 1.$$

The series $\sum_{n=1}^{\infty}\left(\frac{10}{11}\right)^n$ is a convergent geometric series; because L exists, we may therefore conclude by the Limit Comparison Test that the series $\sum_{n=1}^{\infty}\frac{n^{10}+10^n}{n^{11}+11^n}$ also converges.

49. Determine the convergence of $\sum_{n=1}^{\infty}\frac{2^n+n}{3^n-2}$ using the Limit Comparison Test with $b_n=\left(\frac{2}{3}\right)^n$.

SOLUTION With $a_n=\frac{2^n+n}{3^n-2}$, we have

$$L=\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{2^n+n}{3^n-2}\cdot\frac{3^n}{2^n}=\lim_{n\to\infty}\frac{6^n+n3^n}{6^n-2^{n+1}}=\lim_{n\to\infty}\frac{1+n\left(\frac{1}{2}\right)^n}{1-2\left(\frac{1}{3}\right)^n}=1$$

Since $L=1$, the two series either both converge or both diverge. Since $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ is a convergent geometric series, the Limit Comparison Test tells us that $\sum_{n=1}^{\infty}\frac{2^n+n}{3^n-2}$ also converges.

51. Let $a_n=1-\sqrt{1-\frac{1}{n}}$. Show that $\lim_{n\to\infty}a_n=0$ and that $\sum_{n=1}^{\infty}a_n$ diverges. *Hint:* Show that $a_n\geq\frac{1}{2n}$.

SOLUTION

$$1-\sqrt{1-\frac{1}{n}}=1-\sqrt{\frac{n-1}{n}}=\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n}}=\frac{n-(n-1)}{\sqrt{n}(\sqrt{n}+\sqrt{n-1})}=\frac{1}{n+\sqrt{n^2-n}}$$
$$\geq\frac{1}{n+\sqrt{n^2}}=\frac{1}{2n}.$$

The series $\sum_{n=2}^{\infty}\frac{1}{2n}$ diverges, so the series $\sum_{n=2}^{\infty}\left(1-\sqrt{1-\frac{1}{n}}\right)$ also diverges by the Comparison Test.

53. Let $S=\sum_{n=1}^{\infty}\frac{n}{(n^2+1)^2}$.

(a) Show that S converges.
(b) CAS Use Eq. (4) in Exercise 83 of Section 10.3 with $M=99$ to approximate S. What is the maximum size of the error?

SOLUTION
(a) For $n\geq 1$,

$$\frac{n}{(n^2+1)^2}<\frac{n}{(n^2)^2}=\frac{1}{n^3}.$$

The series $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a convergent p-series, so the series $\sum_{n=1}^{\infty}\frac{n}{(n^2+1)^2}$ also converges by the Comparison Test.
(b) With $a_n=\frac{n}{(n^2+1)^2}$, $f(x)=\frac{x}{(x^2+1)^2}$ and $M=99$, Eq. (4) in Exercise 83 of Section 10.3 becomes

$$\sum_{n=1}^{99}\frac{n}{(n^2+1)^2}+\int_{100}^{\infty}\frac{x}{(x^2+1)^2}\,dx\leq S\leq\sum_{n=1}^{100}\frac{n}{(n^2+1)^2}+\int_{100}^{\infty}\frac{x}{(x^2+1)^2}\,dx,$$

or

$$0\leq S-\left(\sum_{n=1}^{99}\frac{n}{(n^2+1)^2}+\int_{100}^{\infty}\frac{x}{(x^2+1)^2}\,dx\right)\leq\frac{100}{(100^2+1)^2}.$$

Now,

$$\sum_{n=1}^{99}\frac{n}{(n^2+1)^2}=0.397066274;\ \text{ and}$$

$$\int_{100}^{\infty}\frac{x}{(x^2+1)^2}\,dx=\lim_{R\to\infty}\int_{100}^{R}\frac{x}{(x^2+1)^2}\,dx=\frac{1}{2}\lim_{R\to\infty}\left(-\frac{1}{R^2+1}+\frac{1}{100^2+1}\right)$$
$$=\frac{1}{20002}=0.000049995;$$

thus,

$$S \approx 0.397066274 + 0.000049995 = 0.397116269.$$

The bound on the error in this approximation is

$$\frac{100}{(100^2+1)^2} = 9.998 \times 10^{-7}.$$

In Exercises 54–57, determine whether the series converges absolutely. If it does not, determine whether it converges conditionally.

55. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1.1}\ln(n+1)}$

SOLUTION Consider the corresponding positive series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.1}\ln(n+1)}$. Because

$$\frac{1}{n^{1.1}\ln(n+1)} < \frac{1}{n^{1.1}}$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ is a convergent p-series, we can conclude by the Comparison Test that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1.1}\ln(n+1)}$ also converges. Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1.1}\ln(n+1)}$ converges absolutely.

57. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi}{4}+2\pi n\right)}{\sqrt{n}}$

SOLUTION $\cos\left(\frac{\pi}{4}+2\pi n\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, so

$$\sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi}{4}+2\pi n\right)}{\sqrt{n}} = \frac{\sqrt{2}}{2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}.$$

This is a divergent p-series, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi}{4}+2\pi n\right)}{\sqrt{n}}$ diverges.

59. Catalan's constant is defined by $K = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$.

(a) How many terms of the series are needed to calculate K with an error of less than 10^{-6}?

(b) CAS Carry out the calculation.

SOLUTION Using the error bound for an alternating series, we have

$$|S_N - K| \le \frac{1}{(2(N+1)+1)^2} = \frac{1}{(2N+3)^2}.$$

For accuracy to three decimal places, we must choose N so that

$$\frac{1}{(2N+3)^2} < 5 \times 10^{-3} \quad \text{or} \quad (2N+3)^2 > 2000.$$

Solving for N yields

$$N > \frac{1}{2}\left(\sqrt{2000}-3\right) \approx 20.9.$$

Thus,

$$K \approx \sum_{k=0}^{21} \frac{(-1)^k}{(2k+1)^2} = 0.915707728.$$

61. Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series. Determine whether the following series are convergent or divergent:

(a) $\sum_{n=1}^{\infty} \left(a_n + \frac{1}{n^2}\right)$

(b) $\sum_{n=1}^{\infty} (-1)^n a_n$

(c) $\sum_{n=1}^{\infty} \frac{1}{1+a_n^2}$

(d) $\sum_{n=1}^{\infty} \frac{|a_n|}{n}$

SOLUTION Because $\sum_{n=1}^{\infty} a_n$ converges absolutely, we know that $\sum_{n=1}^{\infty} a_n$ converges and that $\sum_{n=1}^{\infty} |a_n|$ converges.

(a) Because we know that $\sum_{n=1}^{\infty} a_n$ converges and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, the sum of these two series, $\sum_{n=1}^{\infty} \left(a_n + \frac{1}{n^2}\right)$ also converges.

(b) We have,

$$\sum_{n=1}^{\infty} \left|(-1)^n a_n\right| = \sum_{n=1}^{\infty} |a_n|$$

Because $\sum_{n=1}^{\infty} |a_n|$ converges, it follows that $\sum_{n=1}^{\infty} (-1)^n a_n$ converges absolutely, which implies that $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

(c) Because $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$. Therefore,

$$\lim_{n\to\infty} \frac{1}{1+a_n^2} = \frac{1}{1+0^2} = 1 \neq 0,$$

and the series $\sum_{n=1}^{\infty} \frac{1}{1+a_n^2}$ diverges by the Divergence Test.

(d) $\frac{|a_n|}{n} \le |a_n|$ and the series $\sum_{n=1}^{\infty} |a_n|$ converges, so the series $\sum_{n=1}^{\infty} \frac{|a_n|}{n}$ also converges by the Comparison Test.

In Exercises 63–70, apply the Ratio Test to determine convergence or divergence, or state that the Ratio Test is inconclusive.

63. $\sum_{n=1}^{\infty} \frac{n^5}{5^n}$

SOLUTION With $a_n = \frac{n^5}{5^n}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^5}{5^{n+1}} \cdot \frac{5^n}{n^5} = \frac{1}{5}\left(1 + \frac{1}{n}\right)^5,$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^5 = \frac{1}{5} \cdot 1 = \frac{1}{5}.$$

Because $\rho < 1$, the series converges by the Ratio Test.

65. $\sum_{n=1}^{\infty} \frac{1}{n2^n + n^3}$

SOLUTION With $a_n = \frac{1}{n2^n+n^3}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n2^n + n^3}{(n+1)2^{n+1} + (n+1)^3} = \frac{n2^n\left(1 + \frac{n^2}{2^n}\right)}{(n+1)2^{n+1}\left(1 + \frac{(n+1)^2}{2^{n+1}}\right)} = \frac{1}{2} \cdot \frac{n}{n+1} \cdot \frac{1 + \frac{n^2}{2^n}}{1 + \frac{(n+1)^2}{2^{n+1}}},$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}.$$

Because $\rho < 1$, the series converges by the Ratio Test.

67. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$

SOLUTION With $a_n = \frac{2^{n^2}}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{(n+1)^2}}{(n+1)!} \cdot \frac{n!}{2^{n^2}} = \frac{2^{2n+1}}{n+1} \quad \text{and} \quad \rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty.$$

Because $\rho > 1$, the series diverges by the Ratio Test.

69. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2}\right)^n \frac{1}{n!}$

SOLUTION With $a_n = \left(\frac{n}{2}\right)^n \frac{1}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n+1}{2}\right)^{n+1} \frac{1}{(n+1)!} \cdot \left(\frac{2}{n}\right)^n n! = \frac{1}{2}\left(\frac{n+1}{n}\right)^n = \frac{1}{2}\left(1 + \frac{1}{n}\right)^n,$$

and

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}e.$$

Because $\rho = \frac{e}{2} > 1$, the series diverges by the Ratio Test.

In Exercises 71–74, apply the Root Test to determine convergence or divergence, or state that the Root Test is inconclusive.

71. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n}$

SOLUTION With $a_n = \frac{1}{4^n}$,

$$L = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{4^n}} = \frac{1}{4}.$$

Because $L < 1$, the series converges by the Root Test.

73. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{4n}\right)^n$

SOLUTION With $a_n = \left(\frac{3}{4n}\right)^n$,

$$L = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{3}{4n}\right)^n} = \lim_{n\to\infty} \frac{3}{4n} = 0.$$

Because $L < 1$, the series converges by the Root Test.

In Exercises 75–92, determine convergence or divergence using any method covered in the text.

75. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$

SOLUTION This is a geometric series with ratio $r = \frac{2}{3} < 1$; hence, the series converges.

77. $\displaystyle\sum_{n=1}^{\infty} e^{-0.02n}$

SOLUTION This is a geometric series with common ratio $r = \frac{1}{e^{0.02}} \approx 0.98 < 1$; hence, the series converges.

79. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n} + \sqrt{n+1}}$

SOLUTION In this alternating series, $a_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$. The sequence $\{a_n\}$ is decreasing, and

$$\lim_{n\to\infty} a_n = 0;$$

therefore the series converges by the Leibniz Test.

81. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

SOLUTION The sequence $a_n = \frac{1}{\ln n}$ is decreasing for $n \geq 10$ and

$$\lim_{n\to\infty} a_n = 0;$$

therefore, the series converges by the Leibniz Test.

83. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+\ln n}}$

SOLUTION For $n \geq 1$,

$$\frac{1}{n\sqrt{n+\ln n}} \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+\ln n}}$ converges by the Comparison Test.

85. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

SOLUTION This series telescopes:

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \dots$$

so that the n^{th} partial sum S_n is

$$S_n = \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \cdots + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = 1 - \frac{1}{\sqrt{n+1}}$$

and then

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = \lim_{n\to\infty} S_n = 1 - \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 1$$

87. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

SOLUTION For $n \geq 1$, $\sqrt{n} \leq n$, so that

$$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{2n}$$

which diverges since it is a constant multiple of the harmonic series. Thus $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ diverges as well, by the Comparison Test.

89. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{\ln n}}$

SOLUTION For $n \geq N$ large enough, $\ln n \geq 2$ so that

$$\sum_{n=N}^{\infty} \frac{1}{n^{\ln n}} \leq \sum_{n=N}^{\infty} \frac{1}{n^2}$$

which is a convergent p-series. Thus by the Comparison Test, $\displaystyle\sum_{n=N}^{\infty} \frac{1}{n^{\ln n}}$ also converges; adding back in the terms for $n < N$ does not affect convergence.

91. $\sum_{n=1}^{\infty} \sin^2 \frac{\pi}{n}$

SOLUTION For all $x > 0$, $\sin x < x$. Therefore, $\sin^2 x < x^2$, and for $x = \frac{\pi}{n}$,

$$\sin^2 \frac{\pi}{n} < \frac{\pi^2}{n^2} = \pi^2 \cdot \frac{1}{n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, so the series $\sum_{n=1}^{\infty} \sin^2 \frac{\pi}{n}$ also converges by the Comparison Test.

In Exercises 93–98, find the interval of convergence of the power series.

93. $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$

SOLUTION With $a_n = \frac{2^n x^n}{n!}$,

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n x^n} \right| = \lim_{n\to\infty} \left| x \cdot \frac{2}{n} \right| = 0$$

Then $\rho < 1$ for all x, so that the radius of convergence is $R = \infty$, and the series converges for all x.

95. $\sum_{n=0}^{\infty} \frac{n^6}{n^8 + 1}(x-3)^n$

SOLUTION With $a_n = \frac{n^6(x-3)^n}{n^8+1}$,

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^6(x-3)^{n+1}}{(n+1)^8 - 1} \cdot \frac{n^8+1}{n^6(x-3)^n} \right|$$

$$= \lim_{n\to\infty} \left| (x-3) \cdot \frac{(n+1)^6(n^8+1)}{n^6((n+1)^8+1)} \right|$$

$$= \lim_{n\to\infty} \left| (x-3) \cdot \frac{n^{14} + \text{terms of lower degree}}{n^{14} + \text{terms of lower degree}} \right| = |x-3|$$

Then $\rho < 1$ when $|x-3| < 1$, so the radius of convergence is 1, and the series converges absolutely for $|x-3| < 1$, or $2 < x < 4$. For the endpoint $x = 4$, the series becomes $\sum_{n=0}^{\infty} \frac{n^6}{n^8+1}$, which converges by the Comparison Test comparing with the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. For the endpoint $x = 2$, the series becomes $\sum_{n=0}^{\infty} \frac{n^6(-1)^n}{n^8+1}$, which converges by the Leibniz Test. The series $\sum_{n=0}^{\infty} \frac{n^6(x-3)^n}{n^8+1}$ therefore converges for $2 \le x \le 4$.

97. $\sum_{n=0}^{\infty} (nx)^n$

SOLUTION With $a_n = n^n x^n$, and assuming $x \neq 0$,

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1} x^{n+1}}{n^n x^n} \right| = \lim_{n\to\infty} \left| x(n+1) \cdot \left(\frac{n+1}{n} \right)^n \right| = \infty$$

since $\left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n$ converges to e and the $(n+1)$ term diverges to ∞. Thus $\rho < 1$ only when $x = 0$, so the series converges only for $x = 0$.

99. Expand $f(x) = \frac{2}{4-3x}$ as a power series centered at $c = 0$. Determine the values of x for which the series converges.

SOLUTION Write

$$\frac{2}{4-3x} = \frac{1}{2} \frac{1}{1 - \frac{3}{4}x}.$$

Substituting $\frac{3}{4}x$ for x in the Maclaurin series for $\frac{1}{1-x}$, we obtain

$$\frac{1}{1-\frac{3}{4}x} = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n x^n.$$

This series converges for $\left|\frac{3}{4}x\right| < 1$, or $|x| < \frac{4}{3}$. Hence, for $|x| < \frac{4}{3}$,

$$\frac{2}{4-3x} = \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n x^n.$$

101. Let $F(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k \cdot k!}$.

(a) Show that $F(x)$ has infinite radius of convergence.

(b) Show that $y = F(x)$ is a solution of

$$y'' = xy' + y, \qquad y(0) = 1, \qquad y'(0) = 0$$

(c) CAS Plot the partial sums S_N for $N = 1, 3, 5, 7$ on the same set of axes.

SOLUTION

(a) With $a_k = \frac{x^{2k}}{2^k \cdot k!}$,

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{|x|^{2k+2}}{2^{k+1}\cdot(k+1)!}\cdot\frac{2^k\cdot k!}{|x|^{2k}} = \frac{x^2}{2(k+1)},$$

and

$$\rho = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = x^2\cdot 0 = 0.$$

Because $\rho < 1$ for all x, we conclude that the series converges for all x; that is, $R = \infty$.

(b) Let

$$y = F(x) = \sum_{k=0}^{\infty}\frac{x^{2k}}{2^k\cdot k!}.$$

Then

$$y' = \sum_{k=1}^{\infty}\frac{2kx^{2k-1}}{2^k k!} = \sum_{k=1}^{\infty}\frac{x^{2k-1}}{2^{k-1}(k-1)!},$$

$$y'' = \sum_{k=1}^{\infty}\frac{(2k-1)x^{2k-2}}{2^{k-1}(k-1)!},$$

and

$$xy' + y = x\sum_{k=1}^{\infty}\frac{x^{2k-1}}{2^{k-1}(k-1)!} + \sum_{k=0}^{\infty}\frac{x^{2k}}{2^k k!} = \sum_{k=1}^{\infty}\frac{x^{2k}}{2^{k-1}(k-1)!} + 1 + \sum_{k=1}^{\infty}\frac{x^{2k}}{2^k k!}$$

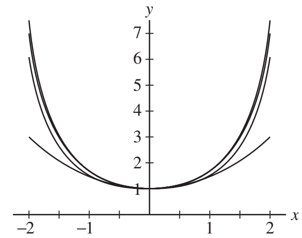
$$= 1 + \sum_{k=1}^{\infty}\frac{(2k+1)x^{2k}}{2^k k!} = \sum_{k=0}^{\infty}\frac{(2k+1)x^{2k}}{2^k k!} = \sum_{k=1}^{\infty}\frac{(2k-1)x^{2k-2}}{2^{k-1}(k-1)!} = y''.$$

Moreover,

$$y(0) = 1 + \sum_{k=1}^{\infty}\frac{0^{2k}}{2^k k!} = 1 \quad\text{and}\quad y'(0) = \sum_{k=1}^{\infty}\frac{0^{2k-1}}{2^{k-1}(k-1)!} = 0.$$

Thus, $\sum_{k=0}^{\infty}\frac{x^{2k}}{2^k k!}$ is the solution to the equation $y'' = xy' + y$ satisfying $y(0) = 1$, $y'(0) = 0$.

(c) The partial sums S_1, S_3, S_5 and S_7 are plotted in the figure below.



In Exercises 103–112, find the Taylor series centered at c.

103. $f(x) = e^{4x}, \quad c = 0$

SOLUTION Substituting $4x$ for x in the Maclaurin series for e^x yields

$$e^{4x} = \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} = \sum_{n=0}^{\infty} \frac{4^n}{n!} x^n.$$

105. $f(x) = x^4, \quad c = 2$

SOLUTION We have

$$f'(x) = 4x^3 \quad f''(x) = 12x^2 \quad f'''(x) = 24x \quad f^{(4)}(x) = 24$$

and all higher derivatives are zero, so that

$$f(2) = 2^4 = 16 \quad f'(2) = 4 \cdot 2^3 = 32 \quad f''(2) = 12 \cdot 2^2 = 48 \quad f'''(2) = 24 \cdot 2 = 48 \quad f^{(4)}(2) = 24$$

Thus the Taylor series centered at $c = 2$ is

$$\sum_{n=0}^{4} \frac{f^{(n)}(2)}{n!}(x-2)^n = 16 + \frac{32}{1!}(x-2) + \frac{48}{2!}(x-2)^2 + \frac{48}{3!}(x-2)^3 + \frac{24}{4!}(x-2)^4$$

$$= 16 + 32(x-2) + 24(x-2)^2 + 8(x-2)^3 + (x-2)^4$$

107. $f(x) = \sin x, \quad c = \pi$

SOLUTION We have

$$f^{(4n)}(x) = \sin x \quad f^{(4n+1)}(x) = \cos x \quad f^{(4n+2)}(x) = -\sin x \quad f^{(4n+3)}(x) = -\cos x$$

so that

$$f^{(4n)}(\pi) = \sin \pi = 0 \quad f^{(4n+1)}(\pi) = \cos \pi = -1 \quad f^{(4n+2)}(\pi) = -\sin \pi = 0 \quad f^{(4n+3)}(\pi) = -\cos \pi = 1$$

Then the Taylor series centered at $c = \pi$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n = \frac{-1}{1!}(x-\pi) + \frac{1}{3!}(x-\pi)^3 + \frac{-1}{5!}(x-\pi)^5 + \frac{1}{7!}(x-\pi)^7 - \dots$$

$$= -(x-\pi) + \frac{1}{6}(x-\pi)^3 - \frac{1}{120}(x-\pi)^5 + \frac{1}{5040}(x-\pi)^7 - \dots$$

109. $f(x) = \dfrac{1}{1-2x}, \quad c = -2$

SOLUTION Write

$$\frac{1}{1-2x} = \frac{1}{5-2(x+2)} = \frac{1}{5} \frac{1}{1-\frac{2}{5}(x+2)}.$$

Substituting $\frac{2}{5}(x+2)$ for x in the Maclaurin series for $\frac{1}{1-x}$ yields

$$\frac{1}{1-\frac{2}{5}(x+2)} = \sum_{n=0}^{\infty} \frac{2^n}{5^n}(x+2)^n;$$

hence,

$$\frac{1}{1-2x} = \frac{1}{5}\sum_{n=0}^{\infty}\frac{2^n}{5^n}(x+2)^n = \sum_{n=0}^{\infty}\frac{2^n}{5^{n+1}}(x+2)^n.$$

111. $f(x) = \ln\frac{x}{2}, \quad c = 2$

SOLUTION Write

$$\ln\frac{x}{2} = \ln\left(\frac{(x-2)+2}{2}\right) = \ln\left(1+\frac{x-2}{2}\right).$$

Substituting $\frac{x-2}{2}$ for x in the Maclaurin series for $\ln(1+x)$ yields

$$\ln\frac{x}{2} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(\frac{x-2}{2}\right)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-2)^n}{n\cdot 2^n}.$$

This series is valid for $|x-2| < 2$.

In Exercises 113–116, find the first three terms of the Maclaurin series of $f(x)$ and use it to calculate $f^{(3)}(0)$.

113. $f(x) = (x^2 - x)e^{x^2}$

SOLUTION Substitute x^2 for x in the Maclaurin series for e^x to get

$$e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \dots$$

so that the Maclaurin series for $f(x)$ is

$$(x^2-x)e^{x^2} = x^2 + x^4 + \frac{1}{2}x^6 + \dots - x - x^3 - \frac{1}{2}x^5 - \dots = -x + x^2 - x^3 + x^4 + \dots$$

The coefficient of x^3 is

$$\frac{f'''(0)}{3!} = -1$$

so that $f'''(0) = -6$.

115. $f(x) = \dfrac{1}{1+\tan x}$

SOLUTION Substitute $-\tan x$ in the Maclaurin series for $\frac{1}{1-x}$ to get

$$\frac{1}{1+\tan x} = 1 - \tan x + (\tan x)^2 - (\tan x)^3 + \dots$$

We have not yet encountered the Maclaurin series for $\tan x$. We need only the terms up through x^3, so compute

$$\tan'(x) = \sec^2 x \quad \tan''(x) = 2(\tan x)\sec^2 x \quad \tan'''(x) = 2(1+\tan^2 x)\sec^2 x + 4(\tan^2 x)\sec^2 x$$

so that

$$\tan'(0) = 1 \quad \tan''(0) = 0 \quad \tan'''(0) = 2$$

Then the Maclaurin series for $\tan x$ is

$$\tan x = \tan 0 + \frac{\tan'(0)}{1!}x + \frac{\tan''(0)}{2!}x^2 + \frac{\tan'''(0)}{3!}x^3 + \dots = x + \frac{1}{3}x^3 + \dots$$

Substitute these into the series above to get

$$\begin{aligned}\frac{1}{1+\tan x} &= 1 - \left(x+\frac{1}{3}x^3\right) + \left(x+\frac{1}{3}x^3\right)^2 - \left(x+\frac{1}{3}x^3\right)^3 + \dots \\ &= 1 - x - \frac{1}{3}x^3 + x^2 - x^3 + \text{higher degree terms} \\ &= 1 - x + x^2 - \frac{4}{3}x^3 + \text{higher degree terms}\end{aligned}$$

The coefficient of x^3 is

$$\frac{f'''(0)}{3!} = -\frac{4}{3}$$

so that

$$f'''(0) = -6 \cdot \frac{4}{3} = -8$$

117. Calculate $\frac{\pi}{2} - \frac{\pi^3}{2^3 3!} + \frac{\pi^5}{2^5 5!} - \frac{\pi^7}{2^7 7!} + \cdots$.

SOLUTION We recognize that

$$\frac{\pi}{2} - \frac{\pi^3}{2^3 3!} + \frac{\pi^5}{2^5 5!} - \frac{\pi^7}{2^7 7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{(\pi/2)^{2n+1}}{(2n+1)!}$$

is the Maclaurin series for $\sin x$ with x replaced by $\pi/2$. Therefore,

$$\frac{\pi}{2} - \frac{\pi^3}{2^3 3!} + \frac{\pi^5}{2^5 5!} - \frac{\pi^7}{2^7 7!} + \cdots = \sin\frac{\pi}{2} = 1.$$