where $n'(t)$ is the growth rate; the relation between the cost and marginal cost functions:
\[
\int_{t_1}^{t_2} C'(t) \, dt = C(t_2) - C(t_1);
\]
and similarly for many other quantities.

### 35. The Substitution Rule

An indefinite integral of the derivative $F'(x)$ is the function $F(x)$ itself. Let $u = F(x)$, where $u$ is a new variable defined as a differentiable function of $x$. Consider the differential $du = F'(x) \, dx$. Then the following equalities hold:
\[
\int F'(x) \, dx = F(x) + C = u + C = \int du,
\]
where $C$ is an arbitrary constant and the last equality follows from the fact that an indefinite integral of $f(u) = 1$ is $u$. So we can conclude that $\int F'(x) \, dx = \int du$, provided the variables $u$ and $x$ are related as $u = F(x)$. This also shows that it is permissible to operate with $dx$ and $du$ after the integral sign as if they were differentials. This observation leads to a neat technical trick to calculate indefinite integrals. For example,
\[
\int \frac{1}{\sqrt{x} + 1} \, dx = \int \frac{d(2\sqrt{x} + 1)}{2\sqrt{x} + 1} = 2\sqrt{x} + 1 + C,
\]
where the substitution $u = 2\sqrt{x} + 1$ has been used. This trick can be generalized.

Let $F(u)$ be an indefinite integral of a continuous function $f(u)$ on an interval $I$. Let $u = g(x)$, where $g$ is differentiable and its range is the interval $I$. By the chain rule,
\[
\left(F(g(x))\right)' = F'(g(x))g'(x) = f(g(x))g'(x).
\]
In other words, $F(g(x)) + C$ is an indefinite integral of $f(g(x))g'(x)$. On an interval, the most general indefinite integral of $f(u)$ is $\int f(u) \, du = F(u) + C$. Therefore, $F(g(x))$ and $\int f(u) \, du$ can differ at most by an additive constant. This proves the following theorem.

**Theorem 42.** (The Substitution Rule). If $u = g(x)$ is a differentiable function whose range is an interval $I$ and $f$ is continuous on $I$, then
\[
(5.19) \quad \int f(g(x))g'(x) \, dx = \int f(g(x)) \, dg(x) = \int f(u) \, du.
\]
The substitution rule is often referred to as a *change of the integration variable*. It is a powerful method to calculate indefinite integrals.

**Example 66.** Find $\int x \sin(x^2 + 1) \, dx$.

**Solution:**

\[
\int x \sin(x^2 + 1) \, dx = \int \sin(x^2 + 1) \frac{1}{2} d(x^2 + 1) = \frac{1}{2} \int \sin(u) \, du = -\frac{1}{2} \cos(u) + C = -\frac{1}{2} \cos(x^2 + 1) + C,
\]

where the substitution $u = x^2 + 1$ has been used. \hfill \Box

**Example 67.** Find $\int \tan(x) \, dx$.

**Solution:**

\[
\int \tan(x) \, dx = \int \frac{\sin(x)}{\cos(x)} \, dx = -\int \frac{d(\cos(x))}{\cos(x)} = -\int \frac{du}{u} = -\ln |u| + C = -\ln |\cos(x)| + C = \ln |\sec(x)| + C,
\]

where the substitution $u = \cos(x)$ and the logarithm property $\ln(1/a) = -\ln(a)$ have been used. \hfill \Box

The substitution rule can be used to evaluate definite integrals by means of the fundamental theorem of calculus.

**Example 68.** Evaluate $\int_0^2 xe^{x^2} \, dx$.

**Solution:** First, find an indefinite integral:

\[
F(x) = \int xe^{x^2} \, dx = \frac{1}{2} \int e^{x^2} \, dx^2 = \frac{1}{2} \int e^u \, du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C.
\]

where $u = x^2$. By the fundamental theorem of calculus,

\[
\int_0^1 xe^{x^2} \, dx = F(2) - F(0) = \frac{1}{2} (e^4 - 1).
\]

Note that, when evaluating the integral, the original variable $x$ has been restored in the indefinite integral in order to apply the fundamental theorem of calculus. The fundamental theorem of calculus can also be applied directly in the new variable $u$, provided the range of $u$ is properly changed. Indeed, in the previous example, the answer could have been recovered from the indefinite integral $\frac{1}{2} e^u + C$ if $u = x^2$ ranges from $0 = 0^2$ to $4 = 2^2$ as $x$ ranges from 0 to 2. This is especially useful when a calculation of a definite integral requires several changes of the integration variable.
Theorem 43. (The Substitution Rule for Definite Integrals). If $g'$ is continuous on $[a, b]$ and $f$ is continuous on the range of $u = g(x)$, then

\begin{equation}
\int_{a}^{b} f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.
\end{equation}

Proof. Let $F$ be an antiderivative of $f$. Then $F(g(x))$ is an antiderivative of $(F(g(x)))' = F'(g(x))g'(x) = f(g(x))g'(x)$. By the fundamental theorem of calculus,

\[ \int_{a}^{b} f(g(x))g'(x) \, dx = F(g(x))|_{a}^{b} = F(g(b)) - F(g(a)). \]

On the other hand, since $F(u)$ is an antiderivative of $f(u)$, the fundamental theorem of calculus yields

\[ \int_{g(a)}^{g(b)} f(u) \, du = F(u)|_{g(a)}^{g(b)} = F(g(b)) - F(g(a)). \]

Since the right-hand sides of these equalities coincide, so must their left-hand sides, which implies (5.20).

Example 69. Evaluate $\int_{1}^{e} \ln(x)/x \, dx$.

Solution: The integrand can be transformed as

\[ \frac{\ln(x)}{x} \, dx = \ln(x) \, d\ln(x). \]

So the substitution $u = \ln(x)$ can be made. The range of the new integration variable $u$ is determined by the range of the old one: $u = 0$ when $x = 1$ and $u = 1$ when $x = e$. Thus,

\[ \int_{1}^{e} \frac{\ln(x)}{x} \, dx = \int_{0}^{1} u \, du = \frac{u^2}{2} \bigg|_{0}^{1} = \frac{1}{2}. \]

35.1. Symmetry. The calculation of a definite integral over a symmetric interval can be simplified if the integrand possesses symmetry properties.

Theorem 44. Suppose $f$ is continuous on a symmetric interval $[-a, a]$. Then

\begin{align*}
(5.21) \quad & \int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx \quad \text{if } f(-x) = f(x) \ (f \text{ is even}), \\
(5.22) \quad & \int_{-a}^{a} f(x) \, dx = 0 \quad \text{if } f(-x) = -f(x) \ (f \text{ is odd}).
\end{align*}
PROOF. The integral can be split into two integrals:

\[ \int_{-a}^{a} f(x) \, dx = \left( \int_{-a}^{0} + \int_{0}^{a} \right) f(x) \, dx = - \int_{0}^{-a} f(x) \, dx + \int_{0}^{a} f(x) \, dx. \]

In the first integral on the very right-hand side, the substitution \( u = -x \) is made so that \( u = 0 \) when \( x = 0 \) and \( u = a \) when \( x = -a \) and \( dx = -du \). Hence,

\[ - \int_{0}^{-a} f(x) \, dx = \int_{0}^{a} f(-u) \, du \]

and

\[ \int_{-a}^{a} f(x) \, dx = \int_{0}^{a} f(-u) \, du + \int_{0}^{a} f(x) \, dx. \]

Now, if \( f \) is even, then \( f(-u) = f(u) \) and (5.21) follows. If \( f \) is odd, then \( f(-u) = -f(u) \) and (5.22) follows.

The geometrical interpretation of this theorem is transparent. Suppose \( f(x) \geq 0 \) for \( 0 \leq x \leq a \). The integral \( \int_{0}^{a} f(x) \, dx = A \) is the area under the graph of \( f \) on \([0,a]\). If \( f \) is even, then, by symmetry, the graph of \( f \) on \([-a,0]\) is obtained from that on \([0,a]\) by a reflection about the \( y \) axis. Therefore, the area \( \int_{0}^{-a} f(x) \, dx \) must coincide with \( A \). If \( f \) is odd, then its graph on \([-a,0]\) is obtained by the mirror reflection about the origin so that the area \( A \) appears beneath the \( x \) axis. Hence, \( \int_{-a}^{0} f(x) \, dx = -A \).

**Example 70.** Evaluate \( \int_{-\pi}^{\pi} \sin(x^3) \, dx \).

**Solution:** Unfortunately, an antiderivative of \( \sin(x^3) \) cannot be expressed in elementary functions, and the fundamental theorem of calculus cannot be used. One can always evaluate the integral by taking the limit of the sequence of Riemann sums. An alternative solution is due to a simple symmetry argument. Note that \( \sin(x^3) \) is an odd function, \( \sin((-x)^3) = \sin(-x^3) = -\sin(x^3) \). The integration interval is also symmetric, \([-\pi, \pi]\). Thus, by property (5.22),

\[ \int_{-\pi}^{\pi} \sin(x^3) \, dx = 0. \]
**Remark.** In the previous example, take a partition of \([-\pi, \pi]\) by points \(x_k = k \Delta x, \ k = -N, -N+1, ..., -1, 0, 1, ..., N-1, N\), where \(\Delta x = \pi/N\). Consider the Riemann sum with sample points being the midpoints. It is then straightforward to show that the Riemann sum vanishes because \(\sin(x_{-k}^*) = -\sin(x_k^*)\) for \(k = 1, 2, ..., N\).