We begin with the Cauchy integral formula. The idea in the next theorem is this: If $f$ is analytic in a simply connected domain and $z_0$ is any point $D$, then the quotient $f(z)/(z - z_0)$ is not analytic in $D$. As a consequence, the integral of $f(z)/(z - z_0)$ around a simple closed contour $C$ that contains $z_0$ is not necessarily zero but has, as we shall now see, the value $2\pi i f(z_0).$ This remarkable result indicates that the values of an analytic function $f$ at points inside a simple closed contour $C$ are determined by the values of $f$ on the contour $C.$

**Theorem 18.4.1 Cauchy's Integral Formula**

Let $f$ be analytic in a simply connected domain $D$, and let $C$ be a simple closed contour lying entirely within $D$. If $z_0$ is any point within $C$, then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} \, dz.$$  

(1)

**PROOF**

Let $D$ be a simply connected domain, $C$ a simple closed contour in $D$, and $z_0$ an interior point of $C$. In addition, let $C_1$ be a circle centered at $z_0$ with radius small enough that it is interior to $C$. By the principle of deformation of contours, we can write

$$\oint_C \frac{f(z)}{z - z_0} \, dz = \oint_{C_1} \frac{f(z)}{z - z_0} \, dz.$$  

(2)

18.4 Cauchy's Integral Formulas

---

**Introduction** In the last two sections we saw the importance of the Cauchy–Goursat theorem in the evaluation of contour integrals. In this section we are going to examine several more consequences of the Cauchy–Goursat theorem. Unquestionably, the most significant of these is the following result:

**The value of an analytic function $f$ at any point $z_0$ in a simply connected domain can be represented by a contour integral.**

After establishing this proposition we shall use it to further show that

**An analytic function $f$ in a simply connected domain possesses derivatives of all orders.**

The ramifications of these two results alone will keep us busy not only for the remainder of this section but in the next chapter as well.

**First Formula** We begin with the Cauchy integral formula. The idea in the next theorem is this: If $f$ is analytic in a simply connected domain and $z_0$ is any point $D$, then the quotient $f(z)/(z - z_0)$ is not analytic in $D$. As a consequence, the integral of $f(z)/(z - z_0)$ around a simple closed contour $C$ that contains $z_0$ is not necessarily zero but has, as we shall now see, the value $2\pi i f(z_0).$ This remarkable result indicates that the values of an analytic function $f$ at points inside a simple closed contour $C$ are determined by the values of $f$ on the contour $C.$

**Theorem 18.4.1 Cauchy's Integral Formula**

Let $f$ be analytic in a simply connected domain $D$, and let $C$ be a simple closed contour lying entirely within $D$. If $z_0$ is any point within $C$, then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} \, dz.$$  

(1)

**PROOF**

Let $D$ be a simply connected domain, $C$ a simple closed contour in $D$, and $z_0$ an interior point of $C$. In addition, let $C_1$ be a circle centered at $z_0$ with radius small enough that it is interior to $C$. By the principle of deformation of contours, we can write

$$\oint_C \frac{f(z)}{z - z_0} \, dz = \oint_{C_1} \frac{f(z)}{z - z_0} \, dz.$$  

(2)
We wish to show that the value of the integral on the right is $2\pi i f(z_0)$. To this end we add and subtract the constant $f(z_0)$ in the numerator:

$$\oint_{C_1} \frac{f(z)}{z - z_0} \, dz = \oint_{C_1} \left( \frac{f(z) - f(z_0) + f(z)}{z - z_0} \right) \, dz$$

$$= f(z_0) \oint_{C_1} \frac{dz}{z - z_0} + \oint_{C_1} \frac{f(z) - f(z_0)}{z - z_0} \, dz. \tag{3}$$

Now from (4) of Section 18.2 we know that

$$\oint_{C_1} \frac{dz}{z - z_0} = 2\pi i.$$

Thus, (3) becomes

$$\oint_{C_1} \frac{f(z)}{z - z_0} \, dz = 2\pi i f(z_0) + \oint_{C_1} \frac{f(z) - f(z_0)}{z - z_0} \, dz. \tag{4}$$

Since $f$ is continuous at $z_0$ for any arbitrarily small $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. In particular, if we choose the circle $C_1$ to be $|z - z_0| = \delta/2 < \delta$, then by the ML-inequality (Theorem 18.1.3) the absolute value of the integral on the right side of (4) satisfies

$$\left| \oint_{C_1} \frac{f(z) - f(z_0)}{z - z_0} \, dz \right| \leq \frac{\varepsilon}{\delta/2} 2\pi \delta = 2\pi \varepsilon.$$  

In other words, the absolute value of the integral can be made arbitrarily small by taking the radius of the circle $C_1$ to be sufficiently small. This can happen only if the integral is zero. The Cauchy integral formula (1) follows from (4) by dividing both sides by $2\pi i$.  

The Cauchy integral formula (1) can be used to evaluate contour integrals. Since we often work problems without a simply connected domain explicitly defined, a more practical restatement of Theorem 18.4.1 is

\[ \text{If } f \text{ is analytic at all points within and on a simple closed contour } C, \text{ and } z_0 \text{ is any point interior to } C, \text{ then } f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} \, dz. \tag{5} \]

**EXAMPLE 1** Using Cauchy's Integral Formula

Evaluate $\oint_C \frac{z^2 - 4z + 4}{z + i} \, dz$, where $C$ is the circle $|z| = 2$.

**Solution** First, we identify $f(z) = z^2 - 4z + 4$ and $z_0 = -i$ as a point within the circle $C$. Next, we observe that $f$ is analytic at all points within and on the contour $C$. Thus by the Cauchy integral formula we obtain

$$\oint_C \frac{z^2 - 4z + 4}{z + i} \, dz = 2\pi i f(-i) = 2\pi i (3 + 4i) = 2\pi (-4 + 3i).$$  

**EXAMPLE 2** Using Cauchy's Integral Formula

Evaluate $\oint_C \frac{z}{z^2 + 9} \, dz$, where $C$ is the circle $|z - 2i| = 4$.

**Solution** By factoring the denominator as $z^2 + 9 = (z - 3i)(z + 3i)$, we see that $3i$ is the only point within the closed contour at which the integrand fails to be analytic. See FIGURE 18.4.1.
Now by writing
\[
\frac{z}{z^2 + 9} = \frac{z}{z + 3i}
\]
we can identify \(f(z) = z(z + 3i)\). This function is analytic at all points within and on the contour \(C\). From the Cauchy integral formula we then have
\[
\oint_C \frac{z}{z^2 + 9} \, dz = \oint_C \frac{z + 3i}{z - 3i} \, dz = 2\pi i f(3i) = 2\pi i \frac{3i}{6i} = \pi.
\]

**EXAMPLE 3** Flux and Cauchy’s Integral Formula

The complex function \(f(z) = k(\overline{z} - z_1)\), where \(k = a + ib\) and \(z_1\) are complex numbers, gives rise to a flow in the domain \(z \neq z_1\). If \(C\) is a simple closed contour containing \(z = z_1\) in its interior, then from the Cauchy integral formula we have
\[
\oint_C f(z) \, dz = \oint_C \frac{a - ib}{z - z_1} \, dz = 2\pi i(a - ib).
\]

Thus the circulation around \(C\) is \(2\pi b\) and the net flux across \(C\) is \(2\pi a\). If \(z_1\) were in the exterior of \(C\), both the circulation and net flux would be zero by Cauchy’s theorem.

Note that when \(k\) is real, the circulation around \(C\) is zero but the net flux across \(C\) is \(2\pi k\). The complex number \(z_1\) is called a source for the flow when \(k > 0\) and a sink when \(k < 0\). Vector fields corresponding to these two cases are shown in FIGURE 18.4.2(a) and 18.4.2(b).

**Second Formula** We can now use Theorem 18.4.1 to prove that an analytic function possesses derivatives of all orders; that is, if \(f\) is analytic at a point \(z_0\), then \(f, f', f'', \ldots\), and so on are also analytic at \(z_0\). Moreover, the values of the derivatives \(f^{(n)}(z_0), n = 1, 2, 3, \ldots\), are given by a formula similar to (1).

**Theorem 18.4.2** Cauchy’s Integral Formula for Derivatives

Let \(f\) be analytic in a simply connected domain \(D\), and let \(C\) be a simple closed contour lying entirely within \(D\). If \(z_0\) is any point interior to \(C\), then
\[
f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz.
\]

**PARTIAL PROOF**

We will prove (6) only for the case \(n = 1\). The remainder of the proof can be completed using the principle of mathematical induction.

We begin with the definition of the derivative and (1):
\[
f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z} \left[ \oint_C \frac{f(z)}{z - (z_0 + \Delta z)} \, dz - \oint_C \frac{f(z)}{z - z_0} \, dz \right]
\]
\[
= \lim_{\Delta z \to 0} \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)} \, dz.
\]

Before proceeding, let us set up some preliminaries. Since \(f\) is continuous on \(C\), it is bounded; that is, there exists a real number \(M\) such that \(|f(z)| \leq M\) for all points \(z\) on \(C\). In addition,
Example 5

**FIGURE 18.4.3**

Let $L$ be the length of $C$ and let $\delta$ denote the shortest distance between points on $C$ and the point $z_0$. Thus for all points $z$ on $C$, we have

$$|z - z_0| \leq \delta \quad \text{or} \quad \frac{1}{|z - z_0|^2} \leq \frac{1}{\delta^2}.$$ 

Furthermore, if we choose $|\Delta z| = \delta/2$, then

$$|z - z_0 - \Delta z| \geq |z - z_0| - |\Delta z| \geq \delta - |\Delta z| \geq \frac{\delta}{2}$$

and so

$$\frac{1}{|z - z_0 - \Delta z|^2} \leq \frac{2}{\delta^3}.$$

Now,

$$\left| \oint_C \frac{f(z)}{(z - z_0)^2} \, dz - \oint_C \frac{f(z)}{(z - z_0)^2} \, dz \right| = \left| \oint_C \frac{-\Delta z f(z)}{(z - z_0)^3(z - z_0 - \Delta z)} \, dz \right| \leq \frac{2\|ML\|\Delta z}{\delta^3}.$$

Because the last expression approaches zero as $\Delta z \to 0$, we have shown that

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} \, dz.$$ 

If $f(z) = u(x, y) + iv(x, y)$ is analytic at a point, then its derivatives of all orders exist at that point and are continuous. Consequently, from

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y},$$

$$f''(z) = \frac{\partial^2 u}{\partial x^2} + i \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2} - i \frac{\partial^2 u}{\partial y^2},$$

we can conclude that the real functions $u$ and $v$ have continuous partial derivatives of all orders at a point of analyticity.

Like (1), (6) can sometimes be used to evaluate integrals.

**EXAMPLE 4** Using Cauchy's Integral Formula for Derivatives

Evaluate $\oint_C \frac{z + 1}{z^2 + 4z} \, dz$, where $C$ is the circle $|z| = 1$.

**Solution** Inspection of the integrand shows that it is not analytic at $z = 0$ and $z = -4$, but only $z = 0$ lies within the closed contour. By writing the integrand as

$$\frac{z + 1}{z^2 + 4z} = \frac{z + 1}{z(z + 4)},$$

we can identify $z_0 = 0$, $n = 2$, and $f(z) = (z + 1)/(z + 4)$. By the Quotient Rule, $f''(z) = -6/(z + 4)^3$ and so by (6) we have

$$\oint_C \frac{z + 1}{z^2 + 4z} \, dz = \frac{2\pi i}{2!} f''(0) = \frac{3\pi}{32} i.$$

**EXAMPLE 5** Using Cauchy's Integral Formula for Derivatives

Evaluate $\oint_C \frac{z^3 + 3}{z(z - i)^2} \, dz$, where $C$ is the contour shown in **FIGURE 18.4.3**.

**Solution** Although $C$ is not a simple closed contour, we can think of it as the union of two simple closed contours $C_1$ and $C_2$ as indicated in Figure 18.4.3. By writing

**CHAPTER 18** Integration in the Complex Plane
\[ \oint_C \frac{z^3 + 3}{z - \bar{z}} \ dz = \oint_C \frac{z^3 + 3}{z(z - \bar{z})} \ dz + \oint_C \frac{z^3 + 3}{z} \ dz = - \oint_C \frac{(z - \bar{z})^2}{z} \ dz + \oint_C \frac{z}{z} \ dz = - I_1 + I_2, \]

we are in a position to use both (1) and (6).

To evaluate \( I_1 \), we identify \( z_0 = 0 \) and \( f(z) = (z^3 + 3)/(z - i)^2 \). By (1) it follows that

\[ I_1 = \oint_C \frac{z^3 + 3}{z(z - \bar{z})} \ dz = 2\pi i f(0) = -6\pi i. \]

To evaluate \( I_2 \) we identify \( z_0 = i, n = 1, f(z) = (z^3 + 3)/z, \) and \( f'(z) = (2z^3 - 3)/z^2 \). From (6) we obtain

\[ I_2 = \oint_C \frac{z}{z - \bar{z}} \ dz = \frac{2\pi i}{1!} f'(i) = 2\pi i(3 + 2i) = 2\pi(-2 + 3i). \]

Finally we get

\[ \oint_C \frac{z^3 + 3}{z(z - \bar{z})} \ dz = - I_1 + I_2 = 6\pi i + 2\pi(-2 + 3i) = 4\pi(-1 + 3i). \]

■ **Liouville’s Theorem**  If we take the contour \( C \) to be the circle \( |z - \bar{z}| = r \), it follows from (6) and the ML-inequality that

\[ |f^{(n)}(z_0)| = \frac{n!}{2\pi} \left| \oint_C \frac{f(z)}{(z - \bar{z})^{n+1}} \ dz \right| \leq \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r = \frac{n!M}{r^n}, \tag{7} \]

where \( M \) is a real number such that \( |f(z)| \leq M \) for all points \( z \) on \( C \). The result in (7), called **Cauchy’s inequality**, is used to prove the next result.

**Theorem 18.4.3**  **Liouville’s Theorem**

The only bounded entire functions are constants.

**Proof**

Suppose \( f \) is an entire function and is bounded; that is, \( |f(z)| \leq M \) for all \( z \). Then for any point \( z_0 \), (7) gives \( |f^{(n)}(z_0)| \leq M/r^n \). By taking \( r \) arbitrarily large, we can make \( |f^{(n)}(z_0)| \) as small as we wish. This means \( f^{(n)}(z_0) = 0 \) for all points \( z_0 \) in the complex plane. Hence \( f \) must be a constant.

■ **Fundamental Theorem of Algebra**  Liouville’s theorem enables us to prove, in turn, a result that is learned in elementary algebra:

*If \( P(z) \) is a nonconstant polynomial, then the equation \( P(z) = 0 \) has at least one root.*

This result is known as the **Fundamental Theorem of Algebra**. To prove it, let us suppose that \( P(z) \neq 0 \) for all \( z \). This implies that the reciprocal of \( P, f(z) = 1/P(z) \), is an entire function. Now since \( |f(z)| \to 0 \) as \( |z| \to \infty \), the function \( f \) must be bounded for all finite \( z \). It follows from Liouville’s theorem that \( f \) is a constant and therefore \( P \) is a constant. But this is a contradiction to our underlying assumption that \( P \) was not a constant polynomial. We conclude that there must exist at least one number \( z \) for which \( P(z) = 0 \).