21. FIGURE 20.2.18 Dirichlet problem in Problem 21

22. FIGURE 20.2.19 Dirichlet problem in Problem 22

In Problems 23–26, use an appropriate conformal mapping and the harmonic function \( U = (c_0/\pi)\text{Arg}(w - 1) - \text{Arg}(w + 1) \) to solve the given Dirichlet problem.

23. FIGURE 20.2.20 Dirichlet problem in Problem 23

24. FIGURE 20.2.21 Dirichlet problem in Problem 24

25. FIGURE 20.2.22 Dirichlet problem in Problem 25

26. FIGURE 20.2.23 Dirichlet problem in Problem 26

27. A real-valued function \( \phi(x, y) \) is called biharmonic in a domain \( D \) when the fourth-order differential equation

\[
\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0
\]

at all points in \( D \). Examples of biharmonic functions are the Airy stress function in the mechanics of solids and velocity potentials in the analysis of viscous fluid flow.

(a) Show that if \( \phi \) is biharmonic in \( D \), then \( u = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \) is harmonic in \( D \).

(b) If \( g(z) \) is analytic in \( D \) and \( \phi(x, y) = \text{Re}(\bar{z}g(z)) \), show that \( \phi \) is biharmonic in \( D \).

20.3 Linear Fractional Transformations

**Introduction** In many applications that involve boundary-value problems associated with Laplace’s equation, it is necessary to find a conformal mapping that maps a disk onto the half-plane \( v \geq 0 \). Such a mapping would have to map the circular boundary of the disk to the boundary line of the half-plane. An important class of elementary conformal mappings that map circles to lines (and vice versa) are the fractional transformations. In this section we will define and study this special class of mappings.
**Linear Fractional Transformation** If \( a, b, c, \) and \( d \) are complex constants with \( ad - bc \neq 0 \), then the complex function defined by

\[
T(z) = \frac{az + b}{cz + d}
\]

is called a **linear fractional transformation**. Since

\[
T'(z) = \frac{ad - bc}{(cz + d)^2}
\]

\( T \) is conformal at \( z \) provided \( \Delta = ad - bc \neq 0 \) and \( z \neq -d/c \). (If \( \Delta = 0 \), then \( T'(z) = 0 \) and \( T(z) \) would be a constant function.) Linear fractional transformations are circle preserving in a sense that we will make precise in this section, and, as we saw in Example 8 of Section 20.2, they can be useful in solving Dirichlet problems in regions bounded by circles.

Note that when \( c \neq 0 \), \( T(z) \) has a simple pole at \( z_0 = -d/c \) and so

\[
\lim_{z \to z_0} |T(z)| = \infty.
\]

We will write \( T(z_0) = \infty \) as shorthand for this limit. In addition, if \( c \neq 0 \), then

\[
\lim_{|z| \to \infty} T(z) = \lim_{|z| \to \infty} \frac{a + b/z}{c + d/z} = \frac{a}{c},
\]

and we write \( T(\infty) = acl \).

**Example 1** A Linear Fractional Transformation

If \( T(z) = (2z + 1)/(z - i) \), compute \( T(0) \), \( T(\infty) \), and \( T(i) \).

**Solution** Note that \( T(0) = 1/(z - i) = i \) and \( T(\infty) = \lim_{z \to \infty} T(z) = 2 \). Since \( z = i \) is a simple pole for \( T(z) \), we have \( \lim_{z \to i} |T(z)| = \infty \) and we write \( T(i) = \infty \).

**Circle-Preserving Property** If \( c = 0 \), the linear fractional transformation reduces to a linear function \( T(z) = Az + B \). We saw in Section 20.1 that such a complex mapping can be considered as the composite of a rotation, magnification, and translation. As such, a linear function will map a circle in the \( z \)-plane to a circle in the \( w \)-plane. When \( c \neq 0 \), we can divide \( cz + d \) into \( az + b \) to obtain

\[
w = \frac{az + b}{cz + d} = \frac{bc - ad}{c} \cdot \frac{1}{cz + d} + \frac{a}{c}.
\]

If we let \( A = (bc - ad)/c \) and \( B = acl \), \( T(z) \) can be written as the composite of transformations:

\[
z_1 = cz + d, \quad z_2 = \frac{1}{z_1}, \quad w = Az_2 + B.
\]

A general linear fractional transformation can therefore be written as the composite of two linear functions and the inversion \( w = 1/z \). Note that if \( |z - z_1| = r \) and \( w = 1/z \), then

\[
\left| \frac{1}{w} - \frac{1}{w_1} \right| = \left| \frac{w - w_1}{|w||w_1|} \right| = r \quad \text{or} \quad |w - w_1| = (r|w_1|)|w - 0|.
\]

It is not hard to show that the set of all points \( w \) that satisfy

\[
|w - w_1| = \lambda|w - w_2|
\]

is a line when \( \lambda = 1 \) and is a circle when \( \lambda > 0 \) and \( \lambda \neq 1 \). It follows from (3) that the image of the circle \( |z - z_1| = r \) under the inversion \( w = 1/z \) is a circle except when \( r = 1/|w_1| = |z_1| \). In the

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latter case, the original circle passes through the origin and the image is a line. See Figure 20.1.3. From (2), we can deduce the following theorem:

**Theorem 20.3.1 Circle-Preserving Property**

A linear fractional transformation maps a circle in the $z$-plane to either a line or a circle in the $w$-plane. The image is a line if and only if the original circle passes through a pole of the linear fractional transformation.

**Proof**

We have shown that a linear function maps a circle to a circle, whereas an inversion maps a circle to a circle or a line. It follows from (2) that a circle in the $z$-plane will be mapped to either a circle or a line in the $w$-plane. If the original circle passes through a pole $z_0$, then $T(z_0) = \infty$, and so the image is unbounded. Therefore, the image of such a circle must be a line. If the original circle does not pass through $z_0$, then the image is bounded and must be a circle.

**Example 2 Images of Circles**

Find the images of the circles $|z| = 1$ and $|z| = 2$ under $T(z) = (z + 2)/(z - 1)$. What are the images of the interiors of these circles?

**Solution** The circle $|z| = 1$ passes through the pole $z_0 = 1$ of the linear fractional transformation and so the image is a line. Since $T(-1) = -\frac{1}{2}$ and $T(i) = -\frac{1}{2} - \frac{1}{2}i$, we can conclude that the image is the line $u = -\frac{1}{2}$. The image of the interior $|z| < 1$ is either the half-plane $u < -\frac{1}{2}$ or the half-plane $u > -\frac{1}{2}$. Using $z = 0$ as a test point, $T(0) = -2$, and so the image is the half-plane $u < -\frac{1}{2}$.

The circle $|z| = 2$ does not pass through the pole and so the image is a circle. For $|z| = 2$,

$$\overline{T(z)} = \frac{\overline{z} + 2}{\overline{z} - 1} = \frac{\overline{z} + 2}{\overline{z} - 1} = T(\overline{z}).$$

Therefore, $\overline{T(z)}$ is a point on the image circle and so the image circle is symmetric with respect to the $u$-axis. Since $T(-2) = 0$ and $T(2) = 4$, the center of the circle is $w = 2$ and the image is the circle $|w - 2| = 2$. See Figure 20.3.1. The image of the interior $|z| < 2$ is either the interior or the exterior of the image circle $|w - 2| < 2$. Since $T(0) = -2$, we can conclude that the image is $|w - 2| > 2$.

**Figure 20.3.1 Images of test points in Example 2**

**Constructing Special Mappings** In order to use linear fractional transformations to solve Dirichlet problems, we must construct special functions that map a given circular region $R$ to a target region $R'$ in which the corresponding Dirichlet problem is solvable. Since a circular boundary is determined by three of its points, we must find a linear fractional transformation $w = T(z)$ that maps three given points $z_1$, $z_2$, and $z_3$ on the boundary of $R$ to three points $w_1$, $w_2$, and $w_3$ on the boundary of $R'$. In addition, the interior of $R'$ must be the image of the interior of $R$. See Figure 20.3.2.
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CHAPTER 20  Conformal Mappings

Matrix Methods  Matrix methods can be used to simplify many of the computations. We can associate the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $T(z) = (az + b)/(cz + d)$.* If $T_1(z) = (a_1z + b_1)/(c_1z + d_1)$ and $T_2(z) = (a_2z + b_2)/(c_2z + d_2)$, then the composite function $T(z) = T_2(T_1(z))$ is given by $T(z) = (az + b)/(cz + d)$, where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

(5)

If $w = T(z) = (az + b)/(cz + d)$, we can solve for $z$ to obtain $z = (dw - b)/(cw + a)$. Therefore, the inverse of the linear fractional transformation $T$ is $T^{-1}(w) = (dw - b)/(cw + a)$ and we associate the matrix

$$\text{adj } A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(6)

with $T^{-1}$. The matrix adj $A$ is the adjoint matrix of $A$ (see Section 8.6), the matrix for $T$.

Example 3  Using Matrices to Find an Inverse Transform

If $T(z) = \frac{2z - 1}{z + 2}$ and $S(z) = \frac{z - i}{iz - 1}$ find $S^{-1}(T(z))$.

Solution  From (5) and (6), we have $S^{-1}(T(z)) = (az + b)/(cz + d)$, where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{adj} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -2 + i & 1 + 2i \\ 1 - 2i & 2 + i \end{pmatrix}$$

Therefore,

$$S^{-1}(T(z)) = \frac{-2 + i}{2}z + \frac{1 + 2i}{2} \quad \frac{1 - 2i}{2}z + \frac{2 + i}{2}.$$

Triplets to Triples  The linear fractional transformation

$$T(z) = \frac{z - z_1 z_2 - z_3}{z - z_3 z_2 - z_1}$$

has a zero at $z = z_1$, a pole at $z = z_3$, and $T(z_3) = 1$. Therefore, $T(z)$ maps three distinct complex numbers $z_1, z_2,$ and $z_3$ to $0, 1,$ and $\infty$, respectively. The term $\frac{z - z_1 z_2 - z_3}{z - z_3 z_2 - z_1}$ is called the cross-ratio of the complex numbers $z, z_1, z_2,$ and $z_3$.

*The matrix $A$ is not unique since the numerator and denominator in $T(z)$ can be multiplied by a nonzero constant.

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Likewise, the complex mapping \( S(w) = \frac{w - w_1 w_2 - w_3}{w - w_3 w_2 - w_1} \) sends \( w_1, w_2, \) and \( w_3 \) to \( 0, 1, \) and \( \infty \), and so \( S^{-1} \) maps \( 0, 1, \) and \( \infty \) to \( w_1, w_2, \) and \( w_3 \). It follows that the linear fractional transformation \( w = S^{-1}(T(z)) \) maps the triple \( z_1, z_2, \) and \( z_3 \) to \( w_1, w_2, \) and \( w_3 \). From \( w = S^{-1}(T(z)) \), we have \( S(w) = T(z) \) and we can conclude that

\[
\frac{w - w_1 w_2 - w_3}{w - w_3 w_2 - w_1} = \frac{z - z_1 z_2 - z_3}{z - z_3 z_2 - z_1}
\]

(7)

In constructing a linear fractional transformation that maps the triple \( z_1, z_2, \) and \( z_3 \) to \( w_1, w_2, \) and \( w_3 \), we can use matrix methods to compute \( w = S^{-1}(T(z)) \). Alternatively, we can substitute into (7) and solve the resulting equation for \( w \).

**EXAMPLE 4** Constructing a Linear Fractional Transformation

Construct a linear fractional transformation that maps the points \( 1, i, \) and \( -1 \) on the circle \(|z| = 1\) to the points \(-1, 0, \) and \( 1 \) on the real axis.

**Solution** Substituting into (7), we have

\[
\frac{w + 1}{w - 1} \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} = \frac{z - 1 + i}{z + 1 - i}
\]

or

\[
\frac{w + 1}{w - 1} = -i \frac{z - 1}{z + 1}.
\]

Solving for \( w \), we obtain \( w = -i(z - i)/(z + i) \). Alternatively, we could use the matrix method to compute \( w = S^{-1}(T(z)) \).

When \( z_3 = \infty \) plays the role of one of the points in a triple, the definition of the cross-ratio is changed by replacing each factor that contains \( z_3 \) by \( 1 \). For example, if \( z_2 = \infty \), both \( z_2 - z_3 \) and \( z_2 - z_1 \) are replaced by \( 1 \), giving \((z - z_1)/(z - z_3)\) as the cross-ratio.

**EXAMPLE 5** Constructing a Linear Fractional Transformation

Construct a linear fractional transformation that maps the points \( \infty, 0, \) and \( 1 \) on the real axis to the points \( 1, i, \) and \( -1 \) on the circle \(|w| = 1\).

**Solution** Since \( z_1 = \infty \), the terms \( z - z_1 \) and \( z_2 - z_1 \) in the cross product are replaced by \( 1 \).

It follows that

\[
\frac{w - 1 + i}{w + 1 - i} = \frac{1}{z - 1}
\]

or

\[
S(w) = -i \frac{w - 1}{w + 1} = \frac{1}{z - 1} = T(z).
\]

If we use the matrix method to find \( w = S^{-1}(T(z)) \), then

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{adj} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -i & 1 + i \\ -i & 1 + i \end{pmatrix}
\]

and so \( w = -iz - 1 + i \) and \( w = z - 1 + i \) if \( z = z_1 \).

**EXAMPLE 6** Solving a Dirichlet Problem

Solve the Dirichlet problem in **FIGURE 20.3.3(a)** using conformal mapping by constructing a linear fractional transformation that maps the given region into the upper half-plane.

**Solution** The boundary circles \(|z| = 1\) and \(|z - \frac{1}{2}| = \frac{1}{2}\) each pass through \( z = 1 \). We can therefore map each boundary circle to a line by selecting a linear fractional transformation that has \( z = 1 \) as a pole. If we further require that \( T(1) = 0 \) and \( T(-1) = 1 \), then

\[
T(z) = \frac{z - i}{z - 1} \quad \text{and} \quad T(z) = \frac{z - i}{z - 1}.
\]

**FIGURE 20.3.3** Image of Dirichlet problem in Example 6

20.3 Linear Fractional Transformations

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In Problems 1–4, a linear fractional transformation is given.

(a) Compute $T(0)$, $T(1)$, and $T(\infty)$.

(b) Find the images of the circles $|z| = 1$ and $|z - 1| = 1$.

(c) Find the image of the disk $|z| \leq 1$.

1. $T(z) = \frac{i}{z}$
2. $T(z) = \frac{1}{z - 1}$
3. $T(z) = \frac{z + 1}{z - 1}$
4. $T(z) = \frac{z - i}{z}$

In Problems 5–8, use the matrix method to compute $S^{-1}(w)$ and $S^{-1}(T(z))$ for each pair of linear fractional transformations.

5. $T(z) = \frac{z}{iz - 1}$ and $S(z) = \frac{iz + 1}{z - 1}$
6. $T(z) = \frac{iz}{z - 2i}$ and $S(z) = \frac{2z + 1}{z + 1}$
7. $T(z) = \frac{2z - 3}{z - 3}$ and $S(z) = \frac{z - 2}{z - 1}$
8. $T(z) = \frac{z - 1 + i}{iz - 2}$ and $S(z) = \frac{2 - iz}{z - 1 - i}$

In Problems 9–16, construct a linear fractional transformation that maps the given triple $z_1, z_2,$ and $z_3$ to the triple $w_1, w_2,$ and $w_3$.

9. $-1, 0, 2$ to $0, 1, \infty$
10. $i, 0, -i$ to $0, 1, \infty$
11. $0, 1, \infty$ to $0, i, 2$
12. $0, 1, \infty$ to $1 + i, 0, 1 - i$
13. $-1, 0, 1$ to $i, \infty, 0$
14. $-1, 0, 1$ to $\infty, -i, 1$
15. $1, i, -i$ to $-1, 0, 3$
16. $1, i, -i$ to $-i, -1$

17. Use the results in Example 2 and the harmonic function $U = (\log_r r)/(\log_r r_0)$ to solve the Dirichlet problem in Figure 20.3.5. Explain why the level curves must be circles.

18. Use the linear fractional transformation that maps $-1, 1, 0$ to $0, 1, \infty$ to solve the Dirichlet problem in Figure 20.3.5. Explain why, with one exception, all level curves must be circles. Which level curve is a line?

19. Derive the conformal mapping $H^{-1}$ in the conformal mappings in Appendix IV.