3.6 Integration with Cylindrical and Spherical Coordinates

In this section, we describe, and give examples of, computing triple integrals in the cylindrical coordinates r, θ, and z, and in spherical coordinates ρ, ϕ, and θ.

In the More Depth portion of this section, we will address how you integrate in $\mathbb{R}^3$ or, more generally, in $\mathbb{R}^n$, using any C^1 change of coordinates.

Basics:

Just as some double integrals don't look very nice in terms of the Cartesian coordinates x and y, many triple integrals don't look particularly nice in terms of x, y, and z. There are two other standard sets of coordinates that are used in space: *cylindrical coordinates* and *spherical coordinates*.

Cylindrical Coordinates

Cylindrical coordinates are easy, given that we already know about polar coordinates in the xy-plane from Section 3.3. Recall that in the context of multivariable integration, we always assume that $r \geq 0$.

Cylindrical coordinates for $\mathbb{R}^3$ are simply what you get when you use polar coordinates r and θ for the xy-plane, and just let z be z. Therefore, we still have that $r = \sqrt{x^2 + y^2}$, but now r is not the distance from the origin; it's the distance from the z-axis.

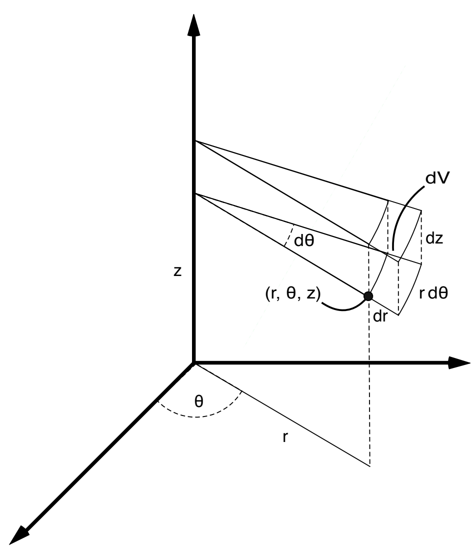

Figure 3.6.1: In cylindrical coordinates, $dV = r\,dr\,d\theta\,dz$.

Our expression for the volume element dV is also easy now; since $dV = dz\,dA$, and $dA = r\,dr\,d\theta$ in polar coordinates, we find that $dV = dz\,r\,dr\,d\theta = r\,dz\,dr\,d\theta$ in cylindrical coordinates.

Thus, to integrate, you use:

Integration in Cylindrical Coordinates:

To perform triple integrals in cylindrical coordinates, and to switch from cylindrical coordinates to Cartesian coordinates, you use:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z, \qquad \text{and} \qquad dV = dz\,dA = r\,dz\,dr\,d\theta.$$

Example 3.6.1. Find the volume of the solid region S which is above the half-cone given by $z = \sqrt{x^2+y^2}$ and below the hemisphere where $z = \sqrt{8-x^2-y^2}$.

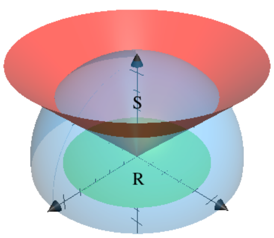

Figure 3.6.2: The "snow cone" S and the projected region R.

Solution:

Note that, in cylindrical coordinates, the half-cone is given by $z = \sqrt{r^2} = r$ and the hemisphere is given by $z = \sqrt{8-r^2}$.

To find the volume, we need to calculate $\displaystyle\int\int\int_S dV$.

The projected region R in the xy-plane, or $r\theta$-plane, is the inside of the circle (thought of as lying in a copy of the xy-plane) along which the two surfaces intersect. To find this circle, we set the two z's equal to each other and find

$$r = \sqrt{8-r^2}, \qquad \text{or, equivalently,} \qquad r^2 = 8 - r^2.$$

We find

$$2r^2 = 8, \qquad \text{so} \qquad r^2 = 4, \qquad \text{and, hence,} \qquad r = 2.$$

Thus, R is the disk in the xy-plane where $r \leq 2$.

For each point $\mathbf{p}$ in R, the corresponding points which lie over it in the solid region S have z-coordinates which start on the half-cone where $z = r$ and end on the hemisphere where $z = \sqrt{8-r^2}$.

Therefore,

$$\int\int\int_S dV = \int\int_R \left[\int_r^{\sqrt{8-r^2}} dz\right] dA = \int_0^{2\pi}\int_0^2\int_r^{\sqrt{8-r^2}} r\,dz\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^2 \left[rz \,\Big|_{z=r}^{z=\sqrt{8-r^2}} \right] dr\,d\theta = \int_0^{2\pi}\int_0^2 \left(r\sqrt{8-r^2} - r^2\right) dr\,d\theta.$$

The inner integral is easy, via the substitution $u = 8 - r^2$ in the first term. We obtain

$$\text{volume} = \int\int\int_S dV = \int_0^{2\pi} \frac{16}{3}\left(\sqrt{2}-1\right) d\theta = \frac{32\pi}{3}\left(\sqrt{2}-1\right).$$

Example 3.6.2. Let R be the region in the xy-plane, or $r\theta$-plane, which is bounded by the curves given by $r = 1+\theta^2$ and $r = 1+\theta+\theta^2$, for $0 \le \theta \le \pi$.

Integrate the function $f(x,y) = 1/\sqrt{x^2+y^2}$ over the solid region S which lies above the region R and is bounded by the plane where $z = 1$ and the half-cone where $z = 1 + 2\sqrt{x^2+y^2}$.

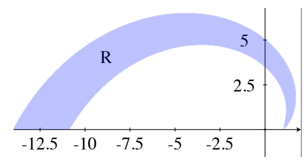

Figure 3.6.3: The spiraling plane region R.

Solution: The problem is given in a mixture of cylindrical and Cartesian coordinates, but the region R is so clearly set up for nice integration in polar coordinates that it should be obvious that you want to use cylindrical coordinates for space.

In Figure 3.6.3 and Figure 3.6.4, we show the region R and the plane and cone. It is difficult to sketch the solid region S, but, fortunately, there's no need to do so. After noting that $f = 1/r$ and that the cone is given by $z = 1+2r$, we can go ahead and calculate:

$$\int\int\int_S \frac{1}{\sqrt{x^2+y^2}}\, dV = \int\int_R \left[\int_1^{1+2r} \frac{1}{r}\, dz\right] dA =$$

$$\int_0^{\pi}\int_{1+\theta^2}^{1+\theta+\theta^2}\int_1^{1+2r} \frac{1}{r}\, dz\, r\, dr\, d\theta = \int_0^{\pi}\int_{1+\theta^2}^{1+\theta+\theta^2}\int_1^{1+2r} dz\, dr\, d\theta =$$

$$\int_0^{\pi}\int_{1+\theta^2}^{1+\theta+\theta^2} 2r\, dr\, d\theta = \int_0^{\pi}\left[r^2\Big|_{r=1+\theta^2}^{r=1+\theta+\theta^2}\right] d\theta = \int_0^{\pi}\left((1+\theta+\theta^2)^2 - (1+\theta^2)^2\right) d\theta =$$

$$\int_0^{\pi} \theta(2+\theta+2\theta^2)\, d\theta = \int_0^{\pi} (2\theta+\theta^2+2\theta^3)\, d\theta = \frac{\pi^2}{6}\left(6+2\pi+3\pi^2\right).$$

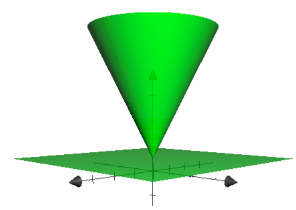

Figure 3.6.4: The plane and cone which form the vertical bounds of the solid region S.

Spherical Coordinates

There is a third common set of coordinates for $\mathbb{R}^3$, other than the Cartesian coordinates x, y, and z, or the cylindrical coordinates r, θ, and z. It is sometimes convenient to use *spherical coordinates* ρ, θ, and ϕ.

Pick a point $\mathbf{p}$ in space, other than the origin, and draw the line segment L from the origin to $\mathbf{p}$. We let ρ denote the length of L, i.e., the distance from the origin to $\mathbf{p}$. We orthogonally project L into the xy-plane, and let θ be the angle, $0 \le \theta < 2\pi$, between the positive x-axis and this projected line segment. If the Cartesian coordinates of $\mathbf{p}$ are (x,y,z), then θ is precisely the polar coordinate angle of the point (x,y). Note that we also have that $\rho^2 = x^2+y^2+z^2$. Finally, we let ϕ, where $0 \le \phi \le \pi$, be the angle between the positive z-axis and L. See Figure 3.6.5.

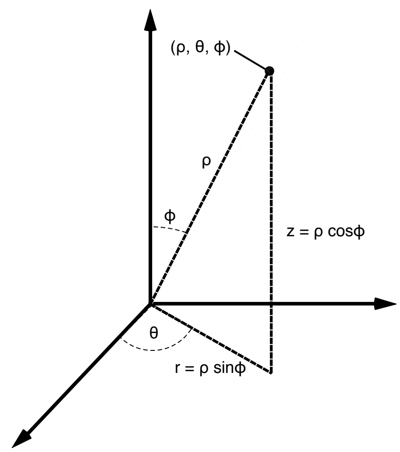

Figure 3.6.5: A point with spherical coordinates (ρ, θ, ϕ).

Figure 3.6.5 makes it clear that the polar coordinate r of the point (x, y) is $\rho \sin \phi$, and that $z = \rho \cos \phi$.

In order to obtain an expression for the infinitesimal volume element dV in spherical coordinates, we need to include the infinitesimal changes in ρ, θ, and ϕ; this makes for the much more complicated diagram in Figure 3.6.6.

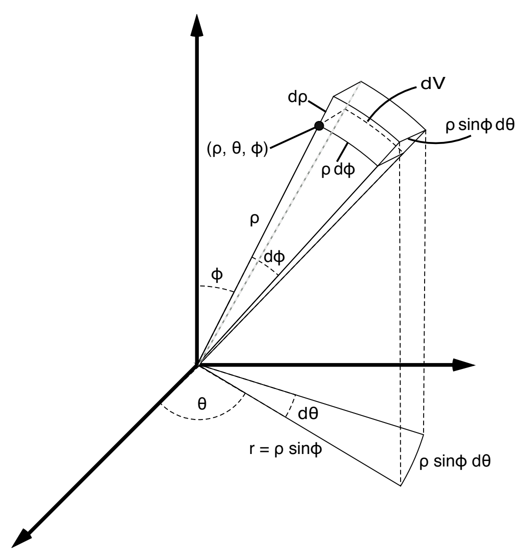

Figure 3.6.6: In spherical coordinates, $dV \;=\; \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$.

In the More Depth portion, we shall derive the formula for dV in spherical coordinates, or in any coordinates, in a more analytic way.

In the diagram, we see that the volume element is given, in spherical coordinates, by

$$dV \;=\; \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

Thus, to integrate in spherical coordinates, you use:

Integration in Spherical Coordinates:

To perform triple integrals in spherical coordinates, and to switch from spherical coordinates to cylindrical or Cartesian coordinates, you use:

$$r = \rho \sin\phi;$$

$$x = r\cos\theta = \rho\sin\phi\cos\theta;$$
$$y = r\sin\theta = \rho\sin\phi\sin\theta;$$

$$z = \rho\cos\phi;$$

and

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta,$$

where $\rho \geq 0$, $0 \leq \theta < 2\pi$, *and* $0 \leq \phi \leq \pi$.

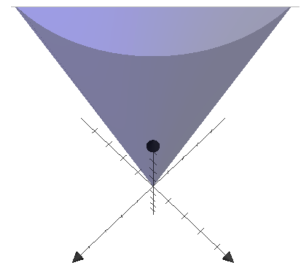

Figure 3.6.7: Graph of $\phi = c$, where $0 < c < \pi/2$.

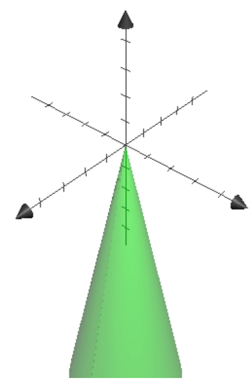

Figure 3.6.8: Graph of $\phi = c$, where $\pi/2 < c < \pi$.

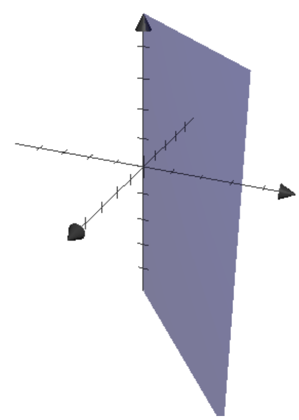

Figure 3.6.9: Graph of $\theta = c$.

Remark 3.6.3. We also make frequent use of the fact that $\rho^2 = x^2 + y^2 + z^2$. This follows from the definition of ρ and the Pythagorean Theorem, but it is a good exercise to take the expressions for x, y and z in spherical coordinates, square them, add them, and use the Fundamental Trigonometric Identity (twice) to verify that you get ρ^2.

Note that $\rho = 0$ corresponds to exactly one point: the origin. What are θ and ϕ for the origin? Anything at all, as long as $0 \leq \theta < 2\pi$, and $0 \leq \phi \leq \pi$.

While $\rho = 0$ describes a single point, if c is a constant, greater than 0, then $\rho = c$ describes a sphere of radius c, centered at the origin.

The equation $\phi = 0$ describes the positive z-axis (plus the origin), while $\phi = \pi$ describes the negative z-axis (plus the origin). The equation $\phi = \pi/2$ describes the xy-plane. If $0 < c < \pi/2$, then $\phi = c$ describes a right circular cone, surrounding the positive z-axis. If $\pi/2 < c < \pi$, then $\phi = c$ describes a right circular cone, surrounding the negative z-axis.

If c is a constant, the equation $\theta = c$ describes a half-plane, which is perpendicular to the xy-plane, and which extends outward from the z-axis at an angle θ with the positive x-axis. In other words, in the xy-plane, you take a ray, which makes an angle of θ with the positive x-axis, and then you move this ray up and down the z-axis to form a plane which gets chopped off on one side by the z-axis.

Example 3.6.4. Integrate 1 over the 1st octant portion of ball of radius R, centered at the origin, to obtain its volume. Verify that you obtain 1/8 of $4\pi R^3/3$, i.e., $\pi R^3/6$.

Solution:

This is really easy in spherical coordinates. Let's call the solid region S. Clearly, the region S is described in spherical coordinates by letting ρ go from 0 to R, θ go from 0 to $\pi/2$, and ϕ go from 0 to $\pi/2$. We calculate

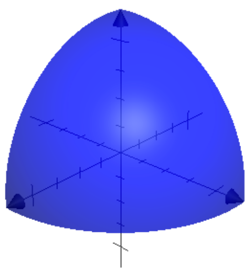

Figure 3.6.10: An eighth of a ball.

$$\text{volume} = \iiint_S dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^R \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta =$$

$$\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\rho^3\sin\phi}{3}\Big|_{\rho=0}^{\rho=R}\right] d\phi\, d\theta = \frac{R^3}{3}\int_0^{\pi/2}\int_0^{\pi/2} \sin\phi\, d\phi\, d\theta =$$

$$\frac{R^3}{3}\int_0^{\pi/2}\left[-\cos\phi\Big|_0^{\pi/2}\right] d\theta = \frac{R^3}{3}\int_0^{\pi/2} 1\, d\theta = \frac{\pi R^3}{6}.$$

Example 3.6.5. Integrate z over the 1st octant portion of ball of radius R, centered at the origin.

Solution:

This is the solid region S from the previous example. As before, the region S is described in spherical coordinates by letting ρ go from 0 to R, θ go from 0 to $\pi/2$, and ϕ go from 0 to $\pi/2$.

We calculate

$$\iiint_S z\, dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^R \rho\cos\phi\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta =$$

$$\frac{R^4}{4}\int_0^{\pi/2}\int_0^{\pi/2}\sin\phi\cos\phi\, d\phi\, d\theta = \frac{R^4}{4}\int_0^{\pi/2}\left[\frac{\sin^2\phi}{2}\Big|_0^{\pi/2}\right] d\theta = \frac{\pi R^4}{16}.$$

Example 3.6.6. Let's redo the problem from Example 3.6.1. We'll find the volume of the solid region S which is above the half-cone given by $z = \sqrt{x^2+y^2}$ and below the hemisphere where $z = \sqrt{8-x^2-y^2}$, but, this time, we'll use spherical coordinates.

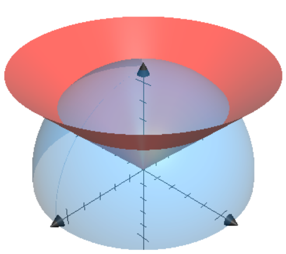

Figure 3.6.11: The "snow cone" S from Example 3.6.1.

It should be clear that ρ goes from 0 to $\sqrt{8}$, and that θ goes all the way around, i.e., from 0 to 2π. But, what about ϕ?

Some people may see quickly that the cone where $z = \sqrt{x^2+y^2}$ is precisely where $\phi = \pi/4$, and so ϕ goes from 0 to $\pi/4$. However, if this isn't so obvious that the cone is where $\phi = \pi/4$, how do you figure it out?

You just switch $z = \sqrt{x^2+y^2}$ into spherical coordinates, passing through cylindrical coordinates along the way. In cylindrical coordinates, $r = \sqrt{x^2+y^2}$; so our equation becomes $z = r$. Now, recalling Figure 3.6.5, we use that $z = \rho\cos\phi$, and that $r = \rho\sin\phi$, so that our equation becomes

$$\rho\cos\phi \;=\; \rho\sin\phi,$$

and so, either $\rho = 0$ and we're at the origin, or $\cos\phi = \sin\phi$, which means $\phi = \pi/4$. We don't have to worry about $\rho = 0$ as a separate case, since the equation $\phi = \pi/4$ already includes the possibility that $\rho = 0$ (since the origin has every possible θ and ϕ).

Now that we know that our cone is where $\phi = \pi/4$, we find that the volume of our snow cone is

$$\text{volume} \;=\; \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{8}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \;=\; \frac{8\sqrt{8}}{3}\int_0^{2\pi}\int_0^{\pi/4}\sin\phi\,d\phi\,d\theta \;=$$

$$\frac{8\sqrt{8}}{3}\int_0^{2\pi}\left[-\cos\phi\,\Big|_0^{\pi/4}\right]d\theta \;=\; \frac{16\sqrt{2}}{3}\cdot 2\pi\cdot\left(-\frac{1}{\sqrt{2}}+1\right) \;=\; \frac{32\pi}{3}\left(\sqrt{2}-1\right),$$

as we found in Example 3.6.1.

+ Linear Algebra:

In the More Depth portion of Section 3.3, we gave a theorem which tells us how changes of coordinates affect integration in $\mathbb{R}^2$. That theorem generalizes in an obvious way to integration in $\mathbb{R}^n$ for any $n \geq 1$ and, in particular, applies to the spherical change of coordinates on $\mathbb{R}^3$.

To understand the statement in $\mathbb{R}^n$, you need to know about determinants of $n \times n$ matrices, which is why this section requires a bit of linear algebra beyond what you may know. Of course, if you care only about $n = 1, 2, 3$, then you're fine without knowing about determinants of larger matrices; you already know how to take determinants of square matrices of those sizes.

Theorem 3.6.7. *Suppose that $\mathbf{\Phi}$ is a C^1 function from an open neighborhood of a point $\mathbf{p}$ in $\mathbb{R}^n$ into $\mathbb{R}^n$, and suppose that* $\det\left[d_{\mathbf{p}}\mathbf{\Phi}\right] \neq 0$.

Then, $(x_1,\ldots,x_n) = \mathbf{\Phi}(u_1,\ldots,u_n)$ is a C^1 local change of coordinates at $\mathbf{p}$ and, near $\mathbf{p}$, the n-dimensional volume element $dV^n = dx_1dx_2\cdots dx_n$ is given by

$$dV^n \;=\; \left|\det\left[d\mathbf{\Phi}\right]\right| du_1du_2\cdots du_n.$$

This follows from Theorem 7.3.8 of Trench, [8].

Thus, if $f = f(\mathbf{x})$ is Riemann integrable on a region R, contained in an open subset $\mathcal{U}$ of $\mathbb{R}^n$ and $\mathbf{\Phi}$ is a C^1 change of coordinates from an open set $\mathcal{W}$ in $\mathbb{R}^n$ onto $\mathcal{U}$, then $f \circ \mathbf{\Phi}$ is Riemann integrable on the region $S = \mathbf{\Phi}^{-1}(R)$ and

$$\int_R f\, dV^n \;=\; \int_S (f \circ \mathbf{\Phi}) \cdot \Big|\det\big[d\mathbf{\Phi}\big]\Big|\, du_1 du_2 \cdots du_n.$$

Let's apply Theorem 3.6.7 to the case where $n = 3$ and our change of coordinates is given by spherical coordinates, i.e., where

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad \text{and} \qquad z \;=\; \rho\cos\phi,$$

so that

$$(x, y, z) \;=\; \Phi(\rho, \theta, \phi) \;=\; (\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi).$$

Theorem 3.6.7 tells us that to calculate dV in spherical coordinates, we need to calculate $\Big|\det\big[d\mathbf{\Phi}\big]\Big|$. We find

$$\Big|\det\big[d\mathbf{\Phi}\big]\Big| \;=\; \left|\det\begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}\right| \;=$$

$$\Big| -\rho^2\sin^3\phi\cos^2\theta - \rho^2\sin\phi\cos^2\phi\sin^2\theta - \rho^2\sin\phi\cos^2\phi\cos^2\theta - \rho^2\sin^3\phi\sin^2\theta\Big| \;=$$

$$\rho^2\sin\phi\sin^2\phi(\cos^2\theta + \sin^2\theta) + \rho^2\sin\phi\cos^2\phi(\sin^2\theta+\cos^2\theta) \;=$$

$$\rho^2\sin\phi(\sin^2\phi + \cos^2\phi) \;=\; \rho^2\sin\phi.$$

Therefore, we recover what we already knew:

$$dV \;=\; dx\,dy\,dz \;=\; \rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$$