Appendix C: Propagation of Errors

In many experiments, the quantities measured are not the quantities of final interest. Since all measurements have uncertainties associated with them, clearly any calculated quantity will have an uncertainty that is related to the uncertainties of the direct measurements. The procedure used to estimate the error for the calculated quantities is called the propagation of errors.

Consider the general case first. Suppose that the variables A, B, C, ... represent independent measureable quantities that will be used to obtain a value for some calculated quantity U. Since U is a function of A, B, C, ..., it can be written as

$$U = f(A, B, C, ...). \tag{1}$$

The measurements of the quantities A, B, C, ... yield estimates of the true values written as $\overline{A}, \overline{B}, \overline{C}, ...$ and the associated uncertainties, ΔA, ΔB, ΔC, ... for each variable. To find the best estimate for the uncertainty U, each (average) measured value is substituted into the equation for U:

$$\overline{U} = f(\overline{A}, \overline{B}, \overline{C}, ...). \tag{2}$$

If the errors of A, B, C, ... are independent, random, and sufficiently small, it can be shown that the uncertainty for U is given by

$$\Delta U = \sqrt{\left(\frac{\partial U}{\partial A}\right)^2 (\Delta A)^2 + \left(\frac{\partial U}{\partial B}\right)^2 (\Delta B)^2 + \left(\frac{\partial U}{\partial C}\right)^2 (\Delta C)^2 + ...} \tag{3}$$

where the partial derivatives are evaluated using the best estimates $\overline{A}, \overline{B}, \overline{C}, ...$ as the values for the independent variables.

NOTE:

Don't worry if you are not familiar with the concept of partial derivative. All you will need for the lab write-ups are the results given in Equations (5), (7), and (9) of this Appendix.

1. ADDITION AND SUBTRACTION

Suppose two quantities A and B are added. If the uncertainties associated with each variable are ΔA and ΔB, then Equation (3) gives

$$U = A + B$$

$$\frac{\partial U}{\partial A} = 1 \quad \frac{\partial U}{\partial B} = 1$$

$$\Delta U = \sqrt{(1)^2(\Delta A)^2 + (1)^2(\Delta B)^2}$$

$$\Delta U = \sqrt{(\Delta A)^2 + (\Delta B)^2}.$$

If the two values are subtracted then

$$U = A - B$$

$$\frac{\partial U}{\partial A} = 1 \quad \frac{\partial U}{\partial B} = -1$$

$$\Delta U = \sqrt{(1)^2 + (\Delta A)^2 + (-1)^2(\Delta B)^2}$$

$$\Delta U = \sqrt{(\Delta A)^2 + (\Delta B)^2}$$

resulting in an identical expression for ΔU. These equations can be generalized for the addition and/or subtraction of an infinite series of numbers.

$$U = A \pm B \pm C \pm D \pm ... \tag{4}$$

$$\Delta U = \sqrt{(\Delta A)^2 + (\Delta B)^2 + (\Delta C)^2 + ...} \tag{5}$$

2. MULTIPLICATION AND DIVISION

Suppose two quantities A and B are multiplied. Then the uncertainty for the resulting value is given by

$$U = AB$$

$$\frac{\partial U}{\partial A} = B \quad \frac{\partial U}{\partial B} = \mathrm{A}$$

$$\Delta U = \sqrt{(B)^2(\Delta A)^2 + (A)^2(\Delta B)^2}$$

$$\Delta U = AB\sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2}$$

$$\Delta U = U\sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2}.$$

Likewise if the two values are divided, Equation (3) results in an uncertainty

$$U = \frac{A}{B}$$

$$\frac{\partial U}{\partial B} = \frac{1}{B} \quad \frac{\partial U}{\partial B} = -\frac{A}{B^2}$$

$$\Delta U = \sqrt{\left(\frac{1}{B}\right)^2(\Delta A)^2 + \left(\frac{-A}{B}\right)^2(\Delta B)^2}$$

$$\Delta U = \frac{A}{B}\sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2}$$

$$\Delta U = U\sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2}.$$

So the equation for determining the uncertainty for operations involving multiplication and division is the same. This result can be generalized for a large number of variables:

$$U = \frac{ABC...}{XY...} \tag{6}$$

$$\Delta U = U\sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2 + \left(\frac{\Delta C}{C}\right)^2 + ... + \left(\frac{\Delta X}{X}\right)^2 + \left(\frac{\Delta Y}{Y}\right)^2 ...} \tag{7}$$

3. EXPONENTS AND ROOTS

Suppose a calculation involves the use of exponents. If $U = kX^2$, where k is a constant, then the uncertainty of U is given by

$$\frac{\partial U}{\partial x} = 2kX$$

$$\Delta U = \sqrt{(2kX)^2(\Delta X)^2}$$

$$\Delta U = 2kX\Delta X$$

$$\Delta U = U\left(\frac{2\Delta X}{X}\right).$$

Likewise if the calculation for U involves a root, $U = KX^{1/2}$, then

$$\frac{\partial U}{\partial X} = \frac{1}{2}kX^{-1/2}$$

$$\Delta U = \sqrt{\left(\frac{1}{2}KX^{-1/2}\right)^2(\Delta X)^2}$$

$$\Delta U = \frac{1}{2}\frac{k\Delta X}{X^{1/2}}$$

$$\Delta U = U\left(\frac{1}{2}\frac{\Delta X}{X}\right).$$

So the uncertainty for quantities involving the use of exponents or roots can be generalized as:

$$U = kX^n \tag{8}$$

$$\Delta U = U\left(n\frac{\Delta X}{X}\right) \tag{9}$$

where n is any real number.

> **<u>NOTE</u>:**
>
> When uncertainties are presented in numerical form, they should be recorded to the same decimal place as the best estimate (i.e., average). The usual rules for significant figures do not apply when calculating uncertainties.

EXAMPLES

Example 1

The distance between two points on a straight line is found by recording the position of each point on a meterstick. Let point $X \pm \Delta X$ be located at 23.10 ± 0.05 cm and point $Y \pm \Delta Y$ be located at 84.91 ± 0.05 cm. Then the best estimate for the distance, D, is given by

$$D = X - Y.$$

The uncertainty in the distance given by Equation (5) is

$$\Delta D = \sqrt{(\Delta X)^2 + (\Delta Y)^2}.$$

Evaluating the various expressions gives

$$\overline{D} = 84.91 - 23.10 = 61.81$$

$$\Delta D = \sqrt{(0.05)^2 + (0.05)^2}$$

$$\overline{D} \pm \Delta D = 61.81 \pm 0.07 \text{ cm}.$$

Example 2

The dimensions of a tabletop are recorded as $I = 96.12 \pm 0.07$ cm and $w = 25.04 \pm 0.07$ cm. The surface area of the table is given by

$$SA = lw.$$

The uncertainty in the surface area given by Equation (7) is

$$\Delta SA = SA\sqrt{\left(\frac{\Delta l}{l}\right)^2 + \left(\frac{\Delta w}{w}\right)^2}.$$

Evaluating the various expressions gives

$$SA = (96.12)(25.04) = 2407 \text{ cm}^2$$

$$\Delta SA = 2407\sqrt{\left(\frac{0.07}{96.12}\right)^2 + \left(\frac{0.07}{25.04}\right)^2}$$

$$\Delta SA = 7 \text{ cm}^2$$

$$SA \pm \Delta SA = 2407 \pm 7 \text{ cm}^2.$$

Example 3

The mass and volume of an object are measured and reported as $M = 121.03 \pm 0.08$ grams and $V = 13.60 \pm 0.09$ cm^3. The density is given by

$$\rho = \frac{M}{V}$$

and the uncertainty in the density is found using Equation (7)

$$\Delta\rho = \rho\sqrt{\left(\frac{\Delta M}{M}\right)^2 + \left(\frac{\Delta V}{V}\right)^2}$$

$$\rho = \frac{121.03}{13.60} = 8.899 \ \frac{\text{grams}}{\text{cm}^3}$$

$$\Delta\rho = 8.899\sqrt{\left(\frac{0.08}{121.03}\right)^2 + \left(\frac{0.09}{13.60}\right)^2} = 0.0592$$

$$\rho \pm \Delta\rho = 8.899 \pm 0.059 \ \frac{\text{grams}}{\text{cm}^3}.$$

Example 4

The volume of a sphere with radius $r = 0.141 \pm 0.002$ mm is given by the equation

$$V = \frac{4}{3}\pi r^3.$$

The uncertainty in the volume, calculated using Equation (9), is

$$\Delta V = V\left(3\frac{\Delta r}{r}\right).$$

Evaluating these expressions gives

$$V = \frac{4}{3}(\pi)(0.141)^3 = 0.0117 \text{ mm}^3$$

$$\Delta V = (0.0117)(3)\left(\frac{0.0002}{0.141}\right)$$

$$\Delta V = 0.0005 \text{ mm}^3$$

$$V \pm \Delta V = 0.0177 \pm 0.0005 \text{ mm}^3.$$

Example 5

Suppose the acceleration due to gravity can be calculated from experimental data, using the equation

$$a = \frac{2xL}{ht^2}$$

where

$$x = 74.11 \pm 0.05 \text{ cm}$$

$$L = 100.1 \pm 0.2 \text{ cm}$$

$$h = 1.10 \pm 0.1 \text{ cm}$$

$$t = 3.708 \pm 0.003 \text{ sec.}$$

The uncertainty in the acceleration given by Equations (7) and (9) is

$$\Delta a = a\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta L}{L}\right)^2 + \left(\frac{\Delta h}{h}\right)^2 + \left(2\frac{\Delta t}{t}\right)^2}$$

$$a = \frac{(2)(74.11)(100.1)}{(1.10)(3.708)^2} = 981\frac{\text{cm}}{\text{sec}^2}$$

$$\Delta a = 981\sqrt{\left(\frac{0.05}{74.11}\right)^2 + \left(\frac{0.2}{100.1}\right)^2 + \left(\frac{0.01}{1.10}\right)^2 + \left(2 \cdot \frac{0.003}{3.708}\right)^2}$$

$$\Delta a = 9\frac{\text{cm}}{\text{sec}^2}$$

$$a \pm \Delta a = 981 + 9\frac{\text{cm}}{\text{sec}^2}.$$

SUMMARY

$$U = A \pm B \pm C \pm D \pm ...$$

$$\Delta U = \sqrt{(\Delta A)^2 + (\Delta B)^2 + (\Delta C)^2 + (\Delta D)^2}$$

$$U = \frac{ABC...}{XY...}$$

$$\frac{\Delta U}{U} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2 + \left(\frac{\Delta C}{C}\right)^2 + ... + \left(\frac{\Delta X}{X}\right)^2 + \left(\frac{\Delta Y}{Y}\right)^2 + ...}$$

$$U = X^n$$

$$\frac{\Delta U}{U} = n\frac{\Delta X}{X}$$

where n is any real number.

