Chapter 17

The Circular Functions

Suppose Cosmo begins at location R and walks in a counterclockwise direction, always maintaining a tight 20 ft long tether. As Cosmo moves around the circle, how can we describe his location at any given instant?

In one sense, we have already answered this question: The measure of $\angle RPS_1$ exactly pins down a location on the circle of radius 20 feet. But, we really might prefer a description of the horizontal and vertical coordinates of Cosmo; this would tie in better with the coordinate system we typically use. Solving this problem will require NEW functions, called the *circular functions*.

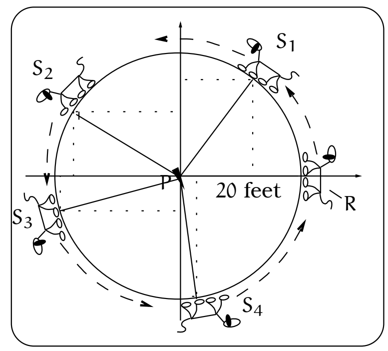


Figure 17.1: Cosmo moves counterclockwise maintaining a tight tether. Where's Cosmo?

17.1 Sides and Angles of a Right Triangle

Example 17.1.1. *You are preparing to make your final shot at the British Pocket Billiard World Championships. The position of your ball is as in Figure 17.2, and you must play the ball off the left cushion into the lower-right corner pocket, as indicated by the dotted path. For the big money, where should you aim to hit the cushion?*

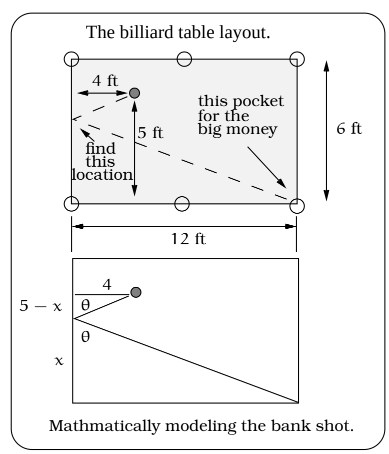


Figure 17.2: A pocket billiard banking problem.

Solution. This problem depends on two basic facts. First, the angles of entry and exit between the path the cushion will be equal. Secondly, the two obvious right triangles in this picture are similar triangles. Let x represent the distance from the bottom left corner to the impact point of the ball's path:

Properties of similar triangles tell us that the ratios of common sides are equal: $\frac{4}{5-x} = \frac{12}{x}$. If we solve this equation for x, we obtain $x = \frac{15}{4} = 3.75$ feet. □

This discussion is enough to win the tourney. But, of course, there are still other questions we can ask about this simple example: What is the angle θ? That is going to require substantially more work; indeed the bulk of this Chapter! It turns out, there is a lot of mathematical mileage in the idea of studying ratios of sides of right triangles. The first step, which will get the ball rolling, is to introduce new functions whose very definition involves relating sides and angles of right triangles.

17.2 The Trigonometric Ratios

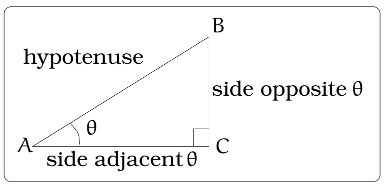


Figure 17.3: Labeling the sides of a right triangle.

From elementary geometry, the sum of the angles of any triangle will equal $180°$. Given a right triangle $\triangle ABC$, since one of the angles is $90°$, the remaining two angles must be *acute angles*; i.e., angles of measure between $0°$ and $90°$. If we specify one of the acute angles in a right triangle $\triangle ABC$, say angle θ, we can label the three sides using this terminology. We then consider the following three ratios of side lengths, referred to as *trigonometric ratios*:

$$\sin(\theta) \stackrel{\text{def}}{=} \frac{\text{length of side opposite } \theta}{\text{length of hypotenuse}} \tag{17.1}$$

$$\cos(\theta) \stackrel{\text{def}}{=} \frac{\text{length of side adjacent } \theta}{\text{length of hypotenuse}} \tag{17.2}$$

$$\tan(\theta) \stackrel{\text{def}}{=} \frac{\text{length of side opposite } \theta}{\text{length of side adjacent to } \theta}. \tag{17.3}$$

For example, we have three right triangles in Figure 17.4; you can verify that the Pythagorean Theorem holds in each of the cases. In the left-hand triangle, $\sin(\theta) = \frac{5}{13}$, $\cos(\theta) = \frac{12}{13}$, $\tan(\theta) = \frac{5}{12}$. In the middle triangle, $\sin(\theta) = \frac{1}{\sqrt{2}}$, $\cos(\theta) = \frac{1}{\sqrt{2}}$, $\tan(\theta) = 1$. In the right-hand triangle, $\sin(\theta) = \frac{1}{2}$, $\cos(\theta) = \frac{\sqrt{3}}{2}$, $\tan(\theta) = \frac{1}{\sqrt{3}}$. The symbols "sin", "cos", and "tan" are abbreviations for the words *sine*, *cosine* and *tangent*, respectively. As we have defined them, the trigonometric ratios depend on the dimensions of the triangle. However, the same ratios are obtained for *any* right triangle with acute angle θ. This follows from the properties of similar triangles. Consider Figure 17.5. Notice $\triangle ABC$ and $\triangle ADE$ are similar. If we use $\triangle ABC$ to compute $\cos(\theta)$, then we find $\cos(\theta) = \frac{|\overline{AC}|}{|\overline{AB}|}$. On the other hand, if we use $\triangle ADE$, we obtain $\cos(\theta) = \frac{|\overline{AE}|}{|\overline{AD}|}$. Since the ratios of common sides of similar triangles must agree, we have $\cos(\theta) = \frac{|\overline{AC}|}{|\overline{AB}|} = \frac{|\overline{AE}|}{|\overline{AD}|}$,

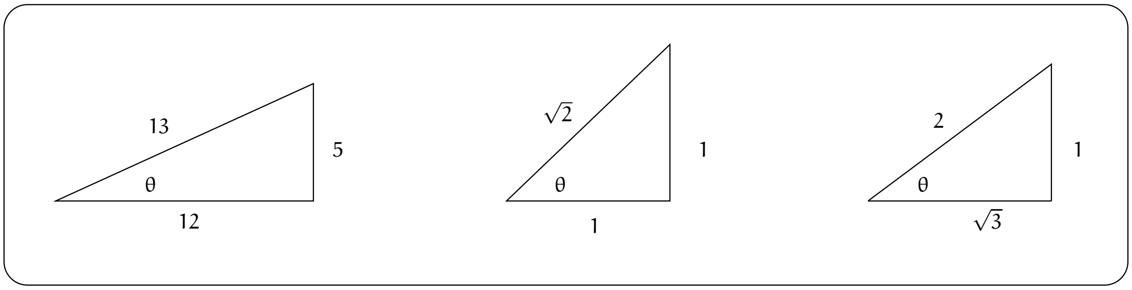


Figure 17.4: Computing trigonometric ratios for selected right triangles.

which is what we wanted to be true. The same argument can be used to show that $\sin(\theta)$ and $\tan(\theta)$ can be computed using *any* right triangle with acute angle θ.

Except for some "rigged" right triangles, it is not easy to calculate the trigonometric ratios. Before the 1970's, approximate values of $\sin(\theta)$, $\cos(\theta)$, $\tan(\theta)$ were listed in long tables or calculated using a slide rule. Today, a scientific calculator saves the day on these computations. Most scientific calculators will give an approximation for the values of the trigonometric ratios. However, it is good to keep in mind we can compute the EXACT values of the trigonometric ratios when $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ radians or, equivalently, when $\theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$.

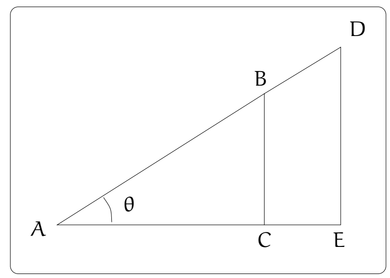


Figure 17.5: Applying trigonometric ratios to any right triangle.

Angle θ		**Trigonometric Ratio**		
Deg	**Rad**	**sin**(θ)	**cos**(θ)	**tan**(θ)
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	Undefined

Table 17.1: Exact Trigonometric Ratios

Some people make a big deal of “approximate” vs. “exact” answers; we won’t worry about it here, unless we are specifically asked for an exact answer. However, here is something we will make a big deal about:

When computing values of $\cos(\theta)$, $\sin(\theta)$, and $\tan(\theta)$ on your calculator, make sure you are using the correct “angle mode” when entering θ; i.e. “degrees” or “radians”.

For example, if $\theta = 1°$, then $\cos(1°) = 0.9998$, $\sin(1°) = 0.0175$, and $\tan(1°) = 0.0175$. In contrast, if $\theta = 1$ radians, then $\cos(1) = 0.5403$, $\sin(1) = 0.8415$, and $\tan(1) = 1.5574$.

17.3 Applications

When confronted with a situation involving a right triangle where the measure of one acute angle θ and one side are known, we can solve for the remaining sides using the appropriate trigonometric ratios. Here is the key picture to keep in mind:

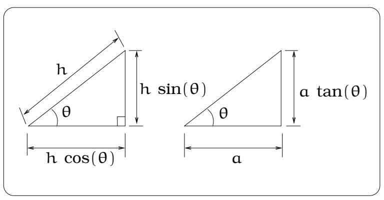


Figure 17.6: What do these ratios mean?

Important Facts 17.3.1 (Trigonometric ratios). *Given a right triangle, the trigonometric ratios relate the lengths of the sides as shown in Figure 17.6.*

Example 17.3.2. *To measure the distance across a river for a new bridge, surveyors placed poles at locations A, B and C. The length $|AB| = 100$ feet and the measure of the angle $\angle ABC$ is $31°18'$. Find the distance to span the river. If the measurement of the angle $\angle ABC$ is only accurate within $\pm 2'$, find the possible error in $|\overline{AC}|$.*

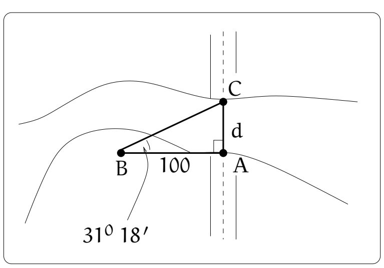


Figure 17.7: The distance spanning a river.

Solution. The trigonometric ratio relating these two sides would be the tangent and we can convert θ into decimal form, arriving at:

$$\tan(31°18') = \tan(31.3°) = \frac{|\overline{AC}|}{|\overline{BA}|} = \frac{d}{100}$$

$$\text{therefore} \quad d = 60.8 \,\text{feet}.$$

This tells us that the bridge needs to span a gap of 60.8 feet. If the measurement of the angle was in error by $+2'$, then $\tan(31°20') = \tan(31.3333°) = 0.6088$ and the span is 60.88 ft. On the other hand, if the measurement of the angle was in error by $-2'$, then $\tan(31°16') = \tan(31.2667°) = 0.6072$ and the span is 60.72 ft. $\square$

Example 17.3.3. *A plane is flying 2000 feet above sea level toward a mountain. The pilot observes the top of the mountain to be* $18°$ *above the horizontal, then immediately flies the plane at an angle of* $20°$ *above horizontal. The airspeed of the plane is 100 mph. After 5 minutes, the plane is directly above the top of the mountain. How high is the plane above the top of the mountain (when it passes over)? What is the height of the mountain?*

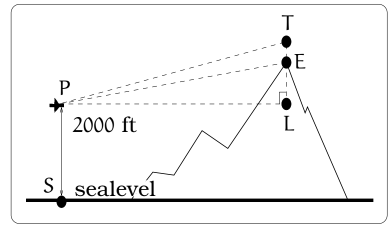


Figure 17.8: Flying toward a mountain.

Solution. We can compute the hypotenuse of $\triangle LPT$ by using the speed and time information about the plane:

$$|\overline{PT}| = (100\,\text{mph})(5\,\text{minutes})(1\,\text{hour}/60\,\text{minutes}) = \frac{25}{3}\,\text{miles.}$$

The definitions of the trigonometric ratios show:

$$|\overline{TL}| = \frac{25}{3}\sin(20°) = 2.850\,\text{miles, and}$$
$$|\overline{PL}| = \frac{25}{3}\cos(20°) = 7.831\,\text{miles.}$$

With this data, we can now find $|\overline{EL}|$:

$$|\overline{EL}| = |\overline{PL}|\tan(18°) = 2.544\,\text{miles.}$$

The height of the plane above the peak is $|\overline{TE}| = |\overline{TL}| - |\overline{EL}| = 2.850 - 2.544 = 0.306\,\text{miles} = 1{,}616\,\text{feet}$. The elevation of the peak above sea level is given by: Peak elevation $=$ plane altitude $+ |\overline{EL}| = |\overline{SP}| + |\overline{EL}| = 2{,}000 + (2.544)(5{,}280) = 15{,}432\,\text{feet}$. □

Example 17.3.4. *A Forest Service helicopter needs to determine the width of a deep canyon. While hovering, they measure the angle* $\gamma = 48°$ *at position* B *(see picture), then descend 400 feet to position* A *and make two measurements of* $\alpha = 13°$ *(the measure of* $\angle EAD$*),* $\beta = 53°$ *(the measure of* $\angle CAD$*). Determine the width of the canyon to the nearest foot.*

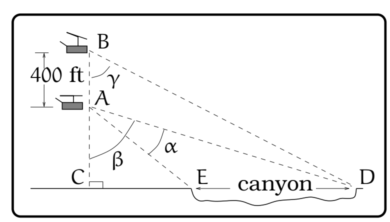


Figure 17.9: Finding the width of a canyon.

Solution. We will need to exploit three right triangles in the picture: $\triangle BCD$, $\triangle ACD$, and $\triangle ACE$. Our goal is to compute $|\overline{ED}| = |\overline{CD}| - |\overline{CE}|$, which suggests more than one right triangle will come into play.

The first step is to use $\triangle BCD$ and $\triangle ACD$ to obtain a system of two equations and two unknowns involving some of the side lengths; we will then solve the system. From the definitions of the trigonometric ratios,

$$\begin{aligned} |\overline{CD}| &= (400 + |\overline{AC}|)\tan(48^\circ) \\ |\overline{CD}| &= |\overline{AC}|\tan(53^\circ). \end{aligned}$$

Plugging the second equation into the first and rearranging we get

$$|\overline{AC}| = \frac{400\tan(48^\circ)}{\tan(53^\circ) - \tan(48^\circ)} = 2{,}053\,\text{feet}.$$

Plugging this back into the second equation of the system gives

$$|\overline{CD}| = (2053)\tan(53^\circ) = 2724\,\text{feet}.$$

The next step is to relate $\triangle ACD$ and $\triangle ACE$, which can now be done in an effective way using the calculations above. Notice that the measure of $\angle CAE$ is $\beta - \alpha = 40^\circ$. We have

$$|\overline{CE}| = |\overline{AC}|\tan(40^\circ) = (2053)\tan(40^\circ) = 1{,}723\,\text{feet}.$$

As noted above, $|\overline{ED}| = |\overline{CD}| - |\overline{CE}| = 2{,}724 - 1{,}723 = 1{,}001$ feet is the width of the canyon. □

17.4 Circular Functions

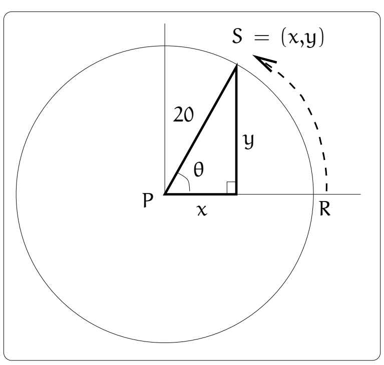


Figure 17.10: Cosmo on a circular path.

If *Cosmo* is located somewhere in the first quadrant of Figure 17.1, represented by the location S, we can use the trigonometric ratios to describe his coordinates. Impose the indicated xy-coordinate system with origin at P and extract the pictured right triangle with vertices at P and S. The radius is 20 ft. and applying Fact 17.3.1 gives

$$S = (x, y) = (20\cos(\theta), 20\sin(\theta)).$$

Unfortunately, we run into a snag if we allow *Cosmo* to wander into the second, third or fourth quadrant, since then the angle θ is no longer acute.

17.4.1 Are the trigonometric ratios functions?

Recall that $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ are defined for acute angles θ inside a right triangle. We would like to say that these three equations actually define functions where the variable is an angle θ. Having said this, it is natural to ask if these three equations can be extended to be defined for ANY angle θ. For example, we need to explain how $\sin\left(\frac{2\pi}{3}\right)$ is defined.

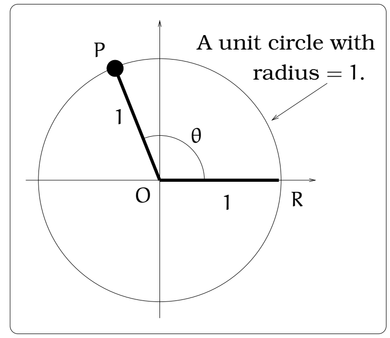


Figure 17.11: Coordinates of points on the unit circle.

To start, we begin with the unit circle pictured in the xy-coordinate system. Let $\theta = \angle ROP$ be the angle in standard central position shown in Figure 17.11. If θ is positive (resp. negative), we adopt the convention that θ is swept out by counterclockwise (resp. clockwise) rotation of the initial side $\overline{OR}$. The objective is to find the coordinates of the point P in this figure. Notice that each coordinate of P (the x-coordinate and the y-coordinate) will depend on the given angle θ. For this reason, we need to introduce two new functions involving the variable θ.

Definition 17.4.1. *Let θ be an angle in standard central position inside the unit circle, as in Figure 17.11. This angle determines a point P on the unit circle. Define two new functions, $\cos(\theta)$ and $\sin(\theta)$, on the domain of all θ values as follows:*

$$\cos(\theta) \stackrel{def}{=} \text{horizontal } x\text{-coordinate of } P \text{ on unit circle}$$
$$\sin(\theta) \stackrel{def}{=} \text{vertical } y\text{-coordinate of } P \text{ on unit circle.}$$

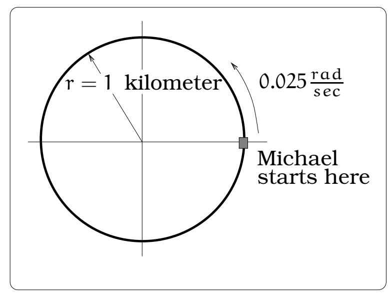


Figure 17.12: A circular driving track.

We refer to $\sin(\theta)$ and $\cos(\theta)$ as the *basic circular functions*. Keep in mind that these functions have variables which are angles (either in degree or radian measure). These functions will be on your calculator. Again, BE CAREFUL to check the angle mode setting on your calculator ("degrees" or "radians") before doing a calculation.

Example 17.4.2. *Michael is test driving a vehicle counterclockwise around a desert test track which is circular of radius 1 kilometer. He starts at the location pictured, traveling $0.025\,\frac{rad}{sec}$. Impose coordinates as pictured. Where is Michael located (in xy-coordinates) after 18 seconds?*

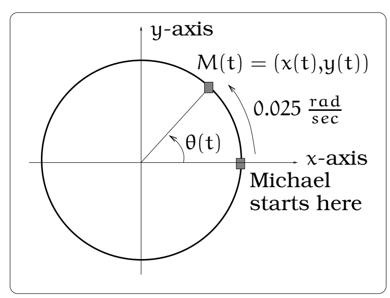


Figure 17.13: Modeling Michael's location.

Solution. Let $M(t)$ be the point on the circle of motion representing Michael's location after t seconds and $\theta(t)$ the angle swept out the by Michael after t seconds. Since we are given the angular speed, we get

$$\theta(t) = 0.025t \text{ radians.}$$

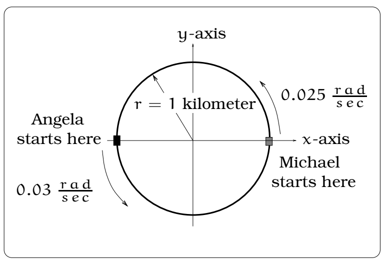
Since the angle $\theta(t)$ is in central standard position, we get

$$M(t) = (\cos(\theta(t)), \sin(\theta(t))) = (\cos(0.025t), \sin(0.025t)).$$

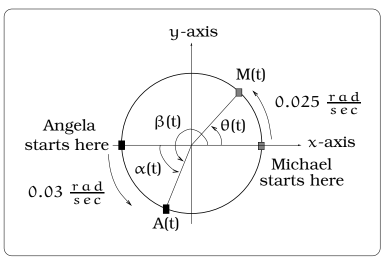
So, after 18 seconds Michael's location will be $M(18) = (0.9004, 0.4350)$. $\square$

iii CAUTION !!!

Interpreting the coordinates of the point $P = (\mathbf{cos}(\theta), \mathbf{sin}(\theta))$ in Figure 17.11 only works if the angle θ is viewed in central standard position. You must do some additional work if the angle is placed in a different position; see the next Example.



(a) Angela and Michael on the same test track.



(b) Modeling the motion of Angela and Michael.

Figure 17.14: Visualizing motion on a circular track.

Example 17.4.3. *Both Angela and Michael are test driving vehicles counterclockwise around a desert test track which is circular of radius 1 kilometer. They start at the locations shown in Figure 17.14(a). Michael is traveling 0.025 rad/sec and Angela is traveling 0.03 rad/sec. Impose coordinates as pictured. Where are the drivers located (in xy-coordinates) after 18 seconds?*

Solution. Let $M(t)$ be the point on the circle of motion representing Michael's location after t seconds. Likewise, let $A(t)$ be the point on the circle of motion representing Angela's location after t seconds. Let $\theta(t)$ be the angle swept out the by Michael and $\alpha(t)$ the angle swept out by Angela after t seconds.

Since we are given the angular speeds, we get

$$\begin{aligned} \theta(t) &= 0.025t \text{ radians, and} \\ \alpha(t) &= 0.03t \text{ radians.} \end{aligned}$$

From the previous Example 17.4.2,

$$\begin{aligned} M(t) &= (\cos(0.025t), \sin(0.025t)), \quad \text{and} \\ M(18) &= (0.9004, 0.4350). \end{aligned}$$

Angela's angle $\alpha(t)$ is NOT in central standard position, so we must observe that $\alpha(t) + \pi = \beta(t)$, where $\beta(t)$ is in central standard position: See Figure 17.14(b). We conclude that

$$\begin{aligned} A(t) &= (\cos(\beta(t)), \sin(\beta(t))) \\ &= (\cos(\pi + 0.03t), \sin(\pi + 0.03t)). \end{aligned}$$

So, after 18 seconds Angela's location will be $A(18) = (-0.8577, -0.5141)$. $\square$

17.4.2 Relating circular functions and right triangles

If the point P on the unit circle is located in the first quadrant, then we can compute $\cos(\theta)$ and $\sin(\theta)$ using trigonometric ratios. In general, it's useful to relate right triangles, the unit circle and the circular functions. To describe this connection, given θ we place it in central standard position in the unit circle, where $\angle ROP = \theta$. Draw a line through P perpendicular to the x-axis, obtaining an inscribed right triangle. Such a right triangle has hypotenuse of length 1, vertical side of length labeled b and horizontal side of length labeled a. There are four cases: See Figure 17.16.

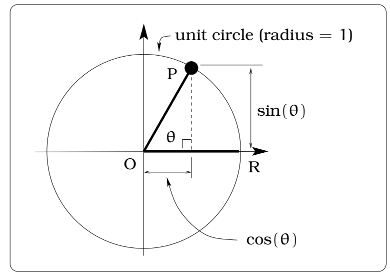


Figure 17.15: The point P in the first quadrant.

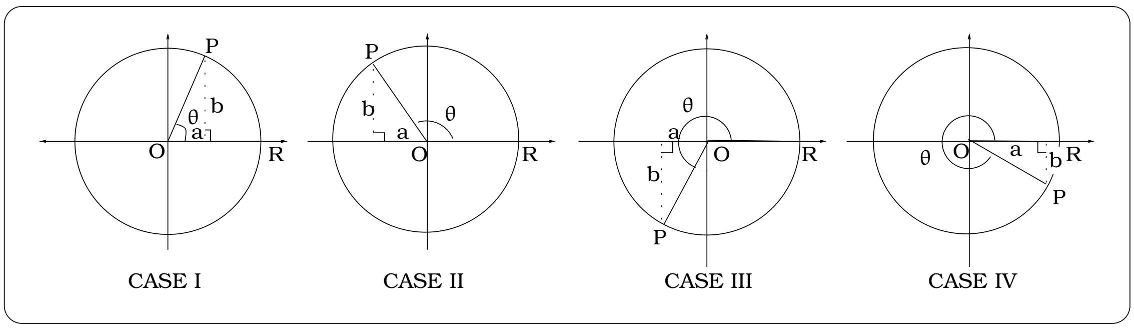


Figure 17.16: Possible positions of θ on the unit circle.

Case I has already been discussed, arriving at $\cos(\theta) = a$ and $\sin(\theta) = b$. In Case II , we can interpret $\cos(\theta) = -a, \sin(\theta) = b$. We can reason similarly in the other Cases III and IV, using Figure 17.16, and we arrive at this conclusion:

Important Facts 17.4.4 (Circular functions and triangles)**.** *View* θ *as in Figure 17.16 and form the pictured inscribed right triangles. Then we can interpret* $\cos(\theta)$ *and* $\sin(\theta)$ *in terms of these right triangles as follows:*

$$\begin{aligned}
\textit{Case I}: &\quad \cos(\theta) = a, &\quad \sin(\theta) = b \\
\textit{Case II}: &\quad \cos(\theta) = -a, &\quad \sin(\theta) = b \\
\textit{Case III}: &\quad \cos(\theta) = -a, &\quad \sin(\theta) = -b \\
\textit{Case IV}: &\quad \cos(\theta) = a, &\quad \sin(\theta) = -b
\end{aligned}$$

17.5 What About Other Circles?

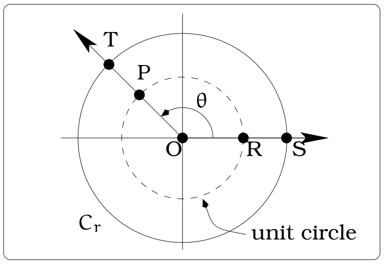


Figure 17.17: Points on other circles.

What happens if we begin with a circle C_r with radius r (possibly different than 1) and want to compute the coordinates of points on this circle?

The circular functions can be used to answer this more general question. Picture our circle C_r centered at the origin in the same picture with unit circle C_1 and the angle θ in standard central position for each circle. As pictured, we can view $\theta = \angle ROP = \angle SOT$. If $P = (x,y)$ is our point on the unit circle corresponding to the angle θ, then the calculation below shows how to compute coordinates on general circles:

$$\begin{aligned} P &= (x,y) \\ &= (\cos(\theta), \sin(\theta)) \in C_1 \quad &\Leftrightarrow \quad & x^2 + y^2 = 1 \\ & &\Leftrightarrow \quad & r^2x^2 + r^2y^2 = r^2 \\ & &\Leftrightarrow \quad & (rx)^2 + (ry)^2 = r^2 \\ & &\Leftrightarrow \quad & T = (rx, ry) \\ & & & = (r\cos(\theta), r\sin(\theta)) \in C_r. \end{aligned}$$

Important Fact 17.5.1. *Let C_r be a circle of radius r centered at the origin and $\theta = \angle SOT$ an angle in standard central position for this circle, as in Figure 17.17. Then the coordinates of $T = (r\cos(\theta), r\sin(\theta))$.*

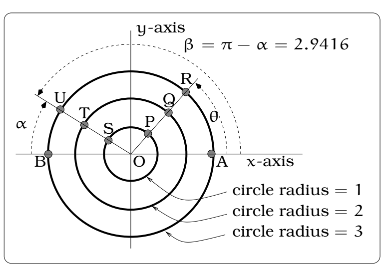


Figure 17.18: Coordinates of points on circles.

Examples 17.5.2. *Consider the picture below, with $\theta = 0.8$ radians and $\alpha = 0.2$ radians. What are the coordinates of the labeled points?*

Solution. The angle θ is in standard central position; α is a central angle, but it is not in standard position. Notice, $\beta = \pi - \alpha = 2.9416$ is an angle in standard central position which locates the same points U, T, S as the angle α. Applying Definition 17.4.1 on page 227:

$$\begin{aligned} P &= (\cos(0.8), \sin(0.8)) &&= (0.6967, 0.7174) \\ Q &= (2\cos(0.8), 2\sin(0.8)) &&= (1.3934, 1.4347) \\ R &= (3\cos(0.8), 3\sin(0.8)) &&= (2.0901, 2.1521) \\ S &= (\cos(2.9416), \sin(2.9416)) &&= (-0.9801, 0.1987) \\ T &= (2\cos(2.9416), 2\sin(2.9416)) &&= (-1.9602, 0.3973) \\ U &= (3\cos(2.9416), 3\sin(2.9416)) &&= (-2.9403, 0.5961). \end{aligned}$$

□

Example 17.5.3. *Suppose Cosmo begins at the position* R *in the figure, walking around the circle of radius 20 feet with an angular speed of* $\frac{4}{5}$ *RPM counterclockwise. After 3 minutes have elapsed, describe Cosmo's precise location.*

Solution. Cosmo has traveled $3\frac{4}{5} = \frac{12}{5}$ revolutions. If θ is the angle traveled after 3 minutes, $\theta = \left(\frac{12}{5}\text{rev}\right)\left(2\pi\,\frac{\text{radians}}{\text{rev}}\right) = \frac{24\pi}{5}$ radians $= 15.08$ radians. By (15.5.1), we have $x = 20\cos\left(\frac{24\pi}{5}\text{rad}\right) = -16.18$ feet and $y = 20\sin\left(\frac{24\pi}{5}\text{rad}\right) = 11.76$ feet. Conclude that Cosmo is located at the point $S = (-16.18, 11.76)$. Using (15.1), $\theta = 864^\circ = 2(360^\circ) + 144^\circ$; this means that Cosmo walks counterclockwise around the circle two complete revolutions, plus 144°. □

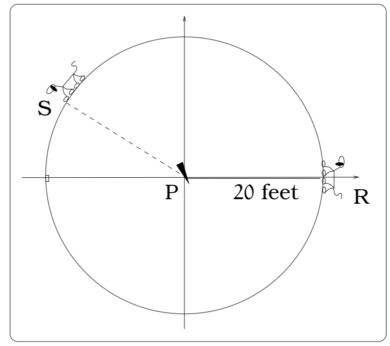


Figure 17.19: Where is Cosmo after 3 minutes?

17.6 Other Basic Circular Function

Given any angle θ, our constructions offer a concrete link between the cosine and sine functions and right triangles inscribed inside the unit circle: See Figure 17.20.

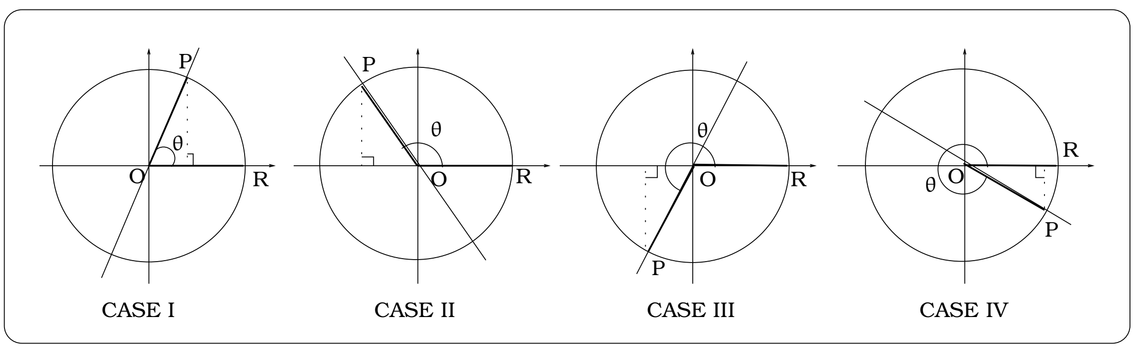


Figure 17.20: Computing the slope of a line using the function $\tan(\theta)$.

The slope of the hypotenuse of these inscribed triangles is just the slope of the line through $\overline{OP}$. Since $P = (\cos(\theta), \sin(\theta))$ and $O = (0, 0)$:

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{\sin(\theta)}{\cos(\theta)};$$

this would be valid as long as $\cos(\theta) \neq 0$. This calculation motivates a new circular function called the *tangent of* θ by the rule

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \quad \text{provided } \cos(\theta) \neq 0.$$

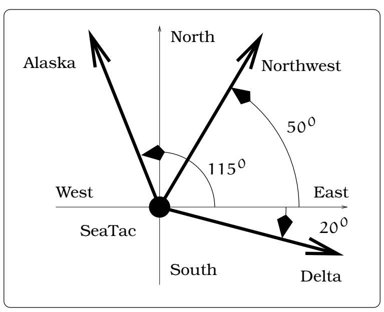
The only time $\cos(\theta) = 0$ is when the corresponding point P on the unit circle has x-coordinate 0. But, this only happens at the positions $(0, 1)$ and $(0, -1)$ on the unit circle, corresponding to angles of the form $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \cdots$. These are the cases when the inscribed right triangle would "degenerate" to having zero width and the line segment $\overline{OP}$ becomes vertical. In summary, we then have this general idea to keep in mind:

Important Fact 17.6.1. ***The slope of a line*** $= \tan(\theta)$, ***where*** θ ***is the angle the line makes with the x-axis (or any other horizontal line)***

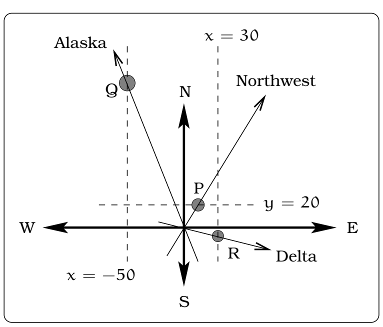
Three other commonly used circular functions come up from time to time. The *cotangent* function $y = \cot(\theta)$, the *secant* function $y = \sec(\theta)$ and the *cosecant* function $y = \csc(\theta)$ are defined by the formulas:

$$\sec(\theta) \stackrel{\text{def}}{=} \frac{1}{\cos(\theta)}, \qquad \csc(\theta) \stackrel{\text{def}}{=} \frac{1}{\sin(\theta)}, \qquad \cot(\theta) \stackrel{\text{def}}{=} \frac{1}{\tan(\theta)}.$$

Just as with the tangent function, one needs to worry about the values of θ for which these functions are undefined (due to division by zero). We will not need these functions in this text.



(a) The flight paths of three airplanes.



(b) Modeling the paths of each flight.

Figure 17.21: Visualizing and modeling departing airplanes.

Example 17.6.2. *Three airplanes depart SeaTac Airport. A NorthWest flight is heading in a direction* $50°$ *counterclockwise from East, an Alaska flight is heading* $115°$ *counterclockwise from East and a Delta flight is heading* $20°$ *clockwise from East. Find the location of the Northwest flight when it is 20 miles North of SeaTac. Find the location of the Alaska flight when it is 50 miles West of SeaTac. Find the location of the Delta flight when it is 30 miles East of SeaTac.*

Solution. We impose a coordinate system in Figure 17.21(a), where "East" (resp. "North") points along the positive x-axis (resp. positive y-axis). To solve the problem, we will find the equation of the three lines representing the flight paths, then determine where they intersect the appropriate horizontal or vertical line. The Northwest and Alaska directions of flight are angles in standard central position; the Delta flight direction will be $-20°$. We can imagine right triangles with their hypotenuses along the directions of flight, then using the tangent function, we have these three immediate conclusions:

$$\begin{aligned} \text{slope NW line} = \tan(50°) &= 1.19, \\ \text{slope Alaska line} = \tan(115°) &= -2.14, \text{ and} \\ \text{slope Delta line} = \tan(-20°) &= -0.364. \end{aligned}$$

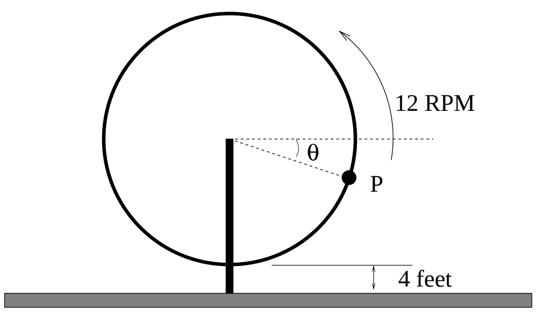
All three flight paths pass through the origin $(0,0)$ of our coordinate system, so the equations of the lines through the flight paths will be:

$$\begin{aligned} \text{NW flight}: &\quad y = 1.19x, \\ \text{Alaska flight}: &\quad y = -2.14x, \\ \text{Delta flight}: &\quad y = -0.364x. \end{aligned}$$

The Northwest flight is 20 miles North of SeaTac when $y = 20$; plugging into the equation of the line of flight gives $20 = 1.19x$, so $x = 16.81$ and the plane location will be $P = (16.81, 20)$. Similarly, the Alaska flight is 50 miles West of SeaTac when $x = -50$; plugging into the equation of the line of flight gives $y = (-2.14)(-50) = 107$ and the plane location will be $Q = (-50, 107)$. Finally, check that the Delta flight is at $R = (30, -10.92)$ when it is 30 miles East of SeaTac. $\square$

17.7 Exercises

Problem 17.1. John has been hired to design an exciting carnival ride. Tiff, the carnival owner, has decided to create the world's greatest ferris wheel. Tiff isn't into math; she simply has a vision and has told John these constraints on her dream: (i) the wheel should rotate counterclockwise with an angular speed of 12 RPM; (ii) the linear speed of a rider should be 200 mph; (iii) the lowest point on the ride should be 4 feet above the level ground. Recall, we worked on this in Exercise 16.5.



(a) Impose a coordinate system and find the coordinates $T(t) = (x(t),y(t))$ of Tiff at time t seconds after she starts the ride.

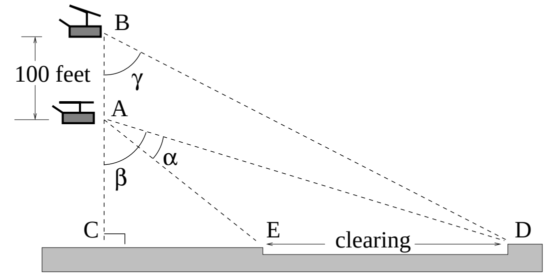
(b) Tiff becomes a human missile after 6 seconds on the ride. Find Tiff's coordinates the instant she becomes a human missile.

(c) Find the equation of the tangential line along which Tiff travels the instant she becomes a human missile. Sketch a picture indicating this line and her initial direction of motion along it when the seat detaches.

Problem 17.2. (a) Find the equation of a line passing through the point (-1,2) and making an angle of 13° with the x-axis. (Note: There are two answers; find them both.)

(b) Find the equation of a line making an angle of 8° with the y-axis and passing through the point $(1,1)$. (Note: There are two answers; find them both.)

Problem 17.3. The crew of a helicopter needs to land temporarily in a forest and spot a flat horizontal piece of ground (a clearing in the forest) as a potential landing site, but are uncertain whether it is wide enough. They make two measurements from A (see picture) finding $\alpha = 25°$ and $\beta = 54°$. They rise vertically 100 feet to B and measure $\gamma = 47°$. Determine the width of the clearing to the nearest foot.



Problem 17.4. Marla is running clockwise around a circular track. She runs at a constant speed of 3 meters per second. She takes 46 seconds to complete one lap of the track. From her starting point, it takes her 12 seconds to reach the northermost point of the track.

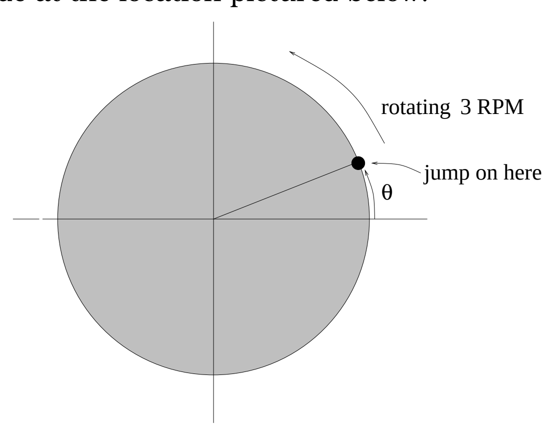
Impose a coordinate system with the center of the track at the origin, and the northernmost point on the positive y-axis.

(a) Give Marla's coordinates at her starting point.

(b) Give Marla's coordinates when she has been running for 10 seconds.

(c) Give Marla's coordinates when she has been running for 901.3 seconds.

Problem 17.5. A merry-go-round is rotating at the constant angular speed of 3 RPM counterclockwise. The platform of this ride is a circular disc of radius 24 feet. You jump onto the ride at the location pictured below.



(a) If $\theta = 34^o$, then what are your xy-coordinates after 4 minutes?

(b) If $\theta = 20^o$, then what are your xy-coordinates after 45 minutes?

(c) If $\theta = -14^o$, then what are your xy-coordinates after 6 seconds? Draw an accurate picture of the situation.

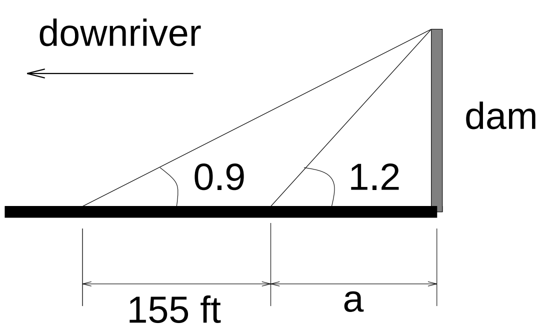
(d) If $\theta = -2.1$ rad, then what are your xy-coordinates after 2 hours and 7 seconds? Draw an accurate picture of the situation.

(e) If $\theta = 2.1$ rad, then what are your xy-coordinates after 5 seconds? Draw an accurate picture of the situation.

Problem 17.6. Shirley is on a ferris wheel which spins at the rate of 3.2 revolutions per minute. The wheel has a radius of 45 feet, and the center of the wheel is 59 feet above the ground. After the wheel starts moving, Shirley takes 16 seconds to reach the top of the wheel.

How high above the ground is she when the wheel has been moving for 9 minutes?

Problem 17.7. The top of the *Boulder Dam* has an angle of elevation of 1.2 radians from a point on the Colorado River. Measuring the angle of elevation to the top of the dam from a point 155 feet farther down river is 0.9 radians; assume the two angle measurements are taken at the same elevation above sea level. How high is the dam?

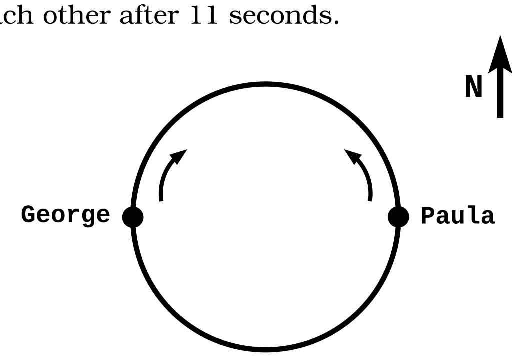


Problem 17.8. A radio station obtains a permit to increase the height of their radio tower on Queen Anne Hill by no more than 100 feet. You are the head of the Queen Anne Community Group and one of your members asks you to make sure that the radio station does not exceed the limits of the permit. After finding a relatively flat area nearby the tower (not necessarily the same altitude as the bottom of the tower), and standing some unknown distance away from the tower, you make three measurements all at the same height above sea level. You observe that the top of the old tower makes an angle of 39° above level. You move 110 feet away from the original measurement and observe that the old top of the tower now makes an angle of 34° above level. Finally, after the new construction is complete, you observe that the new top of the tower, from the same point as the second measurement was made, makes an angle of 40° above the horizontal. All three measurements are made at the same height above sea level and are in line with the tower. Find the height of the addition to the tower, to the nearest foot.

Problem 17.9. Charlie and Alexandra are running around a circular track with radius 60 meters. Charlie started at the westernmost point of the track, and, at the same time, Alexandra started at the northernmost point. They both run counterclockwise. Alexandra runs at 4 meters per second, and will take exactly 2 minutes to catch up to Charlie.

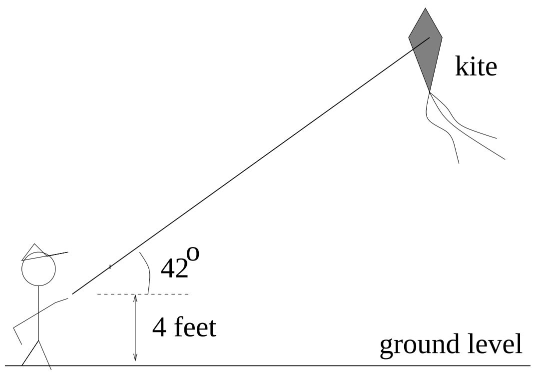
Impose a coordinate system, and give the x- and y-coordinates of Charlie after one minute of running.

Problem 17.10. George and Paula are running around a circular track. George starts at the westernmost point of the track, and Paula starts at the easternmost point. The illustration below shows their starting positions and running directions. They start running toward each other at constant speeds. George runs at 9 feet per second. Paula takes 50 seconds to run a lap of the track. George and Paula pass each other after 11 seconds.



After running for 3 minutes, how far east of his starting point is George?

Problem 17.11. A kite is attached to 300 feet of string, which makes a 42 degree angle with the level ground. The kite pilot is holding the string 4 feet above the ground.



(a) How high above the ground is the kite?

(b) Suppose that power lines are located 250 feet in front of the kite flyer. Is any portion of the kite or string over the power lines?

Problem 17.12. In the pictures below, a bug has landed on the rim of a jelly jar and is moving around the rim. The location where the bug initially lands is described and its angular speed is given. Impose a coordinate system with the origin at the center of the circle of motion. In each of the cases, answer these questions:

(a) Find an angle θ_0 in standard central position that gives the bugs initial location. (In some cases, this is the angle given in the picture; in other cases, you will need to do something.)

(b) The location angle of the bug at time t is given by the formula $\theta(t) = \theta_0 + \omega t$. Plug in the values for θ_0 and ω to explicitly obtain a formula for $\theta(t)$.

(c) Find the coordinates of the bug at time t.

(d) What are the coordinates of the bug after 1 second? After 0 seconds? After 3 seconds? After 22 seconds?