Section 3.1

Introduction to Derivatives Rules

Introduction

Objective 3.1.1 Use the Power Rule to compute the derivative of a function.

Objective 3.1.2 Use the Constant Rule to compute the derivative of a function.

Objective 3.1.3 Compute the derivative of a polynomial.

Objective 3.1.4 Find where the tangent lines of a polynomial are horizontal .

Objective 3.1.5 Given the equation of a polynomial, use the rules of differentiation to determine where the function is increasing, decreasing, concave up, concave down, or has a given slope.

Objective 3.1.6 Find the derivative of a function of the form

$$y = \frac{p(x)}{q(x)}$$

where p and q are polynomials with the degree of q less than or equal to the degree of p, by dividing then using the power rule.

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Derivative of a constant

The graph of a constant function, $f(x) = c$, is a horizontal line; therefore, the derivative equals zero. The slope of all horizontal lines equals zero.

Theorem: If c is any real number then $\dfrac{\mathrm{d}}{\mathrm{d}x}(c) = 0$

Derivative of a non-vertical line.

The function $f(x) = cx$, is a line with slope equal to c for each x. Therefore, the derivative of the function $f(x) = cx$, is c.

Theorem: If c is any real number then $\dfrac{\mathrm{d}}{\mathrm{d}x}(cx) = c$

Derivative of a power function.

Recall: A power function is a function of the form $f(x) = ax^n$, where n is a natural number.

Let us first consider the power function $f(x) = x^4$. By using the definition of the derivative to compute $f'(x)$ we see the following.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x}^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - \cancel{x}^4}{h} \\
&= \lim_{h\to 0} \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3)}{h} (h \neq 0) \\
&= \lim_{h\to 0} \frac{\cancel{h}(4x^3 + 6x^2h + 4xh^2 + h^3)}{\cancel{h}} \\
&= \lim_{h\to 0} (4x^3 + 6x^2h + 4xh^2 + h^3) \\
&= 4x^3
\end{aligned}$$

We can now see that,

$$\frac{d(x^4)}{dx} = 4 \cdot x^{4-1} = 4x^3$$

Theorem (Power Rule): Given the power function $f(x) = x^n$ where n is a natural number,

$$\frac{d(x^n)}{dx} = n \cdot x^{n-1}$$

Proof:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[x^n] &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \quad \textit{(Note: We will use the binomial expansion to multiply)} \\
&= \lim_{h\to 0} \frac{[x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n] - x^n}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\
&= \lim_{h\to 0} [nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}] \\
&= nx^{n-1} + 0 + 0 + \cdots + 0 + 0 \\
&= nx^{n-1}
\end{aligned}$$

Theorem (Power Rule for real number powers): Given the function $f(x) = x^r$ where r is a real number,

$$\frac{d(x^r)}{dx} = r \cdot x^{r-1}$$

We will leave this proof for later.

Example 3.1.1

Let's look at using th empower rule when the exponent is a negative number. You can verify that it is true by using the limit definition.

If $f(x) = \frac{1}{x}$, find $f'(x)$.

Answer:

$$f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = -1(x^{-2}) = -\frac{1}{x^2}$$

Derivative of a constant times a function

Theorem: (Constant Multiple Rule) Let c be a real number and $y = f(x)$ a function of x. If f is differentiable at x and $y = c \cdot f(x)$ then

$$\frac{dy}{dx} = \frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx} f(x).$$

Example 3.1.2

Find the derivatives of the following:

a.) $y = 11x^5$
Answer: $y' = 11(5x^4) = 55x^4$

b.) $y = x^3$
Answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$

c.) $y = x^2$
Answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$

d.) $y = \sqrt{x}$
Answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

e.) $f(x) = 4x^{-6}$
Answer: $f'(x) = 4(-6x^{-7}) = \frac{-24}{x^7}$

Each of the above examples (especially parts b, c, and d) can be easily verified using the limit definition of the derivative. For additional practice, try to verify them on your own.

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Derivatives of the sum and difference of functions

Theorem: Sum/Difference Rule The derivative of the sum (respectively difference) of functions is the sum (respectively difference) of the derivatives:

$$\text{If } y = f(x) \pm g(x), \text{ then } y' = f'(x) \pm g'(x).$$

The proof is left as an exercise.

EXAMPLE 3.1.3

Let $f(x) = x^5 + 17x^3 + \frac{1}{3}\sqrt[3]{x} - \dfrac{5}{x^2} + 4$. Find $f'(x)$.

Answer: We can look at this in parts.

function	derivative
x^5	$5x^4$
$17x^3$	$117x^2$
$\frac{1}{3}\sqrt[3]{x}$	$\frac{1}{9}x^{-2/3}$
$-\dfrac{5}{x^2}$	$-\dfrac{10}{x^3}$
4	0

Therefore, we add the pieces together and get

$$f'(x) = 5x^4 + 117x^2 + \frac{1}{9}x^{-2/3} - \frac{10}{x^3}$$

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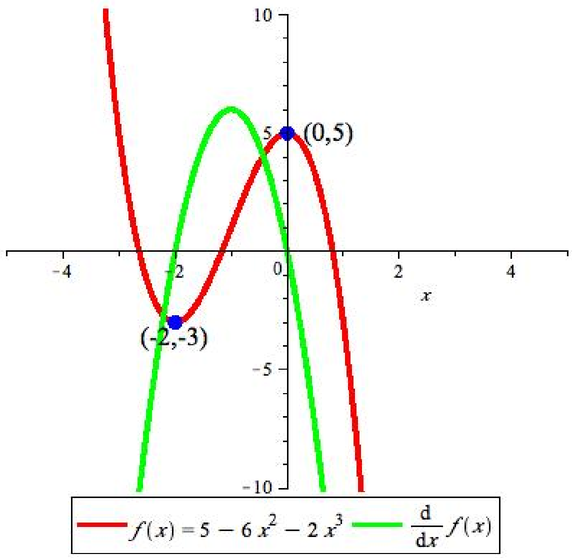
EXAMPLE 3.1.4

Let $f(x) = 5 - 6x^2 - 2x^3$. Find the point(s) where the tangent lines are HORIZONTAL.

Answer:
Note $m_{tan} = f'(x) = -12x - 6x^2 = -6x(2+x)$, so $f'(x) = 0$ when $x = 0$ and $x = -2$. Therefore, the points where the tangent lines are horizontal are

$$(0, f(0)) = (0, 5) \text{ and } (-2, f(-2)) = (-2, -3).$$



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Section 3.2

Derivatives of Exponential Functions

Introduction

Objective 3.2.1 Differentiate a function of the form $f(x) = a^x$.

Objective 3.2.2 Determine the derivative of $f(x) = e^x$.

Objective 3.2.3 Given the equation of a function with exponential terms, use the rules of differentiation to determine where the function is increasing, decreasing, concave up, concave down, or has a given slope.

Objective 3.2.4 Find the equation of a tangent line for a function f that includes a natural exponential function in its definition.

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Let $a > 0$ be a real number and $a \neq 1$. To find the derivative of $f(x) = a^x$ we will start with the limit definition of the derivative.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} && \textit{Definition of Derivative} \\
&= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} && \textit{Rules of Exponents} \\
&= \lim_{h\to 0} a^x \left(\frac{a^h - 1}{h}\right) && \textit{factor } a^x. \\
&= a^x \left[\lim_{h\to 0} \left(\frac{a^h - 1}{h}\right)\right] && \textit{limit rules.}
\end{aligned}$$

It turns out that evaluating the limit rigorously is not such an easy thing to do. We will need a technique from a later section to evaluate the limit for any value of a.

By calculating an approximation to each of the following limits numerically, we see that

$$\text{if } a = 2, \lim_{h\to 0} \frac{2^h - 1}{h} \approx 0.693, \text{ and if} a = 3 \lim_{h\to 0} \frac{3^h - 1}{h} \approx 1.099$$

The limit depends on what the base a is. It follows that there is a number between 2 and 3 where the limit equals 1. It turns out that the irrational number e is that number. Note: we will verify these limits in a later section. For now, e is defined as follows:

Definition: e is the real number such that $\lim_{h\to 0} \left(\frac{e^h - 1}{h}\right) = 1$

We can now find the derivative of the natural exponential function, $f(x) = e^x$ as follows.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x e^h - e^x}{h} \\
&= \lim_{h\to 0} e^x \left(\frac{e^h - 1}{h}\right) \\
&= e^x(1) \\
&= e^x
\end{aligned}$$

Using the limit definition and the definition of e, we get $\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x$. Note: $y = e^x$ is the only exponential function that is its own derivative.

Example 3.2.1

Given the function $y = x^3 + x + 5e^x$. Find where it is increasing.

Answer: Recall, when the function is increasing the graph of the derivative of the function is above the x-axis. Therefore, we will look at the derivative to determine where it is positive. $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 1 + 5e^x$ We will need to solve the inequality, $3x^2 + 1 + 5e^x > 0$ to find where y is increasing. Notice, $3x^2 \geq 0$ for every real number x, which implies that $3x^2 + 1 > 0$ for every real number x. Also, $5e^x > 0$. Thus, $3x^2 + 1 + 5e^x > 0$ for every real number x. Thus, $y = x^3 + x + 5e^x$ is always increasing for all real numbers.

Example 3.2.2

Differentiate the function $y = 4e^x - \dfrac{7}{\sqrt{x}}$.

Answer: First we will rewrite the function so we can use the power rule on the second term.

$$y = 4e^x - 7x^{-\frac{1}{2}}$$

Now we will find the derivative and simplify.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 4e^x - 7(\frac{1}{2})x^{(-\frac{1}{2}-1)} \\
&= 4e^x - \frac{7}{2}x^{-\frac{3}{2}} \\
&= 4e^x - \frac{7}{2x\sqrt{x}}
\end{aligned}$$

Example 3.2.3

Differentiate the function $y = Ax^3 + Be^x$.

Answer: A and B are constants with respect to x, therefore, we treat them the same as we would any real number when finding the derivative.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = A(3)x^{3-1} + Be^x = 3Ax^2 + Be^x$$

EXAMPLE 3.2.4

Find the equation of the tangent line to the graph of the function $f(x) = 5e^x + x^2$ at $x = 0$.

Answer: First we will find the slope of the tangent line by finding the derivative function then evaluate it at $x = 0$.

$$f'(x) = 5e^x + 2x, \text{evaluated at } x = 0 \text{ is } f'(0) = 5e^0 + 2(0) = 5$$

To write the equation of the tangent line, we must first find the point that is on the graph at $x = 0$. $f(0) = 5$, so the point at which you want to find the equation of the tangent line is $(0, 5)$ and the slope of the tangent line is $m_{TAN} = f'(0) = 5$. Using the point and slope we find

$$y - 5 = 5(x - 0) \text{ which simplifies to the equation } y = 5x + 5$$

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Section 3.3

Product and Quotient Rules

Introduction

Objective 3.3.1 Derive and state the Product Rule.

Objective 3.3.2 Use the Product Rule to compute the derivative of a function.

Objective 3.3.3 Derive and state the Quotient Rule.

Objective 3.3.4 Use the Quotient Rule to compute the derivative of a function.

Objective 3.3.5 Differentiate a function that contains arbitrary constants.

Objective 3.3.6 Write the equation of a tangent line of a function that is a product or quotient of two or more functions.

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Our goal in this section is to determine how to differentiate functions that are the product or quotient of other functions.

The Product Rule

Defining a rule that will allow us to find the derivative of the product of functions is, unfortunately, not as straight forward as the finding the sum or difference of of functions. We might guess that the derivative of the product is obtained by multiplying the derivatives of the individual derivatives of the factors, but that is not the case. For example, if we let $f(x) = x$ and $g(x) = x^2$, then $(fg)(x) = x^3$. Then $(fg)'(x) = 3x^2$ by the power rule; however, $f'(x)g'(x) = (1)(x^2) = x^2$. Our initial guess is incorrect. The proper way to differentiate a product is given in the following formula.

Theorem (*Product Rule)* If f and g are differentiable at x, then (fg) is also differentiable at x and

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f(x)\frac{\mathrm{d}}{\mathrm{d}x}[g(x)] + g(x)\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]$$

Proof: We will use the definition of the derivative to show the product rule is true.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} f(x+h)\cdot\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\cdot\frac{f(x+h) - f(x)}{h} \\
&= [\lim_{h\to 0} f(x+h)]\cdot\frac{\mathrm{d}}{\mathrm{d}x}[g(x)] + [\lim_{h\to 0} g(x)]\cdot\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] \\
&= f(x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}[g(x)] + g(x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]
\end{aligned}$$

Note: $f(x+h) \to f(x)$ because f is continuous at x. $g(x) \to g(x)$ as $h \to 0$ because g does not involve h and is therefore treated as a constant.

Now let us see a few examples involving differentiating the product of functions.

Example 3.3.1

Let $f(x) = (x+4)(3x-5)$. Find $f'(x)$.

Answer:
First using the Product Rule:

$$f'(x) = (x+4)\frac{d}{dx}\left[(3x-5)\right] + (3x-5)\frac{d}{dx}\left[(x+4)\right] = (x+4)(3) + (3x-5)(1) = 6x+7.$$

We can check this by expanding and using the power and constant rules:

$$f(x) = (x+4)(3x-5) = 3x^2 + 7x - 20 \Rightarrow f'(x) = 6x+7.$$

Example 3.3.2

Given $f(x) = x^2 \cdot e^x$, find $f'(x)$.

Answer:

$$f'(x) = x^2\frac{d}{dx}[e^x] + e^x\frac{d}{dx}[x^2] = x^2e^x + 2xe^x = xe^x[x+2]$$

Example 3.3.3

Given $F(x) = f(x) \cdot g(x) \cdot h(x)$. Derive the product rule for these three terms.

Answer: Rewrite $F(x)$ as $F(x) = [f(x)g(x)]h(x)$. Then

$$F'(x) = [f(x)g(x)]h'(x) + [f(x)g(x)]'h(x)$$

Noting that

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x),$$

we have

$$F'(x) = f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x)$$

The Quotient Rule

The derivative of a product of functions is not the product of the derivatives. Similarly, the derivative of a quotient of functions is not the quotient of the derivatives.

Theorem (*Quotient Rule)* If f and g are differentiable at x and $g(x) \neq 0$, then $\frac{f}{g}$ is also differentiable at x and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] - f(x)\frac{\mathrm{d}}{\mathrm{d}x}[g(x)]}{[g(x)]^2}$$

Proof: We will use the definition of the derivative to show the quotient rule is true.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] &= \lim_{h\to 0}\left[\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}\right] \\
&= \lim_{h\to 0}\frac{f(x+h)\cdot g(x) - f(x)\cdot g(x+h)}{h\cdot g(x)\cdot g(x+h)} \\
&= \lim_{h\to 0}\frac{f(x+h)\cdot g(x) - f(x)\cdot g(x) - f(x)\cdot g(x+h) + f(x)\cdot g(x)}{h\cdot g(x)\cdot g(x+h)} \\
&= \lim_{h\to 0}\frac{\left[g(x)\cdot\dfrac{f(x+h)-f(x)}{h}\right] - \left[f(x)\cdot\dfrac{g(x+h)-g(x)}{h}\right]}{g(x)\cdot g(x+h)} \\
&= \frac{\left[\lim_{h\to 0} g(x)\cdot\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}\right] - \left[\lim_{h\to 0} f(x)\cdot\lim_{h\to 0}\dfrac{g(x+h)-g(x)}{h}\right]}{\lim_{h\to 0} g(x)\cdot\lim_{h\to 0} g(x+h)} \\
&= \frac{[\lim_{h\to 0} g(x)]\cdot\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] - [\lim_{h\to 0} f(x)]\cdot\frac{\mathrm{d}}{\mathrm{d}x}[g(x)]}{\lim_{h\to 0} g(x)\cdot\lim_{h\to 0} g(x+h)} \\
&= \frac{g(x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] - f(x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}[g(x)]}{g(x)\cdot g(x)} \\
&= \frac{g(x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] - f(x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}[g(x)]}{[g(x)]^2}
\end{aligned}$$

Note: In the third step above, I added and subtracted $f(x)\cdot g(x)$ in the numerator. To understand how I evaluated the limits in the next to the last step, see the comments at the end of the proof for the product rule.

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Now let us see a few examples involving differentiating the quotient of functions and all other rules you have seen thus far.

EXAMPLE 3.3.4

Given $h(x) = \dfrac{x+4}{3x-5}$, use the quotient rule to find $h'(x)$.

Answer: If we think of $h(x)$ as the quotient of the two functions, $f(x) = x + 4$ and $g(x) = 3x - 5$ we see that $h'(x)$ can be computed as follows.

$$\begin{aligned}
h'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] \\
&= \frac{g(x)\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] - f(x)\frac{\mathrm{d}}{\mathrm{d}x}[g(x)]}{[g(x)]^2} \\
&= \frac{[3x-5]\cdot[1] - [x+4]\cdot[3]}{[3x-5]^2} \\
&= \frac{3x-5-[3x+12]}{[3x-5]^2} \\
&= \frac{-17}{(3x-5)^2}
\end{aligned}$$

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Example 3.3.5

In each of the following, A new function is expressed in terms of $f(x)$ and $g(x)$. Given that $f(3) = -1$, $g(3) = 4$, $f'(3) = 2$,$g'(3) = -7$,

a.) if $F(x) = f(x) \cdot g(x)$, find $F'(3)$.

b.) if $H(x) = \dfrac{f(x)}{g(x)}$, find $H'(3)$.

Answer:

a.)

$$\begin{aligned}
F'(3) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[f(3) \cdot g(3)\right] \\
&= f(3) \cdot g'(3) + g(3) \cdot f'(3) \\
&= (-1)(-7) + (4)(2) \\
&= 15
\end{aligned}$$

b.)

$$\begin{aligned}
H'(3) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(3)}{g(3)}\right] \\
&= \frac{g(3) \cdot f'(3) - f(x) \cdot g'(3)}{[g(3)]^2} \\
&= \frac{[(4)(2)] - [(-1)(-7)]}{[4]^2} \\
&= \frac{1}{16}
\end{aligned}$$

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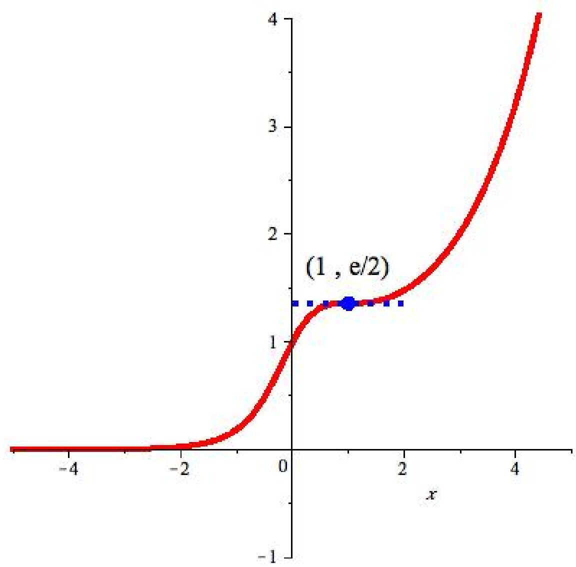
EXAMPLE 3.3.6

Find the equation of the tangent line to the function $f(x) = \dfrac{e^x}{1+x^2}$ at the point $p = \left(1, \dfrac{e}{2}\right)$.

Answer:
The derivative of $f(x)$ is

$$f'(x) = \frac{(1+x^2)(e^x) - e^x(2x)}{(1+x^2)^2} = \frac{e^x(x^2-2x+1)}{(x^2+1)^2}.$$

Plugging in the x-value of p yields the slope of the function at that point, namely, $f'(1) = 0$. Thus the equation of the tangent line is a constant function. The value of this constant is the y-value of p, namely $\frac{e}{2}$. The equation of the tangent line to the function is $t(x) = \frac{e}{2}$.



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Example 3.3.7

Given $f(x) = \dfrac{x+4}{3x-5}$

a.) find $f'(x)$.

b.) What does $f'(x)$ tell us about the graph of f?

Answer:

a.)

$$\begin{aligned} f'(x) &= \frac{-(3x-5)\frac{\mathrm{d}}{\mathrm{d}x}(x+4) - (x+4)\frac{\mathrm{d}}{\mathrm{d}x}(3x-5)}{(3x-5)^2} \\ &= \frac{-(3x-5)(1) - (x+4)(3)}{(3x-5)^2} \\ &= \frac{-17}{(3x-5)^2} \end{aligned}$$

b.) The derivative is always negative because the numerator is negative and the denominator is always positive. Therefore we know that f will be decreasing on it's whole domain.

Section 3.4

Derivatives of Trigonometric Functions

Introduction

Objective 3.4.1 Use the limit definition of the derivative to find the derivatives of $y = \sin x$, $y = \cos x$, and $y = \tan x$.

Objective 3.4.2 State the derivative of the basic trig functions: $y = \sin x$ and $y = \cos x$.

Objective 3.4.3 Using the definitions of the trig functions and the product and quotient rules, derive the derivatives of the other four trig functions, $y = \tan x$, $y = \sec x$, $y = \csc x$, and $y = \cot x$.

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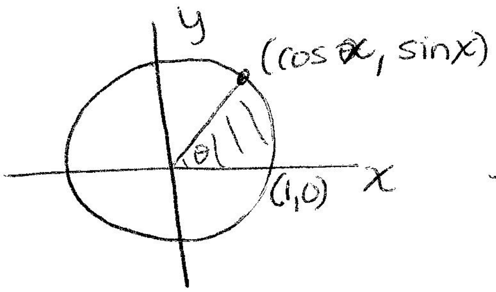
To determine the formula for the derivative of $y = \sin x$, we will need to know the value of the following limits.

a.) $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta}$

b.) $\lim_{\theta\to 0} \dfrac{1-\cos\theta}{\theta}$

First we will determine the value of $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta}$. To begin, we will assume that θ is an angle in radian measure with $0 < \theta < \frac{\pi}{2}$. Consider the graph of the unit circle below. **(Insert Graphs)**



Area of a sector

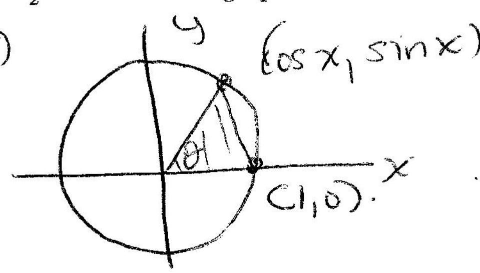


figure 1.

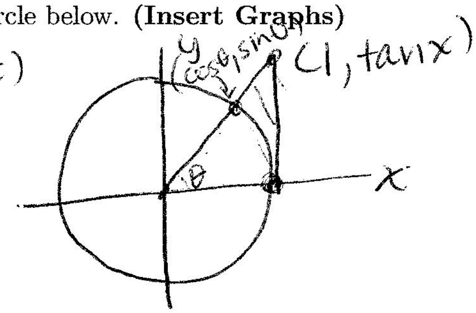


figure 2.

Notice that the area of the sector of the circle with central angle θ is between the area of the triangle in figure 1 and the triangle in figure 2.
The area of the smaller of the two triangles is $\frac{1}{2}\sin\theta$ and the area of the larger of the two triangles is $\frac{1}{2}\tan\theta$. The area of the sector is $\frac{1}{2}\theta$. It follows that

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

Multiplying through by $\frac{2}{\sin\theta}$ yields,

$$1 \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}.$$

Now, take the reciprocal of each fraction and switch the inequalities.

$$1 \geq \frac{\sin\theta}{\theta} \geq \cos\theta$$

Even though we derived the inequality by assuming that $0 < \theta < \frac{\pi}{2}$, it is still true when $\frac{\pi}{2} < \theta < 0$. We know this since replacing θ with $-\theta$ and using the identities, $\sin(-\theta) = -(\sin\theta)$ and $\cos(-\theta) = (\cos\theta)$ does not change the final inequality.

$$\lim_{\theta\to 0}\cos\theta = 1 \text{ and } \lim_{\theta\to 0} 1 = 1$$

By using the Squeeze Theorem, we conclude that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Next we will find the value of $\lim_{\theta\to 0} \dfrac{1-\cos\theta}{\theta}$

By the half-angle formula,

$$\sin^2(\frac{\theta}{2}) = \frac{1-\cos\theta}{2}$$

Therefore,

$$\frac{1-\cos\theta}{\theta} = \frac{2\sin^2(\frac{\theta}{2})}{\theta} = \frac{2\sin^2(\frac{\theta}{2})}{\frac{\theta}{2}}$$

Now let $z = \dfrac{\theta}{2}$. Since $z \to 0$ as $\theta \to 0$

$$\begin{aligned}
\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} &= \lim_{\theta\to 0} \frac{\sin^2(\frac{\theta}{2})}{\frac{\theta}{2}} \\
&= \lim_{z\to 0} \frac{\sin^2(z)}{z} \\
&= \lim_{z\to 0} [(\sin z)\frac{\sin z}{z}] \\
&= \lim_{z\to 0} [(\sin z)] \cdot \lim_{z\to 0} [\frac{\sin z}{z}] \\
&= (0)(1) \\
&= 0
\end{aligned}$$

Now we are ready to find the derivative of $f(x) = \sin x$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} && + \quad \lim_{h\to 0} \frac{\cos x \sin h}{h} \\
&= \sin x \lim_{h\to 0} \frac{(\cos h - 1)}{h} && + \quad \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= -(\sin x) \lim h \to 0 \frac{(1 - \cos h)}{h} && + \quad \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= -(\sin x)(0) && + \quad (\cos x)(1) \\
&= \cos x
\end{aligned}
$$

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Given $f(x) = \cos x$, we will find $f'(x)$ algebraically using the definition of the derivative.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \cos h - \cos x}{h} - \frac{\sin x \sin h}{h} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h\to 0} \frac{\sin x \sin h}{h} \\
&= \cos x \lim_{h\to 0} \frac{(\cos h - 1)}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} \\
&= (\cos x)(0) - (\sin x)(1) \\
&= -\sin x
\end{aligned}$$

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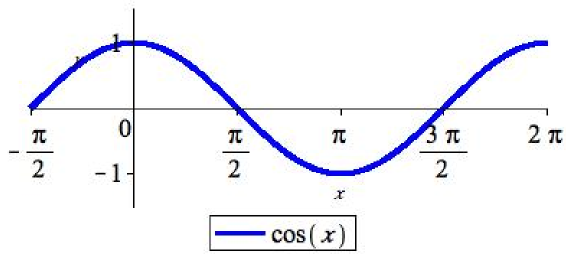
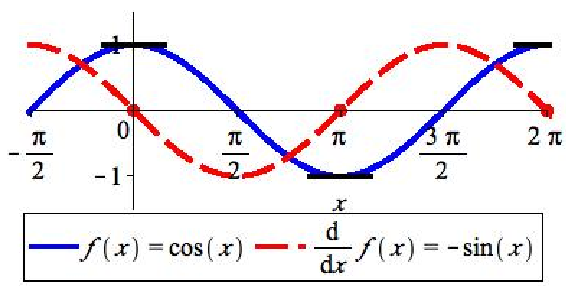
Example 3.4.1

Given the graph of $f(x) = \cos x$, graph $f'(x)$ to verify that your formula for $f'(x)$ from above is correct. In other words, check that where $f(x)$ is increasing, $f'(x)$ is positive and where $f(x)$ is decreasing, $f'(x)$ is negative.



Answer: First we identify the places where $f'(x) = 0$, and then we can sketch in the rest of its graph using the slope of $f(x)$:



We will determine the derivatives of the remaining trig functions by rewriting each of them in terms of sine and cosine and using the Quotient Rule.

In order to find the derivative of $f(x) = \tan x$, we need to rewrite it as $f(x) = \dfrac{\sin x}{\cos x}$.

Starting with $f(x) = \tan x = \dfrac{\sin x}{\cos x}$, we apply the quotient rule:

$$
\begin{aligned}
f'(x) &= \frac{\cos x \frac{\mathrm{d}}{\mathrm{d}x}(\sin x) - \sin x \frac{\mathrm{d}}{\mathrm{d}x}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}
$$

Note that the fact that the derivative of $f(x) = \tan x$ is squared means that the slope of tangent is always greater than or equal to zero. Therefore, the tangent function is always increasing from left to right.

Similarly, it can be shown that $\frac{d}{dx}(\cot x) = -\csc^2 x$. This problem is left as an exercise.

Find the derivative of $f(x) = \sec x$ by rewriting it in terms of $\cos x$.

Answer: Staring with $f(x) = \sec x = \dfrac{1}{\cos x}$, we apply the quotient rule:

$$\begin{aligned} f'(x) = & \frac{(\cos x)\frac{\mathrm{d}}{\mathrm{d}x}(1) - (1)\frac{\mathrm{d}}{\mathrm{d}x}(\cos x)}{\cos^2 x} \\ = & \frac{0 - (-\sin x)}{\cos^2 x} \\ = & \frac{\sin x}{\cos^2 x} \\ = & \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \\ = & \sec x \tan x \end{aligned}$$

Similarly it can be shown that if $f(x) = \csc x$, then $f'(x) = -\cot x \csc x$. This problem is left as an exercise.

In summary, the derivatives of all 6 trig functions are as follows:

a.) $\frac{d}{dx}(\sin x) = \cos x$

b.) $\frac{d}{dx}(\cos x) = -\sin x$

c.) $\frac{d}{dx}(\sec x) = \tan x \sec x$

d.) $\frac{d}{dx}(\csc x) = -\cot x \csc x$

e.) $\frac{d}{dx}(\tan x) = \sec^2 x$

f.) $\frac{d}{dx}(\cot x) = \csc^2 x$

Now let us do some examples using the new rules for trigonometric functions along with all of the derivative rules we have learned so far.

Example 3.4.2

If $y = x^2\cos(x)$, what is $\frac{\mathrm{d}y}{\mathrm{d}x}$?

Answer:

We must use the product rule and the rule for finding the derivative of $y = \cos x$:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= x^2\frac{d}{dx}[\cos(x)] + \frac{d}{dx}[x^2]\cos(x) \\ &= -x^2\sin(x) + 2x\cos(x) \\ &= x[2\cos(x) - x\sin(x)]\end{aligned}$$

Example 3.4.3

If $f(x) = \dfrac{\sec(x)}{1+\tan(x)}$, for what values of x does the graph of $f(x)$ have a horizontal tangent line?

Answer:

$$\begin{aligned}f'(x) &= \frac{(1+\tan(x))\frac{d}{dx}[\sec(x)] - \sec(x)\frac{d}{dx}[1+\tan(x)]}{(1+\tan(x))^2} \\ &= \frac{(1+\tan(x))(\tan(x)\sec(x)) - \sec(x)(1+\sec^2(x))}{(1+\tan(x))^2} \\ &= \frac{\sec(x)[\tan(x)+\tan^2(x) - (1+\tan^2(x))]}{(1+\tan(x))^2} \\ &= \frac{\sec(x)[\tan(x)-1]}{(1+\tan(x))^2}\end{aligned}$$

Setting this equation equal to zero yields $sec(x) = 0$ and $tan(x) = 1$. Notice that there is no solution to $sec(x) = 0$. Places where $f'(x) = 0$ are where $\tan x = 1$ which implies that $\frac{\sin x}{\cos x} = 1$; therefore, $\sin(x) = \cos(x)$, namely when $x = \frac{\pi}{4} + \pi n$ for n an integer.

Section 3.5

Differentiating Composite Functions

Introduction

Objective 3.5.1 State the chain rule.

Objective 3.5.2 Use the chain rule to find the derivative of a function that is the composition of two other functions.

Objective 3.5.3 Use the chain rule to find the derivative of a function that is the composition of three or more functions.

Objective 3.5.4 Use the chain rule to find the equations of lines that are tangent to parametric curves.

Objective 3.5.5 State the derivative of $y = a^x$ for any $a > 0$, $a \neq 1$.

When we want to take the derivative of a function like $y = (3x+4)^2$, we can do one of two things.

Method 1 We can expand the function by multiplying, to get

$$y = 9x^2 + 24x + 16$$

Then we can find y'.

$$y = 18x + 24$$

As you might imagine, the larger the exponent the less willing we will be to expand the function before finding the derivative.

Method 2 We can use the Chain Rule. The Chain Rule is the technique we will use to find derivatives of functions that are the composition of other functions. It is a technique that will work in situations in which Method I will not work or is too cumbersome. For example, we have no way to expand $g(x) = \sqrt{x+4}$ as we did in the case of the polynomial function above; therefore we cannot use Method I to determine g'. We might also note that none of the other methods we have previously discussed will allow us to compute g'. So let's consider this problem in a different way.

$$y = (3x+4)^2$$

can be written as

$$y = u^2 \text{ where } u = 3x + 4$$

Then.

$$\frac{dy}{du} = 2u \quad \frac{du}{dx} = 3$$

We find the derivative by multiplying.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the Chain Rule} \\
&= (2u)(3) \\
&= 2(3x+4) \cdot (3) && \text{Always put back in terms of the variable front the original problem.} \\
&= 18x + 24
\end{aligned}$$

We will not state a formal proof of the Chain Rule, as it is a bit beyond the scope of this class. For now, we can see that both methods lead to the same answer. Let's discuss the Chain Rule as it applies to rates of change. Assume that we express our composition as y in terms of u and u is in terms of x. If we know that y changes 4 times as fast as u and u changes 2 times as fast as x, then we can predict that y will change $(4) \cdot (2) = 8$ times as fast at x. So another way to think of the derivative of the composition is as the product of the derivatives.

The Chain Rule is presented below using different types of notation.

1. If $y = f(u)$ is differentiable at $u = f(x)$ and u is differentiable at x, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{dy}{du} \cdot \frac{du}{dx}$$

2. If g is differentiable at x and f is differentiable at $g(x)$, then $F = (f \circ g)$ is differentiable at x.

$$F'(x) = f'(g(x)) \cdot g'(x)$$

3. If f and g are differentiable, then $f \circ g$ is differentiable.

$$f' = f'(g) \cdot g'$$

4. In words: The derivative of the composition of two functions is the derivative of the outside function evaluated at the inside function times the derivative of the inside function.

Let us consider examples where we are using the Chain Rule.

Example 3.5.1

Use the chain rule to find $\frac{dy}{dx}$ for $y = \sqrt{x^2 + x - 3}$.

Answer: E xpress the composition in terms of u and y and x..

Let $u = x^2 + x - 3$. Then $y = \sqrt{u}$.

First we will find the derivative of y (the outside function) and the derivative of u (the inside function)

$$\begin{aligned} \frac{dy}{du} &= \frac{1}{2\sqrt{u}} \\ \frac{du}{dx} &= 2x + 1 \end{aligned}$$

Then we will substitute into the formula of the chain rule in 1 above.

Thus,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot (2x + 1)$$

Our final step is to replace u.

$$\frac{1}{2\sqrt{u}} \cdot (2x + 1) = \frac{2x + 1}{2\sqrt{x^2 + x - 3}}$$

Finally , we see

$$\frac{dy}{dx} = \frac{2x+1}{2\sqrt{x^2+x-3}}$$

Example 3.5.2

Use the chain rule to find $\frac{dy}{dx}$ for $y = (x^4+x+1)^{55}$.

Answer: E xpress the composition in terms of u and y.

Let $u = x^4+x+1$. Then $y = u^{55}$, so
Find the derivative of y (the outside function) and the derivative of u (the inside function)
.

$$\begin{aligned}\frac{dy}{du} &= 55u^{54}\\ \frac{du}{dx} &= 4x^3+1\end{aligned}$$

Then we will substitute into the formula of the chain rule in 1 above and replace u.

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 55u^{54}\cdot(4x^3+1) = 55(x^4+x+1)^{54}(4x^3+1)$$

Example 3.5.3

For $y = \left(\frac{x-1}{2x+3}\right)^5$, find $\frac{dy}{dx}$.

Answer: I n this example we will use the same notation as above, but we will find the derivatives as we go along. We will only use the variable u as a placeholder; it will not appear in the computations. Notice, the outside function is $f = u^2$ and the inside function is $u = \frac{x-1}{2x+3}$, so when we take the derivative of u we will use the quotient rule.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\ &= 5\left(\frac{x-1}{2x+3}\right)^4\frac{d}{dx}\left(\frac{x-1}{2x+3}\right)\\ &= 5\left(\frac{x-1}{2x+3}\right)^4\left[\frac{(2x+3)(1)-(x-1)(2)}{(2x+3)^2}\right]\\ &= 5\left(\frac{x-1}{2x+3}\right)^4\left[\frac{5}{(2x+3)^2}\right]\\ &= \frac{25(x-1)^4}{(2x+3)^6}\end{aligned}$$

Example 3.5.4

Let $y = (3x-1)^2(4x^2+x-5)^4$, find y'.

Answer:

We use the product and chain rule. Again we will use the simplified process as in the last example.

$$\begin{aligned}
y' &= (3x-1)^2\frac{d}{dx}[(4x^2+x-5)^4] + (4^2+x-5)^4\frac{d}{dx}[(3x-1)^2] \\
&= (3x-1)^2 \cdot 4(4x^2+x-5)^3 \cdot (8x+1) + (4x^2+x-5)^4 \cdot 2(3x-1)^1 \cdot 3 \\
&= 2(4x^2+x-5)^3(3x-1)[(3x-1)(2)(8x+1) + (4x^2+x-5)(3)] \\
&= 2(4x^2+x-5)^3(3x-1)[60x^2-7x-17]
\end{aligned}$$

Example 3.5.5

Let $y = e^{\tan(x)}$, find y'.

Answer: H ere the inner function is g(x) = $\tan x$ and the outer function is f(x) = e^x

$$y' = e^{\tan(x)}\frac{d}{dx}[\tan(x)] = \sec^2(x)e^{\tan(x)}$$

Example 3.5.6

Find y' for each of the following:

a.) $y = \tan(x^2)$

Answer: H ere the inner function is $g(x) = x^2$ and the outer function is $f(x) = \tan x$. Using the chain rule, we have

$$y' = \sec^2(x^2)\frac{d}{dx}[x^2] = 2x\sec^2(x^2).$$

b.) $y = \tan^2(x)$

Answer: H ere the inner function is $g(x) = \tan(x)$ and the outer function is $f(x) = x^2$. Using the chain rule, we have

$$y' = 2(\tan(x))^1\frac{d}{dx}[\tan(x)] = 2\tan(x)\sec^2(x).$$

EXAMPLE 3.5.7

$f(x) = 5\sec(5x)$ Find $f'(x)$.

Answer:

In this case, the inside function is $g(x) = 5x$ and the outside function is $f(x) = 5\sec x$.

$$\begin{aligned} f'(x) &= 5\big(\sec(5x)\tan(5x)\big)\frac{d}{dx}[5x] \\ &= 25\sec(5x)\tan(5x) \end{aligned}$$

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The Chain Rule for Three Functions.

We are not limited to using the chain rule when only two functions are composed. We can extend the rule to as many functions as we like. We will consider the composition of three functions below.

Let $y = (f \circ g \circ h)(x) = f((g \circ h)(x))) = f(g(h(x)))$ The inside function for f is $(g \circ h)(x)$ and the inside function of g is $h(x)$
Therefore by applying the chain rule twice we get the following.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= [f'(g \circ h)(x))] \cdot [g \circ h)'(x)] \\ &= [f'(g \circ h)(x))] \cdot [g'(h(x)] \cdot [h'(x)] \end{aligned}$$

$\boxed{\textsc{Example } 3.5.8}$

If $y = e^{\sin\left(x^2+x\right)}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Answer: I n this example, $f(x) = e^x$, $g(x) = \sin x$, and $h(x) = x^2 + x$

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= e^{\sin\left(x^2+x\right)} \frac{d}{dx}\left[\sin\left(x^2+x\right)\right] \\ &= e^{\sin\left(x^2+x\right)} \cos\left(x^2+x\right) \frac{d}{dx}\left[x^2+x\right] \\ &= \left[e^{\sin\left(x^2+x\right)}\right] \left[\cos\left(x^2+x\right)\right] [2x+1] \end{aligned}$$

Using the Chain Rule to Differentiate $y = a^x$.

In section 3.2 we talked about finding the derivative of exponential functions, $f(x) = a^x$ where $a > 0, a \neq 1$. We are not ready to evaluate the limit from that section, but we can calculate the derivative using the chain rule. (Note: We will also see what the limit can eventually be shown to equal).

Given that $y = a^x, a > 0, a \neq 1$ we will use the chain rule to find y'

Recall $e^{\ln x} = x$ so,

$$a^x = e^{\ln(a^x)} = e^{x\ln(a)}$$

Therefore $y = a^x = e^{x\ln(a)}$ so

$$\begin{aligned} y' &= e^{x\ln(a)} \frac{d}{dx}[x\ln(a)] \\ &= e^{x\ln(a)} \ln(a) \\ &= a^x \ln(a) \end{aligned}$$

Now we have the formula for finding the derivative of a general exponential function. Assuming $a > 0, a \neq 1$,

$$y = a^x \text{ implies } \frac{\mathrm{d}y}{\mathrm{d}x} = (\ln a)a^x$$

Section 3.6

Implicit Differentiation

Introduction

Objective 3.6.1 Use implicit differentiation to find dy/dx.

Objective 3.6.2 Use implicit differentiation to find d^2y/dx^2.

Objective 3.6.3 Use implicit differentiation to find the equation of the tangent line to a curve at a given point.

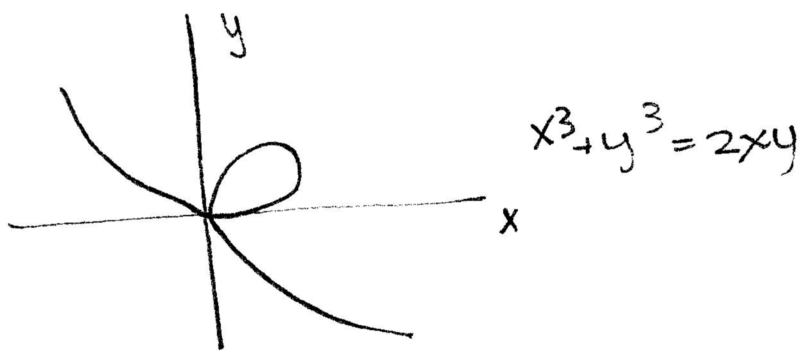
In the previous sections we have only been concerned with finding the rates of change and derivatives of functions that can be written explicitly in terms of one variable. For example, in the case of the function $y = x^2 + 1$, y is expressed as a formula only in x. Contrast that with the curve, $x^2 + y^2 = 9$. We cannot write y in terms of x or write x in terms of y as a single equation. When we attempt to do it, we end up with the following two equations instead of one.

$$\begin{aligned} y &= -\sqrt{9 - x^2} \\ y &= \sqrt{9 - x^2} \end{aligned}$$

In order to find derivatives at certain points on the circle, we have to find the two equations then determine which one to differentiate, which is not too hard in this case; however, consider the following case.

$$x^3 + y^3 = 6xy$$

The curve is called **Folium of Descartes**. In this case we cannot express y in terms of x. However, I included a graph of the curve and a graph of each of ~~the~~ three pieces of the curve. y is said to be defined *implicitly* as 3 functions of x, represented by ~~the~~ three graphs that make up the curve.



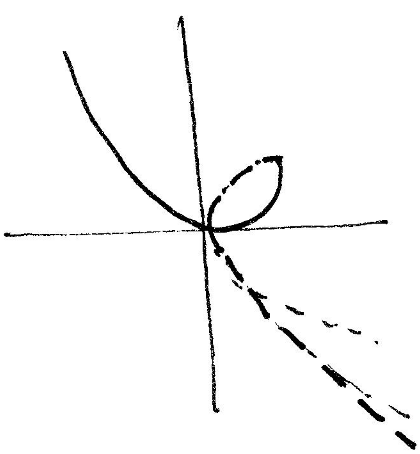


Figure 1

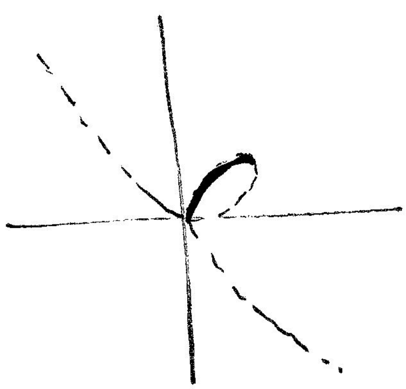


Figure 2

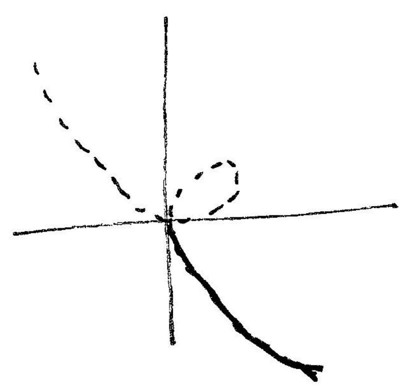


Figure 3.

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It is not necessary to express one variable explicitly in terms of another to find a derivative. We have a technique that allows us to differentiate y when it is defined implicitly. It is called **Implicit Differentiation**.

Consider the equation $x^3 + y^3 = 4$. To find $\dfrac{dy}{dx}$ using implicit differentiation,

first we will differentiate each side with respect to x. Recall that y is a function of x so we must use the chain rule when finding the derivative.

$$3x^2 \frac{\mathrm{d}}{\mathrm{d}x}[x] + 3y^2 \frac{\mathrm{d}}{\mathrm{d}x}[y] = 0$$

Differentiating gives us.

$$3x^2 + 3y^2 \frac{dy}{dx} = 0$$

Now solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$\begin{aligned} 3y^2 \frac{dy}{dx} &= -3x^2 \\ \frac{dy}{dx} &= -\frac{x^2}{y^2} \end{aligned}$$

$\boxed{\text{Example 3.6.1}}$

Given the equation $x^3 \cdot y^3 = 8$, find $\dfrac{dy}{dx}$.

Answer:

First, differentiate both sides with respect to x.

$$x^3 \frac{d}{dx}\left[y^3\right] + y^3 \frac{d}{dx}\left[x^3\right] = 0$$

$$x^3 \cdot 3y^2 \frac{dy}{dx} + y^3 \cdot 3x^2 = 0$$

Solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$ by moving all terms that contain $\frac{\mathrm{d}y}{\mathrm{d}x}$ in them to one side of the equation and every other term to the opposite side. Then simplify the answer.

$$\begin{aligned} 3x^3y^2 \frac{dy}{dx} &= -3x^2y^3 \\ \frac{dy}{dx} &= -\frac{3x^2y^3}{3x^3y^2} \\ \frac{dy}{dx} &= -\frac{y}{x} \end{aligned}$$

EXAMPLE 3.6.2

Given the equation $x^3 \cdot y^3 = 8$, find $\dfrac{d^2y}{dx^2}$.

Answer: F rom the previous example we saw that,

$$\frac{dy}{dx} = -\frac{y}{x}$$

To find $\dfrac{d^2y}{dx^2}$ we will need to take the derivative with respect to x one more time. We need to differentiate both sides with respect to x.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{x\frac{\mathrm{d}y}{\mathrm{d}x} - y\frac{\mathrm{d}}{\mathrm{d}x}(x)}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{x\frac{\mathrm{d}y}{\mathrm{d}x} - y}{x^2}$$

Now we replace $\frac{\mathrm{d}y}{\mathrm{d}x}$ to get

$$\frac{d^2y}{dx^2} = \frac{x(-\frac{y}{x}) - y}{x^2}$$

Which simplifies to

$$\frac{d^2y}{dx^2} = \frac{-2y}{x^2}$$

Example 3.6.3

Given the equation $x^2 - 2xy + y^3 = C$ where C is a real number, find $\frac{dy}{dx}$.

Answer: D ifferentiate both sides with respect to x.. Remember to use the product rule when differentiating the term $2xy$.

$$2x - \left[2x\frac{dy}{dx} + 2y\right] + 3y^2\frac{dy}{dx} = 0$$

Solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$ by moving all terms that contain $\frac{\mathrm{d}y}{\mathrm{d}x}$ to one side of the equation and every other term to the opposite side. Simplify the answer.

$$\begin{aligned}
-2x\frac{dy}{dx} + 3y^2\frac{dy}{dx} &= -2x + 2y \\
\frac{dy}{dx}\left[3y^2 - 2x\right] &= -2x + 2y \\
\frac{dy}{dx} &= \frac{-2x + 2y}{3y^2 - 2x}
\end{aligned}$$

Example 3.6.4

Given the equation $\cos(x - y) = x \cdot e^x$, find $\dfrac{dy}{dx}$.

Answer: D ifferentiate both sides with respect to x.

$$-\sin(x-y)\frac{d}{dx}[x-y] = xe^x + e^x$$

Solve for $\frac{dy}{dx}$ by moving all terms that contain $\frac{dy}{dx}$ to one side of the equation and every other term to the opposite side. Simplify the answer.

$$\begin{aligned}
-\sin(x-y)\left[1-\frac{dy}{dx}\right] &= xe^x + e^x \\
\left(1-\frac{dy}{dx}\right) &= \frac{xe^x + e^x}{-\sin(x-y)} \\
-\frac{dy}{dx} &= \frac{xe^x + e^x}{-\sin(x-y)} - 1 \\
\frac{dy}{dx} &= \frac{xe^x + e^x}{\sin(x-y)} + 1
\end{aligned}$$

Example 3.6.5

Given $y^5 + x^2y^3 = 1 + ye^{x^2}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Answer: D ifferentiate both sides with respect to x.

$$5y^4\frac{\mathrm{d}y}{\mathrm{d}x} + \left[x^2 \cdot 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 \cdot 2x\right] = 0 + \left[y \cdot e^{x^2}(2x) + e^{x^2} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}\right]$$

Solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$ by moving all terms that contain $\frac{\mathrm{d}y}{\mathrm{d}x}$ to one side of the equation and every other term to the opposite side. Simplify the answer.

$$\begin{aligned}
5y^4\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2y^2\frac{\mathrm{d}y}{\mathrm{d}x} - e^{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} &= -2xy^2 + 2xye^{x^2} \\
\frac{\mathrm{d}y}{\mathrm{d}x}\left[5y^4 + 3x^2y^2 - e^{x^2}\right] &= -2xy^2 + 2xye^{x^2} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-2xy^2 + 2xye^{x^2}}{5y^4 + 3x^2y^2 - e^{x^2}}
\end{aligned}$$

EXAMPLE 3.6.6

Given $y\cos(x^2) = x\tan(y^2)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Answer: F ind the derivative of both sides with respect to x.

$$y \cdot \frac{\mathrm{d}(\cos(x^2))}{\mathrm{d}x} + \cos(x^2) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = x \cdot \frac{\mathrm{d}(\tan(y^2))}{\mathrm{d}x} + \tan(y^2) \cdot (1)$$

Solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$ by moving all terms that contain $\frac{\mathrm{d}y}{\mathrm{d}x}$ to one side of the equation and every other term to the opposite side. Simplify the answer.

$$\begin{aligned}
y \cdot [-\sin(x^2)](2x) + \cos(x^2)\frac{\mathrm{d}y}{\mathrm{d}x} &= x\left[[\sec^2(y^2)] \cdot 2y \cdot \frac{\mathrm{d}y}{\mathrm{d}x}\right] + \tan(y^2) \\
\cos(x^2)\frac{\mathrm{d}y}{\mathrm{d}x} - 2xy\sec^2(y^2)\frac{\mathrm{d}y}{\mathrm{d}x} &= \tan(y^2) + 2xy\sin(x^2) \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\tan(y^2) + 2xy\sin(x^2)}{\cos(x^2) - 2xy\sec^2(y^2)}
\end{aligned}$$

Last update: September 19, 2014

Example 3.6.7

Given the equation $x^{2/3} + y^{2/3} = 1$, find $\frac{dy}{dx}$ and the equation of the tangent line at the point $\left(-\frac{1}{8}, \frac{3\sqrt{3}}{8}\right)$.

Answer:

Recall that the slope of the tangent line is given by $\frac{dy}{dx}$.

$$\begin{aligned}
\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} &= 0 \\
\frac{2}{3}y^{-1/3}\frac{dy}{dx} &= -\frac{2}{3}x^{-1/3} \\
\frac{dy}{dx} &= -\frac{x^{-1/3}}{y^{-1/3}} \\
\frac{dy}{dx} &= -\frac{y^{1/3}}{x^{1/3}}
\end{aligned}$$

To find the slope at the point given:

$$\frac{dy}{dx} = \frac{-\sqrt[3]{\frac{3\sqrt{3}}{8}}}{\sqrt[3]{-\frac{1}{8}}} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$

Therefore, the equation of the tangent line at the point given is

$$y - \frac{3\sqrt{3}}{8} = \sqrt{3}\left(x + \frac{1}{8}\right).$$

Section 3.7

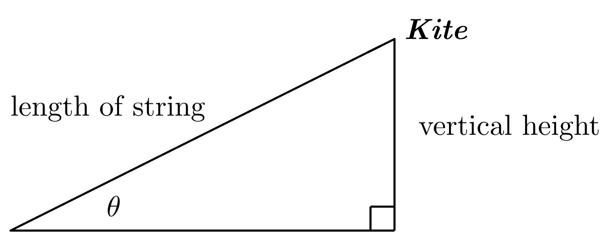
Related Rates

Introduction

Objective 3.7.1 Given a word problem, find an equation that relates two given quantities.

Objective 3.7.2 Solve related rates word problems of various types.

In related rates problems, we often see that two different quantities are changing over time and that the changes in the rates are related. For example, when flying a kite, the rate at which the string is being played out and the rate at which the vertical height of the kite is changing are related to each other. However, the rate at which the string is played out is generally much easier to measure than the rate at which the height of the kite is changing.



We will use the rate that is easier to find and the relationship between the two rates to determine the value of the rate that is more difficult to measure. Now, let us discuss a typical related rates problem.

Sand is falling from a conveyor belt at a rate of 20 ft^3/min. It forms a pile in the shape of a cone whose height is always equal to the diameter of the base. How fast is the height of the pile increasing when the pile is 8 ft high?

After reading through the problem, we draw a diagram.

Next we write an equation that describes the relationship between the variables in the problem. In this case it is the volume of the cone.

$$V = \frac{1}{3}\pi r^2 h$$

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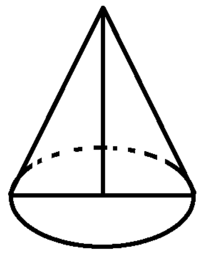

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We are asked to find the rate at which the height is increasing, $\frac{dh}{dt}$, when the height, h, and rate at which the volume is changing, $\frac{dV}{dt}$, are specified. The known values must be used in an equation that describes the relationship between $\frac{dh}{dt}$ and $\frac{dV}{dt}$, so we will need to use implicit differentiation on the volume formula to obtain it. Before using implicit differentiation, we need to replace any other variables, except V and h, in the formula. The height equals the diameter at every time in the process, therefore, $h = d = 2r$. We can replace r with $\frac{h}{2}$ in the volume formula to get the formula

$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h \\ &= \frac{1}{12}\pi h^3. \end{aligned}$$

The equation for volume is in terms of a single variable, h, so we can use implicit differentiation to write $\frac{dV}{dt}$ in terms of h and $\frac{\mathrm{d}h}{\mathrm{d}t}$.

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{d}{dt}\left(\frac{1}{12}\pi h^3\right) \\ &= \left(\frac{\pi}{4}h^2\right)\frac{\mathrm{d}h}{\mathrm{d}t}. \end{aligned}$$

Solving for $\frac{\mathrm{d}h}{\mathrm{d}t}$ gives us the equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t}\left(\frac{4}{\pi h^2}\right).$$

Replacing $\frac{\mathrm{d}V}{\mathrm{d}t}$ with 20 and h with 8 we determine that

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 20\left(\frac{4}{\pi(8)^2}\right) = 0.398 \text{ ft/min}.$$

Last update: September 30, 2014

The process we used can be generalized into the following steps to help you solve the problem.

Procedure for Solving Related Rates Problems

Step 1: Read the problem. Identify quantities involved in the statement of the problem.

Step 2: Draw a diagram which illustrates the relationship among the quantities discussed in the problem. Identify variables.

Step 3: Write an equation that describes a relationship among the variables.

Step 4: Use a constraint equation to replace any variables that do not involve the rates in which we are interested.

Step 5: Differentiate implicitly with respect to time.

Step 6: Substitute the value of the known variables and rates to get an equation with just the desired rate.

Step 7: Solve for desired rate.

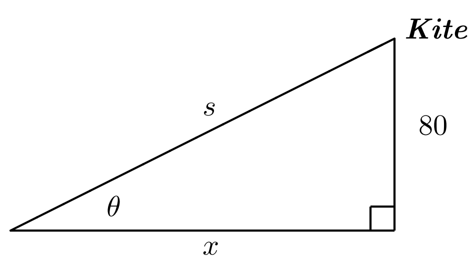
We now consider the related rates problem discussed at the beginning of the section.

Example 3.7.1

A kite 80 feet above the ground moves horizontally at a speed of 6 feet per second. At what rate is the angle between the string and the horizontal decreasing when 160 feet of string have been let out?

Step 1:
We are interested in the rate the angle is changing given the horizontal speed while the height is remaining at 80 feet.

Step 2:



Let s represent the length of the string. Let x represent the horizontal distance from the person flying the kite to the spot on the ground directly under the kite. Let θ represent the angle the string makes with the ground. The horizontal speed is the rate at which the horizontal distance between the kite and the person holding the string is changing is $\frac{dx}{dt}$. The rate the angle θ is changing is given by $\frac{d\theta}{dt}$.

Step 3:
We will use a trigonometric relationship to write an equation that will relate the angle θ to the length x.

$$\tan\theta = \frac{80}{x}.$$

Step 4:
There are no unwanted variables in the formula so we skip to step 5.

Step 5:
Differentiate each side of the equation with respect to time, t.

$$\sec^2(\theta)\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t} = -80x^{-2}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}.$$

Step 6:
We need to replace the values of x, θ and $\frac{\mathrm{d}x}{\mathrm{d}t}$ into the equation from Step 5 to solve for $\frac{\mathrm{d}\theta}{\mathrm{d}t}$. To determine the measurement of x, we can use the fact that $s = 160$ ft of string is let out; therefore, by the Pythagorean Theorem

$$x = \sqrt{160^2 - 80^2} = 138.564 \text{ ft}.$$

To find θ we recall from trigonometry that $\sin\theta = \frac{opp}{hyp} = \frac{80}{160} = \frac{1}{2}$ and $0 < \theta < 90°$. Therefore,

$$\theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

We also know that $\frac{\mathrm{d}x}{\mathrm{d}t} = 6$. Replacing these values in the formula from Step 5, we get the equation

$$\sec^2\left(\frac{\pi}{6}\right)\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{-80}{19200}(6).$$

Step 7:
Solving for $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ we get

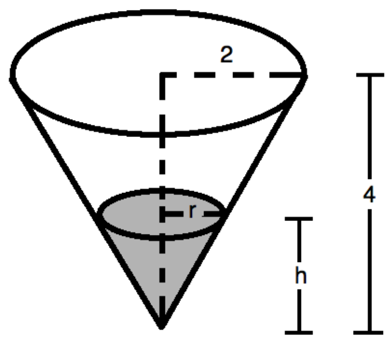
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -0.01875 \text{ radian/sec}.$$

Example 3.7.2

An ice cream cone has the shape of an inverted circular cone with a base of radius 2 inches and height of 4 inches. If soft serve ice cream is being pumped into the cone at a rate of 2 in^3/min, find the rate at which the ice cream level is rising when the ice cream is 3 in deep.

Step 1:
We are interested in finding a relationship between the rate of change of the volume of the ice cream in the cone and the rate of change of the depth of the ice cream.

Step 2:



Let V represent the volume of the ice cream. Let r represent the radius of the base of the cone. Let h represent the depth of the ice cream in the cone. The rate at which the volume of the ice cream is changing is given by $\frac{dV}{dt}$. The rate at which the depth of the ice cream is changing is given by $\frac{dh}{dt}$.

Step 3:
We want an equation that relates volume, height, and radius. We will use the volume formula

$$V = \frac{1}{3}\pi r^2 h.$$

Step 4:
We need to remove the variable r, the radius of the base of the cone, from the formula before differentiating. From the information given, we see from similar triangles that the ratio of the radius of the cone to the height is given by the proportion

$$\frac{r}{h} = \frac{2}{4} \text{ which implies } r = \frac{1}{2}h.$$

The volume equation now becomes

$$V = \frac{1}{2}\pi(\frac{1}{2}h)^2 h.$$

Which simplifies to

$$V = (\frac{\pi}{8})h^3.$$

Step 5:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \left(\frac{\pi}{4}\right)\left(h^2\right)\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right).$$

Step 6:
Replacing $\frac{\mathrm{d}V}{\mathrm{d}t}$ with 2 and h with 3 yields

$$2 = \frac{\pi}{4} \cdot (3)^2 \frac{\mathrm{d}h}{\mathrm{d}t}.$$

Step 7:
Solving for $\frac{\mathrm{d}h}{\mathrm{d}t}$ yields

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 0.2829 \text{ in/min.}$$

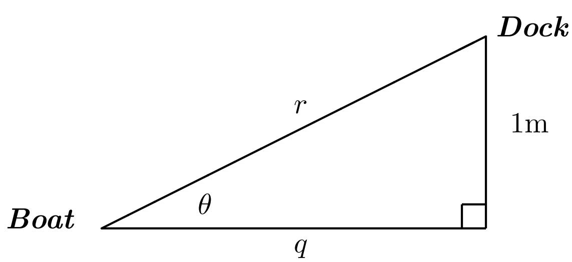
Last update: September 30, 2014

Example 3.7.3

A boat is pulled into a dock by a rope attached to the bow of the boat. The rope passes through a pulley on the dock that is 1 meter higher than the bow of the boat. If the rope is pulled in at a rate of 1 meter per second, how fast is the boat approaching the dock when it is 6 meters from the dock?

Step 1:
We are interested in finding the rate at which the distance from the dock to the boat is changing given information about the length of rope and the rate at which the boat is being pulled to the dock; i.e., how fast the rope is being pulled in.

Step 2:



Letting q represent the horizontal distance from the boat to the dock and r represent the length of the rope, we are interested in finding the rate at which the distance to the dock is changing, $\frac{\mathrm{d}q}{\mathrm{d}t}$, when the rate the length of the rope is changing, $\frac{\mathrm{d}r}{\mathrm{d}t}$, is -1 meters per second. (NOTE: the rate is negative because the length of the rope is decreasing as the boat gets closer to the dock.) and q is 6 feet.

Step 3:
We want an equation that will relate r and q. Using the Pythagorean theorem we see that

$$r^2 = 1 + q^2.$$

Step 4:
There are no extra variables to replace, so go to Step 5.

Step 5:
Differentiating both sides we get,

$$2r \cdot \frac{\mathrm{d}r}{\mathrm{d}t} = 2q \cdot \frac{\mathrm{d}q}{\mathrm{d}t}.$$

Step 6:
When $q = 6$, $r = \sqrt{37}$. Replace $\frac{\mathrm{d}r}{\mathrm{d}t} = -1$ and $q = 6$, $r = \sqrt{37}$ to get,

$$2(\sqrt{37}) \cdot (-1) = 2(6) \cdot \frac{\mathrm{d}q}{\mathrm{d}t}.$$

Step 7:
Solving for $\frac{\mathrm{d}q}{\mathrm{d}t}$ we get,

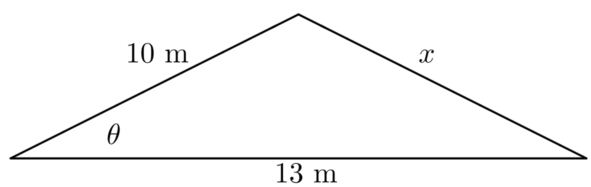
$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{-\sqrt{37}}{6} = -1.0169 \text{ m/s}.$$

Example 3.7.4

Two sides of a triangle have lengths 10 meters and 13 meters. The angle between them is increasing at a rate of 3 degrees per minute. How fast is the length of the third side increasing when the angle between the sides of fixed length is $60°$?

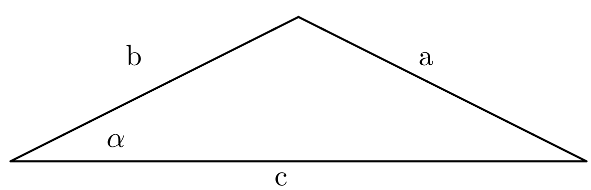
Step 1:
We are interested in finding how fast the length of one side of a triangle is changing when the angle opposite that side is a given measure.

Step 2:



Let two sides of a triangle have length 10 m and 13 m and represent the third side by the variable x. Let the angle between the two known sides be represented by θ. The rate at which the third side is changing is given by $\frac{dx}{dt}$. The rate at which the angle is changing is given by $\frac{d\theta}{dt}$.

Step 3:
We want a formula that relates θ and x. The Law of Cosines states that $a^2 = b^2 + c^2 - 2bc\cos\alpha$, where a, b, c are the three sides of a triangle and α is the angle opposite side a.



Using the Law of Cosines, we have

$$x^2 = 10^2 + 13^2 - 2(10)(13)\cos(\theta).$$

Step 4:
We don't have extra variables we need to replace in the equation, so go on to Step 5.

Step 5:
Implicitly differentiating each side with respect to t yields,

$$2x \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = \left(-2(10)(13)(-\sin\theta)\,)(\,\frac{\mathrm{d}\theta}{\mathrm{d}t} \right).$$

Step 6:
We know that $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 3^\circ = \frac{\pi}{60}$ and recall that $\theta = 60^\circ = \frac{\pi}{3}$. The value of x at this instant is

$$x^2 = 10^2 + 13^2 - 2(10)(13)\cos\left(\frac{\pi}{3}\right) = 139 \text{ implies that } x = \sqrt{139}.$$

Replacing those variables into the equation from Step 5 yields,

$$2(\sqrt{139} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = -2(10)(13)(-\sin\frac{\pi}{3}) \cdot \frac{\pi}{60}.$$

Step 7:
Solving for $\frac{\mathrm{d}x}{\mathrm{d}t}$ yields,

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{130}{\sqrt{139}}(\sin\frac{\pi}{3}) \cdot \frac{\pi}{60} \\ &= .499994 \text{ m/min.} \end{aligned}$$

Last update: September 30, 2014

3.8 Derivatives of Inverse Trigonometric Functions

After completing this section, the learner will be able to...

Objective 3.8.1. *Derive the derivative of* $y = \sin^{-1} x$.

Objective 3.8.2. *Derive the derivative of* $y = \cos^{-1} x$.

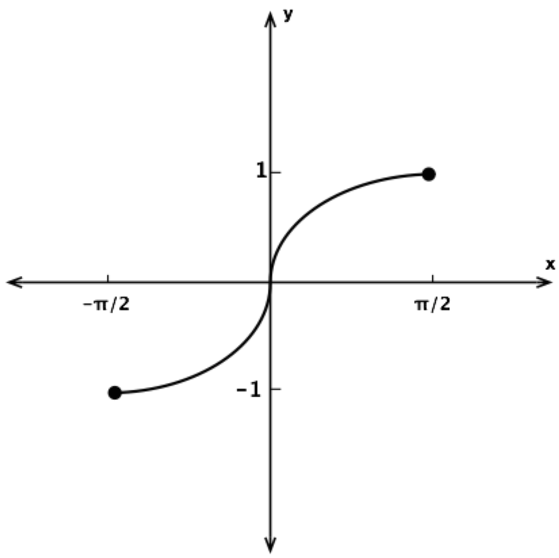
Objective 3.8.3. *Derive the derivative of* $y = \tan^{-1} x$.

Objective 3.8.4. *Use the derivatives of the inverse trigonometric functions to differentiate functions.*

Last update: October 6, 2014

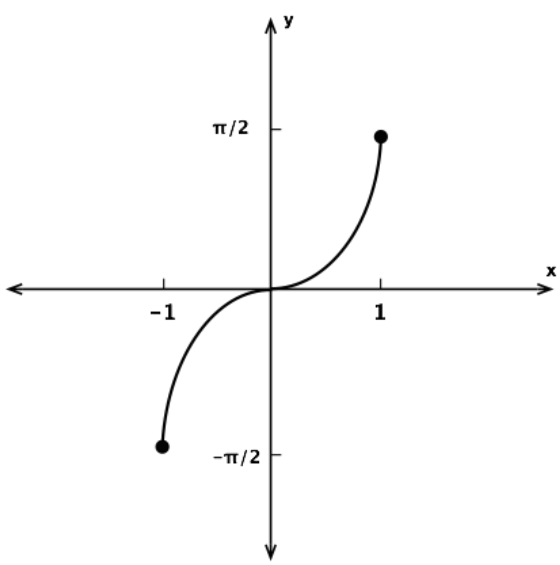
Finding derivatives of inverse trigonometric functions requires careful consideration due to the way the functions are defined. Recall that trigonometric functions are periodic; therefore, they are not one-to-one and do not have inverses that are functions. By restricting the domain of each of the trigonometric functions; we obtain a one-to-one function with the same range as the unrestricted function.

3.8.1 Derivative of the Inverse Sine Function



To define $y = \sin^{-1} x$, we use the part of $y = \sin x$ such that $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$. The new function has domain $[\frac{-\pi}{2}, \frac{\pi}{2}]$ and range $[-1, 1]$. See figure on the left.

This new function is one-to-one; therefore, we can define an inverse function, $y = \sin^{-1} x$ with domain $[-1, 1]$ and range $[\frac{-\pi}{2}, \frac{\pi}{2}]$. See figure on the left.
Note that, $\sin^{-1}(-\frac{1}{2}) = \frac{-\pi}{6}$.



Since the two functions are inverses of each other, we can see that

$$y = \sin^{-1} x$$

implies that

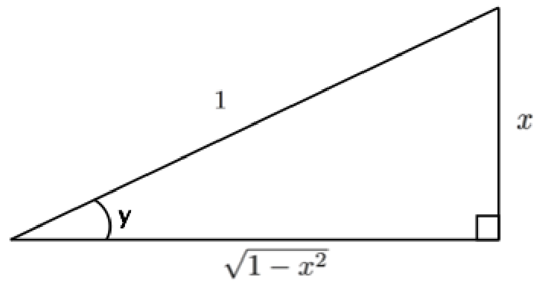
$$x = \sin y.$$

Differentiating implicitly we get

$$\frac{dx}{dx} = (\cos y)\frac{dy}{dx}.$$

We now simplify and solve for $\frac{dy}{dx}$ to get

$$\frac{dy}{dx} = \frac{1}{\cos y}, \quad \text{provided } \cos y \neq 0.$$



When $0 < y < \frac{\pi}{2}$, we can see the relationship between x and y by using the right triangle we get from the unit circle. The triangle is illustrated on the left.

Using the diagram of the triangle we see that

$$\cos y = \frac{\sqrt{1-x^2}}{1}.$$

Replacing $\cos y$ above we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

If $-\frac{\pi}{2} < y < 0$ we can use the identities, $\cos(-y) = \cos(y)$ and $(-x)^2 = x^2$ to get the same derivative formula.

We will now consider the cases where $y = 0$, $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$.

- When $y = 0$, $x = \sin 0 = 0$.

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos 0} = \frac{1}{1} = 1 \text{ and } \frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-0^2}} = 1.$$

- When $y = -\frac{\pi}{2}$, $x = \sin(-\frac{\pi}{2}) = -1$ and $\cos(-\frac{\pi}{2}) = 0 = \sqrt{1-1^2} = \sqrt{1-x^2}$.

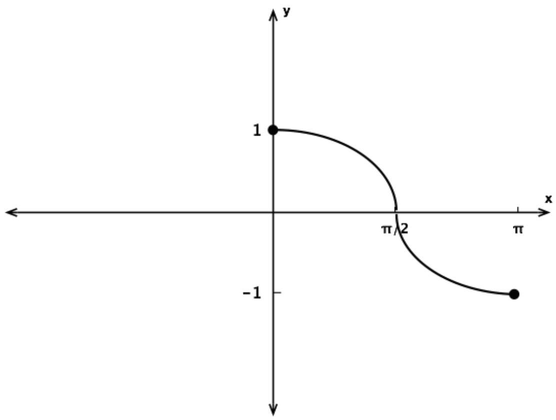
Since $\frac{1}{0}$ is undefined, this formula does not give us the derivative of $\sin^{-1} x$ at $x = -1$. It turns out that $\sin^{-1} x$ is not differentiable at $x = -1$.

A similar result holds when $x = +1$.

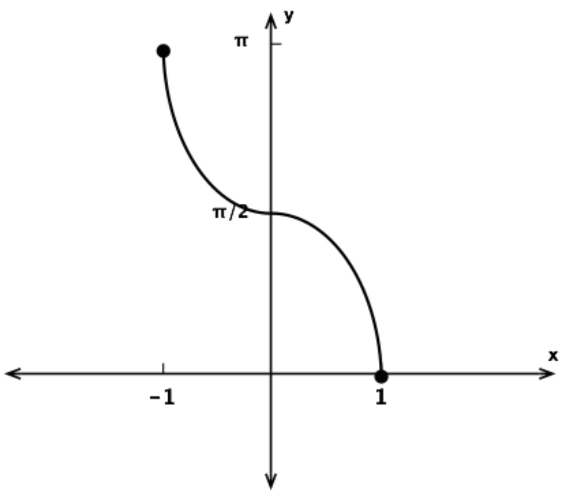
Basic Formulas 1.

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad \textit{for } -1 < x < 1.$$

3.8.2 Derivative of the Inverse Cosine Function



To define $y = \cos^{-1} x$, we use the part of $y = \cos x$ such that $0 \leq x \leq \pi$. The new function has domain $[0, \pi]$ and range $[-1, 1]$. See figure on the left.



This function is one-to-one; therefore, we can define an inverse function, $y = \cos^{-1} x$ with domain $[-1, 1]$ and range $[0, \pi]$. See figure on the left.
Note that, $\cos^{-1}(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$.
Since the two functions are inverses of each other, we can see that

$$y = \cos^{-1} x$$

implies that

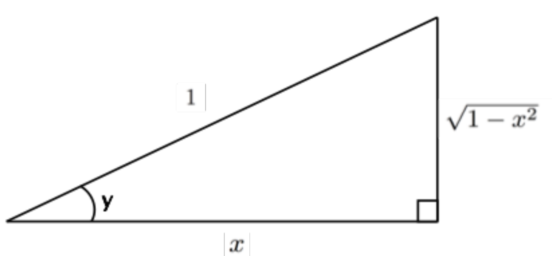
$$x = \cos y.$$

Differentiating implicitly gives

$$\frac{dx}{dx} = (-\sin y)\frac{dy}{dx}.$$

We now simplify and solve for $\frac{dy}{dx}$ to get

$$\frac{dy}{dx} = \frac{-1}{\sin y}, \quad \text{provided } \sin y \neq 0.$$



When $0 < y < \frac{\pi}{2}$, we can see the relationship between x and y by using a right triangle as illustrated on the left.
Using the diagram of the triangle we see that

$$\sin y = \frac{\sqrt{1-x^2}}{1}.$$

Replacing $\sin y$ above we get

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}.$$

If $\frac{\pi}{2} < y < \pi$ we can use the identities, $\sin(-y) = -\sin(y)$ and $(-x)^2 = x^2$ to get the same derivative formula.

We will now consider the cases where $y = 0$, $y = \frac{\pi}{2}$ and $y = \pi$.

Last update: October 6, 2014

- When $y = \frac{\pi}{2}$, $x = \cos\frac{\pi}{2} = 0$.

$$\frac{dy}{dx} = \frac{1}{\sin y} = \frac{1}{\sin(\frac{\pi}{2})} = \frac{1}{1} = 1 \text{ and } \frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-0^2}} = 1.$$

- When $y = 0$, $x = \cos 0 = 1$, $\sin 0 = 0$ and $\sqrt{1-1^1} = 0$.

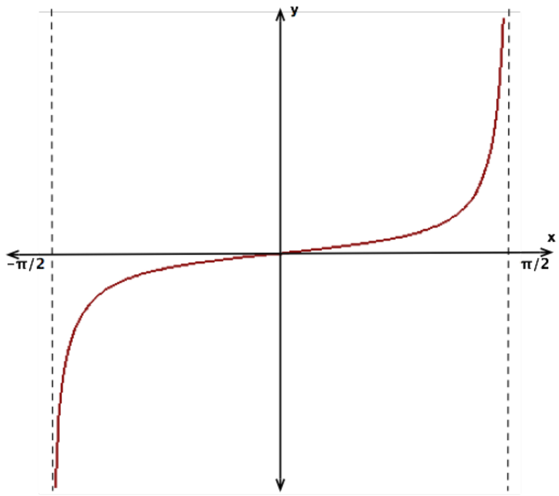
Since $\frac{1}{0}$ is undefined, this formula does not give us the derivative when $y = 0$. It turns out that $\cos^{-1} x$ is not differentiable at $x = 1$.

Similarly, $\cos^{-1} x$ is not differentiable at $x = -1$.

Basic Formulas 2.

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, \quad \textit{for } -1 < x < 1.$$

3.8.3 Derivative of the Inverse Tangent Function



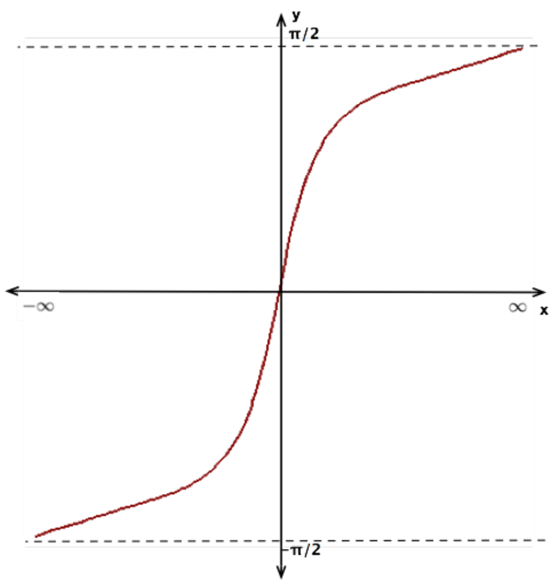
To define $y = \tan^{-1} x$, we use the part of $y = \tan x$ such that $\frac{-\pi}{2} < x < \frac{\pi}{2}$. The new function has domain $(-\frac{\pi}{2}, \frac{\pi}{2})$ and range $(-\infty, \infty)$.

This new function is one-to-one; therefore, we can define an inverse function, $y = \tan^{-1} x$ with domain $(-\infty, \infty)$ and range $(-\frac{\pi}{2}, \frac{\pi}{2})$. Note that, $\tan^{-1}(-1) = \frac{-\pi}{4}$.
Since the two functions are inverses of each other, we can see that

$$y = \tan^{-1} x$$

implies that

$$x = \tan y.$$

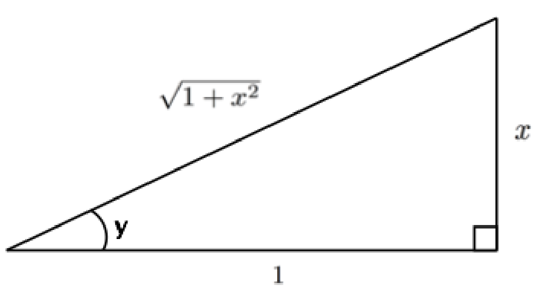


Differentiating implicitly we get

$$\frac{dx}{dx} = (\sec^2 y)\frac{dy}{dx}.$$

Solve for $\frac{dy}{dx}$ to get

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}.$$



When $0 < y < \frac{\pi}{2}$, we can see the relationship between x and y by using the right triangle illustrated on the left.
Using the diagram of the triangle we see that

$$\sec y = \frac{\sqrt{1+x^2}}{1}.$$

Replacing $\sec y$ above we get

$$\frac{dy}{dx} = \frac{1}{(\sqrt{1+x^2})^2} = \frac{1}{1+x^2}.$$

If $-\frac{\pi}{2} < y < 0$ we can use the identities, $\sec(-y) = \sec(y)$ and $(-x)^2 = x^2$ to get the same derivative formula.

We will now consider the case where $y = 0$. When $y = 0$, $x = \tan 0 = 0$.

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\sec^2 0} = \frac{1}{1^2} = 1 \text{ and } \frac{1}{1+x^2} = \frac{1}{1+0^2} = 1.$$

Basic Formulas 3.

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \quad \textit{for } -\infty < x < \infty.$$

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Basic Formulas 4. *The derivatives for* $y = \sin^{-1} x$*,* $y = \cos^{-1} x$*, and* $y = \tan^{-1} x$ *are the ones that are used most often. The derivatives of the remaining trigonometric functions are*

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}, \quad \text{for } x < -1 \text{ or } x > 1.$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}, \quad \text{for } x < -1 \text{ or } x > 1.$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \quad \text{for } -\infty < x < \infty.$$

Example 3.8.1. *Differentiate* $y = (\tan^{-1} x)^2$*.*

Solution:

$$\frac{dy}{dx} = \left(2(\tan^{-1} x)^{2-1}\right)\left(\frac{d}{dx}(\tan^{-1} x)\right) \qquad \textit{Differentiate using the Chain Rule.}$$

$$= 2(\tan^{-1} x)\left(\frac{1}{1+x^2}\right) \qquad \textit{Differentiate } \tan^{-1} x.$$

$$= \frac{2\tan^{-1} x}{1+x^2}. \qquad \textit{Simplify.}$$

Example 3.8.2. *Differentiate* $y = \tan^{-1}(x^2)$.

Solution:

$$\frac{dy}{dx} = \left(\frac{1}{1+(x^2)^2}\right)\left(\frac{d}{dx}(x^2)\right)$$ *Differentiate using the Chain Rule.*

$$= \left(\frac{1}{1+(x^2)^2}\right)(2x)$$ *Differentiate the inside function,* x^2.

$$= \frac{2x}{1+x^4}.$$ *Simplify.*

Example 3.8.3. *Differentiate* $y = \sin^{-1} x + x\sqrt{1-x^2}$.

Solution:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + (x)\left(\frac{d}{dx}(1-x^2)\right) + \left(\frac{d}{dx}(x)\right)\left(\sqrt{1-x^2}\right)$$ *Differentiate* $\sin^{-1} x$ *then use Product Rule.*

$$= \frac{1}{\sqrt{1-x^2}} + (x)(-2x) + (1)(\sqrt{1-x^2})$$ *Differentiate.*

$$= \frac{1}{\sqrt{1-x^2}} - 2x^2 + \sqrt{1-x^2}.$$ *Simplify.*

Find $\frac{dy}{dx}$ for the following functions of x in problems 1- 10. Note: $\arcsin, \arccos,$ and $\arctan$ are alternate notation for $\sin^{-1}, \cos^{-1},$ and $\tan^{-1}$.

1. $y = \tan^{-1} x + 3x^5 + e^2$
2. $y = \frac{2x}{\arcsin(2x)}$
3. $y = x\cos^{-1} x - \sqrt{1-x^2}$
4. $y = \arccos(\tan(3x+1))$
5. $y = (\tan^{-1} x)^3$
6. $y = \tan^{-1}(x^3)$
7. $y = \arcsin x + x\sqrt{1-x^2}$
8. $y = \sec(\tan^{-1} x)$
9. $y = \frac{\arctan(4x)}{e^{4x}}$
10. $y = x\sin^{-1}(3x+1)$

Find the equation of the tangent lines to the following functions at the given points for problems 11-15.

11. $y = \tan^{-1} x$ at the point $(1, \frac{\pi}{4})$
12. $y = \cos^{-1} x$ at the point $(1, 0)$
13. $y = \sin^{-1} x + 2x$ at the point $(1, 2 + \frac{\pi}{2})$
14. $y = \frac{3\cos^{-1} x}{x+1}$ at the point $(0, 3)$
15. $y = x + \sin^{-1} x$ at the point $(0, 0)$

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3.9 Derivatives of Logarithmic Functions

After completing this section the learner will be able to...

Objective 3.9.1. *Derive the derivative of* $y = \log_a x$.

Objective 3.9.2. *Derive the derivative of* $y = \ln x$.

Objective 3.9.3. *Use logarithmic differentiation to find the derivative of a function.*

In this section we will determine the **derivative of the logarithmic function** $y = \log_a x$ for all $a > 0, a \neq 1$, and when $x > 0$. A special case gives the derivative for the natural logarithmic function, $y = \ln x$. Once we have those derivatives, we will take a look at how they can help us determine derivatives we have previously been unable to calculate.

We will first find the derivative for $y = \log_a x$ for any $a > 0, a \neq 1$, and $x > 0$. Notice that $y = \log_a x$ implies

$$a^y = x.$$

Using implicit differentiation, we get

$$(\ln a)(a^y)\left(\frac{dy}{dx}\right) = \frac{dx}{dx}.$$

Replace a^y with x and simplifying, we get

$$(\ln a)(x)\left(\frac{dy}{dx}\right) = 1.$$

Now solve for $\frac{dy}{dx}$ to see that

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \text{ for } x > 0.$$

When the base $a = e$, the derivative is

$$\frac{d}{dx}(\ln x) = \frac{d}{dx}(\log_e x) = \frac{1}{x \ln e} = \frac{1}{x} \text{ for } x > 0.$$

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Example 3.9.1. *Given $y = \log_8(x^2 + x - 1)$, find y'.*

Solution:

$$y' = \frac{1}{(x^2 + x - 1)\ln 8} \cdot \frac{d}{dx}(x^2 + x - 1) = \frac{2x + 1}{(x^2 + x - 1)\ln 8}.$$

Notice that the function above is the composition of two functions. So in general we can see that if

$$f(x) = \log_a(g(x)), \text{ then } f'(x) = \frac{g'(x)}{(g(x))(\ln a)}$$

or in the case that $a = e$ we have

$$f'(x) = \frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}.$$

Example 3.9.2. *Given $y = \ln(2x)$, find y'.*

Solution:

$$y' = \frac{\frac{d}{dx}(2x)}{2x} = \frac{2}{2x} = \frac{1}{x}.$$

Another way to see this is to use log rules to rewrite y as $y = \ln 2 + \ln x$. The first term is a constant, so $y' = 0 + \frac{1}{x} = \frac{1}{x}$, as we found above.

Example 3.9.3. *Given $y = \ln(\sin x)$, find y'.*

Solution:

$$y' = \frac{\frac{d}{dx}(\sin x)}{\sin x} = \frac{\cos x}{\sin x} = \cot x.$$

Example 3.9.4. *Given* $y = \log_5(2x+3)$, *find* y'.

Solution:

$$y' = \frac{\frac{d}{dx}(2x+3)}{(2x+3)(\ln 5)} = \frac{2}{(2x+3)(\ln 5)}.$$

Example 3.9.5. *Given* $y = \ln(-4x)\cos(3x)$, *find* $\frac{dy}{dx}$.

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \ln(-4x)\frac{d}{dx}(\cos(3x)) + (\cos(3x))\frac{d}{dx}(\ln(-4x)) && \textit{Use product rule.}\\
&= \ln(-4x)\left(-\sin(3x)\right)\frac{d}{dx}(3x) + (\cos(3x))\left(\frac{\frac{d}{dx}(-4x)}{-4x}\right) && \textit{Differentiate.}\\
&= -\ln(-4x)\sin(3x)(3) + (\cos(3x))\left(\frac{-4}{-4x}\right) && \textit{Differentiate.}\\
&= -3\ln(-4x)\sin(3x) + \frac{\cos(3x)}{x}. && \textit{Simplify.}
\end{aligned}$$

Example 3.9.6. *Given* $y = \ln\left[\frac{x+1}{\sqrt{3x+4}}\right]$*, find* y'*.*

Solution: *Without using properties of logarithms, we can differentiate to obtain*

$$\begin{aligned} y' &= \frac{1}{\left[\frac{x+1}{\sqrt{3x+4}}\right]} \cdot \frac{1 \cdot \sqrt{3x+4} - (x+1) \cdot \frac{3}{2\sqrt{3x+4}}}{(\sqrt{3x+4})^2} \\ &= \frac{1}{x+1} \cdot \frac{(3x+4) - (x+1) \cdot \frac{3}{2}}{(3x+4)} \\ &= \frac{1}{x+1} - \frac{3}{2(3x+4)}. \end{aligned}$$

Or an alternate method would be to first can rewrite y *using properties of logarithms:*

$$\begin{aligned} y &= \ln\left[\frac{x+1}{\sqrt{3x+4}}\right] \\ &= \ln(x+1) - \ln(\sqrt{3x+4}) \\ &= \ln(x+1) - \ln(3x+4)^{1/2}. \end{aligned}$$

So,

$$y = \ln(x+1) - \frac{1}{2}\ln(3x+4).$$

From here, we can take the derivative of y *more easily than before:*

$$y' = \frac{1}{x+1}(1) - \frac{1}{2} \cdot \frac{1}{3x+4}(3) = \frac{1}{x+1} - \frac{3}{2(3x+4)}.$$

Example 3.9.7. *Given $y = \log_5(xe^x) + \sec^3(5x)$, find y'.*

Solution: *Use logarithm rules to rewrite.*

$$y = \log_5 x + \log_5 e^x + \sec^3(5x),$$

From here we can take the derivative of y:

$$\begin{aligned}
y' &= \frac{d}{dx}(\log_5 x) + \frac{d}{dx}(\log_5 e^x) + 3\left[(\sec^{3-1}(5x)\right]\left[\frac{d}{dx}(\sec(5x))\right] \\
&= \frac{1}{x\ln 5} + \frac{\frac{d}{dx}(e^x)}{e^x \ln 5} + 3[\sec^2(5x)] \cdot [\sec(5x)\tan(5x)] \cdot \left[\frac{d}{dx}(5x)\right] \\
&= \frac{1}{x\ln 5} + \frac{e^x}{e^x \ln 5} + 3[\sec^2(5x)] \cdot [\sec(5x)\tan(5x)] \cdot [5] \\
&= \frac{1}{x\ln 5} + \frac{1}{\ln 5} + 15[\sec^3(5x)]\tan(5x).
\end{aligned}$$

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Example 3.9.8. *Given $y = \ln(\sec x + \tan x)$, find y' and y''.*

Solution: *For the first derivative of y:*

$$\begin{aligned} y' &= \frac{\frac{d}{dx}(\sec x + \tan x)}{\sec x + \tan x} \\ &= \frac{(\sec x \tan x + \sec^2 x)}{\sec x + \tan x} \\ &= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \\ &= \sec x. \end{aligned}$$

For the second derivative of y we find the derivative of $\sec x$.

$$y'' = \sec x \tan x.$$

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Example 3.9.9. *Find $f'(x)$ if $f(x) = \ln|x|$.*

Solution: *Recall that*

$$f(x) = \begin{cases} \ln(x) & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

It follows that

$$f'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{-x} \cdot (-1) = \frac{1}{x} & \text{if } x < 0 \end{cases}$$

Therefore, $f'(x) = \frac{1}{x}$ for all $x \neq 0$. Thus we have shown that,

If $f(x) = \ln|x|$, $f'(x) = \frac{1}{x}$, $x \neq 0$.

Finding derivatives of complex functions involving powers, products, or quotients are often too complicated for the techniques we have seen. We can use logarithms to find the derivatives of some of these complex functions. This method is called **logarithmic differentiation**. The steps are as follows:

1. **Logarithmic Differentiation Procedure.**

    1. *Take the logarithm of both sides.*
    2. *Simplify using properties of logarithms.*
    3. *Differentiate implicitly.*
    4. *Solve for $\frac{dy}{dx}$.*
    5. *Substitute for y and write $\frac{dy}{dx}$ in terms of x (if possible).*

Example 3.9.10. *Given $y = (\cot x)^{\ln x}$, find y'.*

Solution:

$$
\begin{aligned}
y &= (\cot x)^{\ln x} \\
\ln y &= \ln\left((\cot x)^{\ln x}\right) && \textit{Take the logarithm of both sides.} \\
\ln y &= [\ln x][\ln(\cot x)] && \textit{Rewrite using logarithm properties.} \\
\frac{d}{dx}(ln(y)) &= [\ln x]\cdot\frac{d}{dx}[\ln(\cot x)] + [\ln(\cot x)]\cdot\frac{d}{dx}[\ln x] && \textit{Differentiate both sides with respect to } x. \\
\frac{1}{y}\frac{dy}{dx} &= [\ln x]\cdot\left[\frac{1}{\cot x}(-\csc^2 x)\right] + [\ln(\cot x)]\cdot\frac{1}{x} \\
\frac{1}{y}\frac{dy}{dx} &= \frac{(\ln x)(-\csc^2 x)}{\cot x} + \frac{\ln(\cot x)}{x} \\
\frac{dy}{dx} &= \left[\frac{(\ln x)(-\csc^2 x)}{\cot x} + \frac{\ln(\cot x)}{x}\right] y && \textit{Solve for } \tfrac{dy}{dx}. \\
\frac{dy}{dx} &= \left[\frac{(\ln x)(-\csc^2 x)}{\cot x} + \frac{\ln(\cot x)}{x}\right] (\cot x)^{\ln x} && \textit{Substitute for } y.
\end{aligned}
$$

Example 3.9.11. *Given $y^x = x^y$, find $y' = \frac{dy}{dx}$.*

Solution:

$$
\begin{aligned}
y^x &= x^y \\
\ln y^x &= \ln x^y && \textit{Take the logarithm of both sides.} \\
x \ln y &= y \ln x && \textit{Rewrite using logarithm properties.} \\
x \cdot \frac{d(\ln y)}{dx} + \ln y \cdot \frac{d(x)}{dx} &= y \cdot \frac{d(\ln x)}{dx} + (\ln x) \cdot \frac{dy}{dx} && \textit{Differentiate both sides.} \\
x \frac{1}{y} \frac{dy}{dx} + (\ln y)(1) &= y \frac{1}{x} + (\ln x) \frac{dy}{dx} \\
\frac{x}{y} \frac{dy}{dx} - (\ln x) \frac{dy}{dx} &= \frac{y}{x} - \ln y && \textit{Collect the terms with } \tfrac{dy}{dx} \textit{ on one side of equation.} \\
\left(\frac{x}{y} - \ln x \right) \frac{dy}{dx} &= \frac{y}{x} - \ln x && \textit{Factor.} \\
\frac{dy}{dx} &= \frac{\frac{y}{x} - \ln y}{\frac{x}{y} - (\ln x)} && \textit{Divide to get } \tfrac{dy}{dx} \textit{ by itself.} \\
\Rightarrow \frac{dy}{dx} &= \frac{y^2 - xy \ln y}{x^2 - xy \ln x} && \textit{Simplify.}
\end{aligned}
$$

Exercises with Logarithmic Functions

Find the derivatives of the given functions.

1. $H(z) = 5\log_2(z) + 2^z$
2. $G(x) = \frac{\ln x}{4x^2+1}$
3. $F(t) = \ln(5t^2 + 4)$
4. $y = x \ln x - x$
5. $J(s) = e^{\ln s}$
6. $G(t) = (\arctan t)(\ln(4t^2))$
7. $y = \ln(x + \sqrt{3x - 1})$
8. $F(x) = (\log_2(5x))^3$
9. $s = 2t \log(t^4)$
10. $y = \frac{2-\ln x}{2+\ln x}$

Write the equation of the tangent line to the curve for problems 11 and 12.

11. $y = \ln(x^4 - 15)$ at the point $(2, 0)$
12. $y = \ln(xe^x)$ at the point $(1, 1)$

Use logarithmic differentiation to find the derivative of the given functions in problems 13-16.

13. $y = \frac{(2x+1)^3(4x^2+5x+1)^2}{\sqrt{3x-7}}$
14. $y = (\sin x)^{2x}$
15. $y = \sqrt{x}(3x + 1)^4 e^{2x}$
16. $y = x^{\cos x}$

