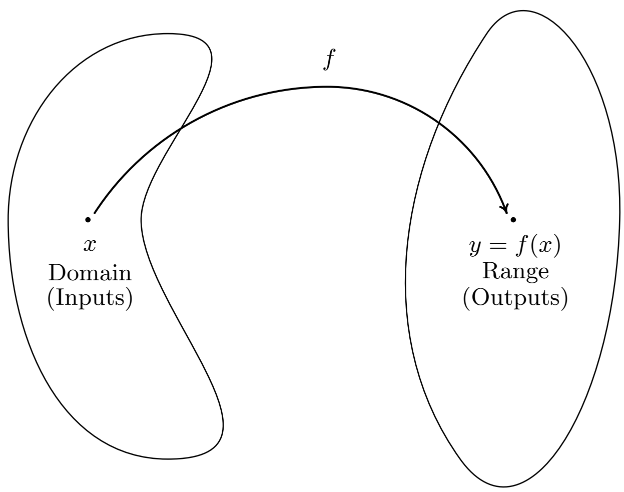
1.5 FUNCTION NOTATION

In Definition 1.4, we described a function as a special kind of relation − one in which each x-coordinate is matched with only one y-coordinate. In this section, we focus more on the **process** by which the x is matched with the y. If we think of the domain of a function as a set of **inputs** and the range as a set of **outputs**, we can think of a function f as a process by which each input x is matched with only one output y. Since the output is completely determined by the input x and the process f, we symbolize the output with **function notation**: '$f(x)$', read 'f **of** x.' In this case, the parentheses here do not indicate multiplication, as they do elsewhere in algebra. This could cause confusion if the context is not clear. In other words, $f(x)$ is the **output** which results by applying the **process** f to the **input** x. This relationship is typically visualized using a diagram similar to the one below.



The value of y is completely dependent on the choice of x. For this reason, x is often called the **independent variable**, or **argument** of f, whereas y is often called the **dependent variable**.

As we shall see, the process of a function f is usually described using an algebraic formula. For example, suppose a function f takes a real number and performs the following two steps, in sequence

1. multiply by 3
2. add 4

If we choose 5 as our input, in step 1 we multiply by 3 to get $(5)(3) = 15$. In step 2, we add 4 to our result from step 1 which yields $15 + 4 = 19$. Using function notation, we would write $f(5) = 19$ to indicate that the result of applying the process f to the input 5 gives the output 19. In general, if we use x for the input, applying step 1 produces $3x$. Following with step 2 produces $3x + 4$ as our final output. Hence for an input x, we get the output $f(x) = 3x + 4$. Notice that to check our formula for the case $x = 5$, we **replace** the occurrence of x in the formula for $f(x)$ with 5 to get $f(5) = 3(5) + 4 = 15 + 4 = 19$, as required.

Example 1.5.1. Suppose a function g is described by applying the following steps, in sequence

1. add 4
2. multiply by 3

Determine $g(5)$ and find an expression for $g(x)$.

Solution. Starting with 5, step 1 gives $5+4=9$. Continuing with step 2, we get $(3)(9)=27$. To find a formula for $g(x)$, we start with our input x. Step 1 produces $x+4$. We now wish to multiply this entire quantity by 3, so we use a parentheses: $3(x+4)=3x+12$. Hence, $g(x)=3x+12$. We can check our formula by replacing x with 5 to get $g(5)=3(5)+12=15+12=27\,\checkmark$. □

Most of the functions we will encounter in College Algebra will be described using formulas like the ones we developed for $f(x)$ and $g(x)$ above. Evaluating formulas using this function notation is a key skill for success in this and many other math courses.

Example 1.5.2. For $f(x)=-x^2+3x+4$, find and simplify

1. $f(-1)$, $f(0)$, $f(2)$
2. $f(2x)$, $2f(x)$
3. $f(x+2)$, $f(x)+2$, $f(x)+f(2)$

Solution.

1. To find $f(-1)$, we replace every occurrence of x in the expression $f(x)$ with -1

$$\begin{aligned} f(-1) &= -(-1)^2+3(-1)+4 \\ &= -(1)+(-3)+4 \\ &= 0 \end{aligned}$$

   Similarly, $f(0)=-(0)^2+3(0)+4=4$, and $f(2)=-(2)^2+3(2)+4=-4+6+4=6$.

2. To find $f(2x)$, we replace every occurrence of x with the quantity $2x$

$$\begin{aligned} f(2x) &= -(2x)^2+3(2x)+4 \\ &= -(4x^2)+(6x)+4 \\ &= -4x^2+6x+4 \end{aligned}$$

   The expression $2f(x)$ means we multiply the expression $f(x)$ by 2

$$\begin{aligned} 2f(x) &= 2\left(-x^2+3x+4\right) \\ &= -2x^2+6x+8 \end{aligned}$$

Note the difference between the answers for $f(2x)$ and $2f(x)$. For $f(2x)$, we are multiplying the **input** by 2; for $2f(x)$, we are multiplying the **output** by 2. As we see, we get entirely different results. Also note the practice of using parentheses when substituting one algebraic expression into another; we highly recommend this practice as it will reduce careless errors.

3. To find $f(x+2)$, we replace every occurrence of x with the quantity $x+2$

$$\begin{aligned} f(x+2) &= -(x+2)^2+3(x+2)+4 \\ &= -\left(x^2+4x+4\right)+(3x+6)+4 \\ &= -x^2-4x-4+3x+6+4 \\ &= -x^2-x+6 \end{aligned}$$

To find $f(x)+2$, we add 2 to the expression for $f(x)$

$$\begin{aligned} f(x)+2 &= \left(-x^2+3x+4\right)+2 \\ &= -x^2+3x+6 \end{aligned}$$

Once again, we see there is a dramatic difference between modifying the input and modifying the output. Finally, in $f(x)+f(2)$ we are adding the value $f(2)$ to the expression $f(x)$. From our work above, we see $f(2)=6$ so that

$$\begin{aligned} f(x)+f(2) &= \left(-x^2+3x+4\right)+6 \\ &= -x^2+3x+10 \end{aligned}$$

Notice that $f(x+2)$, $f(x)+2$ and $f(x)+f(2)$ are three **different** expressions. Even though function notation uses parentheses, as does multiplication, there is no general 'distributive property' of function notation. □

Suppose we wish to find $r(3)$ for $r(x)=\dfrac{2x}{x^2-9}$. Substitution gives

$$r(3)=\frac{2(3)}{(3)^2-9}=\frac{6}{0},$$

which is undefined. The number 3 is not an allowable input to the function r; in other words, 3 is not in the domain of r. Which other real numbers are forbidden in this formula? We think back to arithmetic. The reason $r(3)$ is undefined is because substitution results in a division by 0. To determine which other numbers result in such a transgression, we set the denominator equal to 0 and solve

$$\begin{aligned} x^2-9 &= 0 \\ x^2 &= 9 \\ \sqrt{x^2} &= \sqrt{9} \quad \text{extract square roots} \\ x &= \pm 3 \end{aligned}$$

As long as we substitute numbers other than 3 and -3, the expression $r(x)$ is a real number. Hence, we write our domain in interval notation as $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$. When a formula for a function is given, we assume the function is valid for all real numbers which make arithmetic sense when substituted into the formula. This set of numbers is often called the **implied domain**[1] of the function. At this stage, there are only two mathematical sins we need to avoid: division by 0 and extracting even roots of negative numbers. The following example illustrates these concepts.

Example 1.5.3. Find the domain[2] of the following functions.

1. $f(x)=\dfrac{2}{1-\dfrac{4x}{x-3}}$
2. $g(x)=\sqrt{4-3x}$
3. $h(x)=\sqrt[5]{4-3x}$
4. $r(x)=\dfrac{4}{6-\sqrt{x+3}}$
5. $I(x)=\dfrac{3x^2}{x}$

Solution.

1. In the expression for f, there are two denominators. We need to make sure neither of them is 0. To that end, we set each denominator equal to 0 and solve. For the 'small' denominator, we get $x-3=0$ or $x=3$. For the 'large' denominator

$$\begin{aligned} 1-\frac{4x}{x-3} &= 0 \\ 1 &= \frac{4x}{x-3} \\ (1)(x-3) &= \left(\frac{4x}{\cancel{x-3}}\right)\cancel{(x-3)} \quad \text{clear denominators} \\ x-3 &= 4x \\ -3 &= 3x \\ -1 &= x \end{aligned}$$

So we get two real numbers which make denominators 0, namely $x=-1$ and $x=3$. Our domain is all real numbers **except** -1 and 3: $(-\infty,-1)\cup(-1,3)\cup(3,\infty)$.

[1] or, 'implicit domain'
[2] The word 'implied' is, well, implied.

2. The potential disaster for g is if the radicand[3] is negative. To avoid this, we set $4-3x \geq 0$

$$\begin{aligned} 4-3x &\geq 0 \\ 4 &\geq 3x \\ \frac{4}{3} &\geq x \end{aligned}$$

   Hence, as long as $x \leq \frac{4}{3}$, the expression $4-3x \geq 0$, and the formula $g(x)$ returns a real number. Our domain is $\left(-\infty, \frac{4}{3}\right]$.

3. The formula for $h(x)$ is hauntingly close to that of $g(x)$ with one key difference $-$ whereas the expression for $g(x)$ includes an even indexed root (namely a square root), the formula for $h(x)$ involves an odd indexed root (the fifth root.) Since odd roots of real numbers (even negative real numbers) are real numbers, there is no restriction on the inputs to h. Hence, the domain is $(-\infty, \infty)$.

4. To find the domain of r, we notice that we have two potentially hazardous issues: not only do we have a denominator, we have a square root in that denominator. To satisfy the square root, we set the radicand $x+3 \geq 0$ so $x \geq -3$. Setting the denominator equal to zero gives

$$\begin{aligned} 6-\sqrt{x+3} &= 0 \\ 6 &= \sqrt{x+3} \\ 6^2 &= \left(\sqrt{x+3}\right)^2 \\ 36 &= x+3 \\ 33 &= x \end{aligned}$$

   Since we squared both sides in the course of solving this equation, we need to check our answer. Sure enough, when $x = 33$, $6-\sqrt{x+3} = 6-\sqrt{36} = 0$, and so $x = 33$ will cause problems in the denominator. At last we can find the domain of r: we need $x \geq -3$, but $x \neq 33$. Our final answer is $[-3, 33) \cup (33, \infty)$.

5. It's tempting to simplify $I(x) = \frac{3x^2}{x} = 3x$, and, since there are no longer any denominators, claim that there are no longer any restrictions. However, in simplifying $I(x)$, we are assuming $x \neq 0$, since $\frac{0}{0}$ is undefined.[4] Proceeding as before, we find the domain of I to be all real numbers except 0: $(-\infty, 0) \cup (0, \infty)$. □

It is worth reiterating the importance of finding the domain of a function **before** simplifying, as evidenced by the function I in the previous example. Even though the formula $I(x)$ simplifies to $3x$, it would be inaccurate to write $I(x) = 3x$ without adding the stipulation that $x \neq 0$. It would be analogous to not reporting taxable income or some other sin of omission.

[3]The 'radicand' is the expression 'inside' the radical.

[4]More precisely, the fraction $\frac{0}{0}$ is an 'indeterminant form'. Much time will be spent in Calculus wrestling with such creatures.

Our next example shows how a function can be used to model real-world phenomena.

Example 1.5.4. The height h in feet of a model rocket above the ground t seconds after lift off is given by

$$h(t) = \begin{cases} -5t^2 + 100t, & \text{if} \quad 0 \leq t \leq 20 \\ 0, & \text{if} \quad t > 20 \end{cases}$$

Find and interpret $h(10)$ and $h(60)$.

Solution. There are a few qualities of h which may be off-putting. The first is that, unlike previous examples, the independent variable is t, not x. In this context, t is chosen because it represents time. The second is that the function is broken up into two rules: one formula for values of t between 0 and 20 inclusive, and another for values of t greater than 20. To find $h(10)$, we first notice that 10 is between 0 and 20 so we use the first formula listed: $h(t) = -5t^2 + 100t$. Hence, $h(10) = -5(10)^2 + 100(10) = 500$. In terms of the model rocket, this means that 10 seconds after lift off, the model rocket is 500 feet above the ground. To find $h(60)$, we note that 60 is greater than 20, so we use the rule $h(t) = 0$. This function returns a value of 0 regardless of what value is substituted in for t, so $h(60) = 0$. This means that 60 seconds after lift off, the rocket is 0 feet above the ground; in other words, a minute after lift off, the rocket has already returned to earth. □

The type of function in the previous example is called a **piecewise-defined** function, or 'piecewise' function for short. Many real-world phenomena (e.g. postal rates,[5] income tax formulas[6]) are modeled by such functions. Also note that the domain of h in the above example is restricted to $t \geq 0$. For example, $h(-3)$ is not defined because $t = -3$ doesn't satisfy any of the conditions in any of the function's pieces. There is no inherent arithmetic reason which prevents us from calculating, say, $-5(-3)^2 + 100(-3)$, it's just that in this applied setting, $t = -3$ is meaningless. In this case, we say h has an **applied domain**[7] of $[0, \infty)$

[5] See the United States Postal Service website http://www.usps.com/prices/first-class-mail-prices.htm

[6] See the Internal Revenue Service's website http://www.irs.gov/pub/irs-pdf/i1040tt.pdf

[7] or, 'explicit domain'