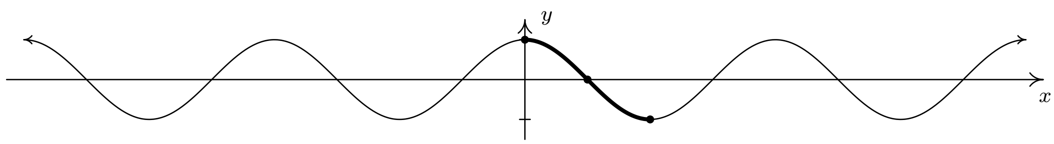
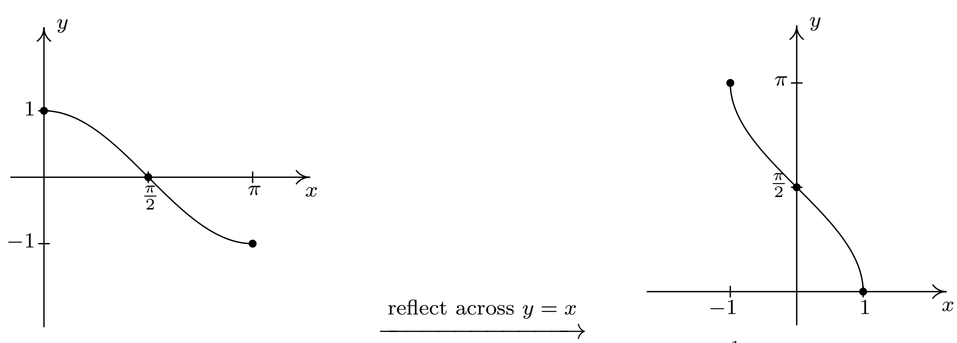
10.6 The Inverse Trigonometric Functions

As the title indicates, in this section we concern ourselves with finding inverses of the (circular) trigonometric functions. Our immediate problem is that, owing to their periodic nature, none of the six circular functions is one-to-one. To remedy this, we restrict the domains of the circular functions in the same way we restricted the domain of the quadratic function in Example 5.2.3 in Section 5.2 to obtain a one-to-one function. We first consider $f(x) = \cos(x)$. Choosing the interval $[0, \pi]$ allows us to keep the range as $[-1, 1]$ as well as the properties of being smooth and continuous.



Restricting the domain of $f(x) = \cos(x)$ to $[0, \pi]$.

Recall from Section 5.2 that the inverse of a function f is typically denoted f^{-1}. For this reason, some textbooks use the notation $f^{-1}(x) = \cos^{-1}(x)$ for the inverse of $f(x) = \cos(x)$. The obvious pitfall here is our convention of writing $(\cos(x))^2$ as $\cos^2(x)$, $(\cos(x))^3$ as $\cos^3(x)$ and so on. It is far too easy to confuse $\cos^{-1}(x)$ with $\frac{1}{\cos(x)} = \sec(x)$ so we will not use this notation in our text.[1] Instead, we use the notation $f^{-1}(x) = \arccos(x)$, read 'arc-cosine of x.' To understand the 'arc' in 'arccosine', recall that an inverse function, by definition, reverses the process of the original function. The function $f(t) = \cos(t)$ takes a real number input t, associates it with the angle $\theta = t$ radians, and returns the value $\cos(\theta)$. Digging deeper,[2] we have that $\cos(\theta) = \cos(t)$ is the x-coordinate of the terminal point on the Unit Circle of an oriented arc of length $|t|$ whose initial point is $(1, 0)$. Hence, we may view the inputs to $f(t) = \cos(t)$ as oriented arcs and the outputs as x-coordinates on the Unit Circle. The function f^{-1}, then, would take x-coordinates on the Unit Circle and return oriented arcs, hence the 'arc' in arccosine. Below are the graphs of $f(x) = \cos(x)$ and $f^{-1}(x) = \arccos(x)$, where we obtain the latter from the former by reflecting it across the line $y = x$, in accordance with Theorem 5.3.

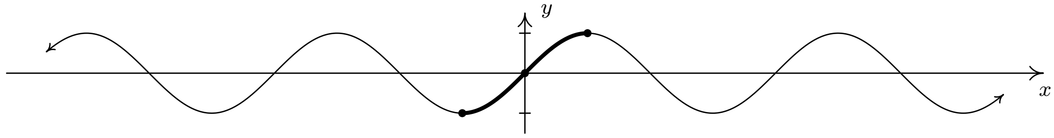


$f(x) = \cos(x)$, $0 \leq x \leq \pi$ switch x and y coordinates $f^{-1}(x) = \arccos(x)$.

We restrict $g(x) = \sin(x)$ in a similar manner, although the interval of choice is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

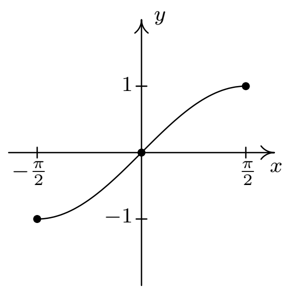
[1] But be aware that many books do! As always, be sure to check the context!

[2] See page 604 if you need a review of how we associate real numbers with angles in radian measure.



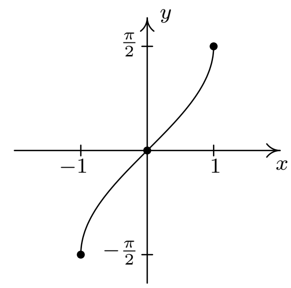
Restricting the domain of $f(x) = \sin(x)$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

It should be no surprise that we call $g^{-1}(x) = \arcsin(x)$, read 'arc-sine of x.'



$g(x) = \sin(x)$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

reflect across $y = x$ $\longrightarrow$ switch x and y coordinates



$g^{-1}(x) = \arcsin(x)$.

We list some important facts about the arccosine and arcsine functions in the following theorem.

THEOREM 10.26. Properties of the Arccosine and Arcsine Functions

- Properties of $F(x) = \arccos(x)$
  - Domain: $[-1, 1]$
  - Range: $[0, \pi]$
  - $\arccos(x) = t$ if and only if $0 \leq t \leq \pi$ and $\cos(t) = x$
  - $\cos(\arccos(x)) = x$ provided $-1 \leq x \leq 1$
  - $\arccos(\cos(x)) = x$ provided $0 \leq x \leq \pi$
- Properties of $G(x) = \arcsin(x)$
  - Domain: $[-1, 1]$
  - Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
  - $\arcsin(x) = t$ if and only if $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ and $\sin(t) = x$
  - $\sin(\arcsin(x)) = x$ provided $-1 \leq x \leq 1$
  - $\arcsin(\sin(x)) = x$ provided $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
  - additionally, arcsine is odd

Everything in Theorem 10.26 is a direct consequence of the facts that $f(x) = \cos(x)$ for $0 \leq x \leq \pi$ and $F(x) = \arccos(x)$ are inverses of each other as are $g(x) = \sin(x)$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and $G(x) = \arcsin(x)$.

It is time for an example.

Example 10.6.1.

1. Find the exact values of the following.

   (a) $\arccos\left(\frac{1}{2}\right)$
   (b) $\arcsin\left(\frac{\sqrt{2}}{2}\right)$
   (c) $\arccos\left(-\frac{\sqrt{2}}{2}\right)$
   (d) $\arcsin\left(-\frac{1}{2}\right)$
   (e) $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right)$
   (f) $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right)$
   (g) $\cos\left(\arccos\left(-\frac{3}{5}\right)\right)$
   (h) $\sin\left(\arccos\left(-\frac{3}{5}\right)\right)$

2. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.

   (a) $\tan\left(\arccos\left(x\right)\right)$
   (b) $\cos\left(2\arcsin(x)\right)$

Solution.

1. (a) To find $\arccos\left(\frac{1}{2}\right)$, we need to find the real number t (or, equivalently, an angle measuring t radians) which lies between 0 and π with $\cos(t) = \frac{1}{2}$. We know $t = \frac{\pi}{3}$ meets these criteria, so $\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

   (b) The value of $\arcsin\left(\frac{\sqrt{2}}{2}\right)$ is a real number t between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\sin(t) = \frac{\sqrt{2}}{2}$. The number we seek is $t = \frac{\pi}{4}$. Hence, $\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

   (c) The number $t = \arccos\left(-\frac{\sqrt{2}}{2}\right)$ lies in the interval $[0, \pi]$ with $\cos(t) = -\frac{\sqrt{2}}{2}$. Our answer is $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

   (d) To find $\arcsin\left(-\frac{1}{2}\right)$, we seek the number t in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin(t) = -\frac{1}{2}$. The answer is $t = -\frac{\pi}{6}$ so that $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

   (e) Since $0 \leq \frac{\pi}{6} \leq \pi$, we could simply invoke Theorem 10.26 to get $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$. However, in order to make sure we understand *why* this is the case, we choose to work the example through using the definition of arccosine. Working from the inside out, $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \arccos\left(\frac{\sqrt{3}}{2}\right)$. Now, $\arccos\left(\frac{\sqrt{3}}{2}\right)$ is the real number t with $0 \leq t \leq \pi$ and $\cos(t) = \frac{\sqrt{3}}{2}$. We find $t = \frac{\pi}{6}$, so that $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$.

   (f) Since $\frac{11\pi}{6}$ does not fall between 0 and π, Theorem 10.26 does not apply. We are forced to work through from the inside out starting with $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right) = \arccos\left(\frac{\sqrt{3}}{2}\right)$. From the previous problem, we know $\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$. Hence, $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right) = \frac{\pi}{6}$.

   (g) To help simplify $\cos\left(\arccos\left(-\frac{3}{5}\right)\right)$ let $t = \arccos\left(-\frac{3}{5}\right)$. Then, by definition, $0 \leq t \leq \pi$ and $\cos(t) = -\frac{3}{5}$. Hence, $\cos\left(\arccos\left(-\frac{3}{5}\right)\right) = \cos(t) = -\frac{3}{5}$.

(h) As in the previous example, we let $t = \arccos\left(-\frac{3}{5}\right)$ so that $0 \leq t \leq \pi$ and $\cos(t) = -\frac{3}{5}$. In terms of t, then, we need to find $\sin\left(\arccos\left(-\frac{3}{5}\right)\right) = \sin(t)$. Using the Pythagorean Identity $\cos^2(t) + \sin^2(t) = 1$, we get $\left(-\frac{3}{5}\right)^2 + \sin^2(t) = 1$ or $\sin(t) = \pm\frac{4}{5}$. Since $0 \leq t \leq \pi$, we choose[3] $\sin(t) = \frac{4}{5}$. Hence, $\sin\left(\arccos\left(-\frac{3}{5}\right)\right) = \frac{4}{5}$.

2. (a) We begin rewriting $\tan(\arccos(x))$ using $t = \arccos(x)$. We know that $0 \leq t \leq \pi$ and $\cos(t) = x$, so our goal is to express $\tan(\arccos(x)) = \tan(t)$ in terms of x. This is where identities come into play, but we must be careful to use identities which are defined for all values of t under consideration. In this situation, we have $0 \leq t \leq \pi$, but since the quantity we are looking for, $\tan(t)$, is undefined at $t = \frac{\pi}{2}$, the identities we choose to need to hold for all t in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. Since $\tan(t) = \frac{\sin(t)}{\cos(t)}$, and we know $\cos(t) = x$, all that remains is to find $\sin(t)$ in terms of x and we'll be done.[4] The identity $\cos^2(t) + \sin^2(t) = 1$ holds for all t, in particular the ones in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, so substituting $\cos(t) = x$, we get $x^2 + \sin^2(t) = 1$. Hence, $\sin(t) = \pm\sqrt{1 - x^2}$ and since t belongs to $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, $\sin(t) \geq 0$, so we choose $\sin(t) = \sqrt{1 - x^2}$. Thus, $\tan(t) = \frac{\sin(t)}{\cos(t)} = \frac{\sqrt{1-x^2}}{x}$. To determine the values of x for which this is valid, we first note that $\arccos(x)$ is valid only for $-1 \leq x \leq 1$. Additionally, as we have already mentioned, $\tan(t)$ is not defined when $t = \frac{\pi}{2}$, which means we must exclude $x = \cos\left(\frac{\pi}{2}\right) = 0$. Hence, $\tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x}$ for x in $[-1, 0) \cup (0, 1]$.

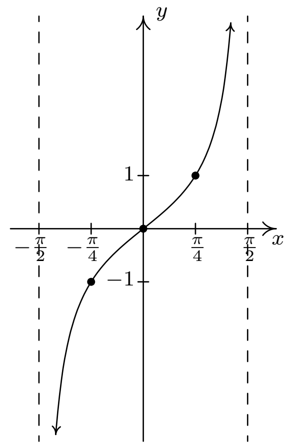
(b) We proceed as in the previous problem by writing $t = \arcsin(x)$ so that t lies in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin(t) = x$. We aim to express $\cos(2\arcsin(x)) = \cos(2t)$ in terms of x. Since $\cos(2t)$ is defined everywhere, we get no additional restrictions on t. We have three choices for rewriting $\cos(2t)$: $\cos^2(t) - \sin^2(t)$, $2\cos^2(t) - 1$ and $1 - 2\sin^2(t)$, each of which is valid for t in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Since we already know $x = \sin(t)$, it is easiest to use the last form. We have $\cos(2\arcsin(x)) = \cos(2t) = 1 - 2\sin^2(t) = 1 - 2x^2$. Since $\arcsin(x)$ is defined only for $-1 \leq x \leq 1$, the equivalence $\cos(2\arcsin(x)) = 1 - 2x^2$ is valid on $[-1, 1]$. □

A few remarks about Example 10.6.1 are in order. Most of the common errors encountered in dealing with the inverse circular functions come from the need to restrict the domains of the original functions so that they are one-to-one. One instance of this phenomenon is the fact that $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right) = \frac{\pi}{6}$ as opposed to $\frac{11\pi}{6}$. This is the exact same phenomenon discussed in Section 5.2 when we saw $\sqrt{(-2)^2} = 2$ as opposed to -2. Additionally, even though the expression $1 - 2x^2$ is defined for all real numbers, the equivalence $\cos(2\arcsin(x)) = 1 - 2x^2$ is valid for only $-1 \leq x \leq 1$. This is akin to the fact that while the expression x is defined for all real numbers, the equivalence $\left(\sqrt{x}\right)^2 = x$ is valid only for $x \geq 0$.

[3] In other words, the angle $\theta = t$ radians is a Quadrant I or II angle where sine is nonnegative.

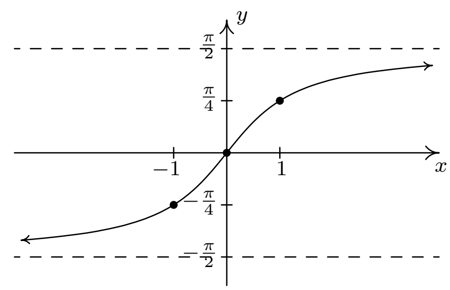
[4] Alternatively, we could use the identity: $1 + \tan^2(t) = \sec^2(t)$. Since we are given $x = \cos(t)$, we know $\sec(t) = \frac{1}{\cos(t)} = \frac{1}{x}$. The reader is invited to work through this approach to see what, if any, difficulties arise.

The next pair of functions we wish to discuss are the inverses of tangent and cotangent. First, we restrict $f(x) = \tan(x)$ to its fundamental cycle on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to obtain $f^{-1}(x) = \arctan(x)$. Among other things, note that the *vertical* asymptotes $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ of the graph of $f(x) = \tan(x)$ become the *horizontal* asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ of the graph of $f^{-1}(x) = \arctan(x)$.



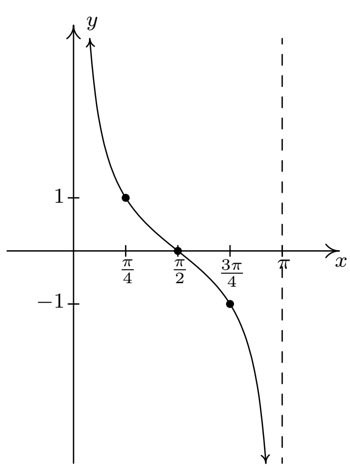
$f(x) = \tan(x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

reflect across $y = x$ $\longrightarrow$ switch x and y coordinates



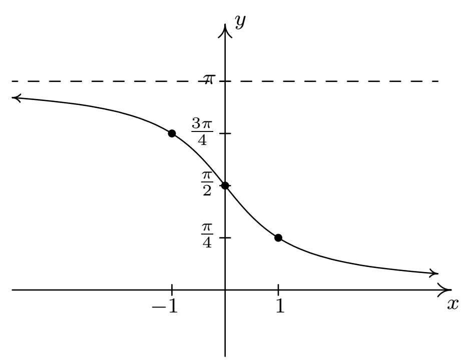
$f^{-1}(x) = \arctan(x)$.

Next, we restrict $g(x) = \cot(x)$ to its fundamental cycle on $(0, \pi)$ to obtain $g^{-1}(x) = \operatorname{arccot}(x)$. Once again, the vertical asymptotes $x = 0$ and $x = \pi$ of the graph of $g(x) = \cot(x)$ become the horizontal asymptotes $y = 0$ and $y = \pi$ in the graph of $g^{-1}(x) = \operatorname{arccot}(x)$.



$g(x) = \cot(x)$, $0 < x < \pi$.

reflect across $y = x$ $\longrightarrow$ switch x and y coordinates



$g^{-1}(x) = \operatorname{arccot}(x)$.

Theorem 10.27. **Properties of the Arctangent and Arcotangent Functions**

- Properties of $F(x) = \arctan(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
  - as $x \to -\infty$, $\arctan(x) \to -\frac{\pi}{2}^{+}$; as $x \to \infty$, $\arctan(x) \to \frac{\pi}{2}^{-}$
  - $\arctan(x) = t$ if and only if $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and $\tan(t) = x$
  - $\arctan(x) = \operatorname{arccot}\left(\frac{1}{x}\right)$ for $x > 0$
  - $\tan(\arctan(x)) = x$ for all real numbers x
  - $\arctan(\tan(x)) = x$ provided $-\frac{\pi}{2} < x < \frac{\pi}{2}$
  - additionally, arctangent is odd
- Properties of $G(x) = \operatorname{arccot}(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $(0, \pi)$
  - as $x \to -\infty$, $\operatorname{arccot}(x) \to \pi^{-}$; as $x \to \infty$, $\operatorname{arccot}(x) \to 0^{+}$
  - $\operatorname{arccot}(x) = t$ if and only if $0 < t < \pi$ and $\cot(t) = x$
  - $\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$ for $x > 0$
  - $\cot(\operatorname{arccot}(x)) = x$ for all real numbers x
  - $\operatorname{arccot}(\cot(x)) = x$ provided $0 < x < \pi$

Example 10.6.2.

1. Find the exact values of the following.

   (a) $\arctan(\sqrt{3})$

   (b) $\operatorname{arccot}(-\sqrt{3})$

   (c) $\cot(\operatorname{arccot}(-5))$

   (d) $\sin\left(\arctan\left(-\frac{3}{4}\right)\right)$

2. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.

   (a) $\tan(2\arctan(x))$

   (b) $\cos(\operatorname{arccot}(2x))$

Solution.

1. (a) We know $\arctan(\sqrt{3})$ is the real number t between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\tan(t) = \sqrt{3}$. We find $t = \frac{\pi}{3}$, so $\arctan(\sqrt{3}) = \frac{\pi}{3}$.

(b) The real number $t = \text{arccot}(-\sqrt{3})$ lies in the interval $(0, \pi)$ with $\cot(t) = -\sqrt{3}$. We get $\text{arccot}(-\sqrt{3}) = \frac{5\pi}{6}$.

(c) We can apply Theorem 10.27 directly and obtain $\cot(\text{arccot}(-5)) = -5$. However, working it through provides us with yet another opportunity to understand why this is the case. Letting $t = \text{arccot}(-5)$, we have that t belongs to the interval $(0, \pi)$ and $\cot(t) = -5$. In terms of t, the expression $\cot(\text{arccot}(-5)) = \cot(t)$, and since $\cot(t) = -5$ by definition, we have $\cot(\text{arccot}(-5)) = -5$.

(d) We start simplifying $\sin\left(\arctan\left(-\frac{3}{4}\right)\right)$ by letting $t = \arctan\left(-\frac{3}{4}\right)$. Then t lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\tan(t) = -\frac{3}{4}$. We seek $\sin\left(\arctan\left(-\frac{3}{4}\right)\right) = \sin(t)$. There are many ways to proceed at this point. The Pythagorean Identity, $1 + \cot^2(t) = \csc^2(t)$ relates the reciprocals of $\sin(t)$ and $\tan(t)$, so this seems a reasonable place to start. Since $\tan(t) = -\frac{3}{4}$, $\cot(t) = \frac{1}{\tan(t)} = -\frac{4}{3}$. We get $1 + \left(-\frac{4}{3}\right)^2 = \csc^2(t)$ so that $\csc(t) = \pm\frac{5}{3}$, and, hence, $\sin(t) = \pm\frac{3}{5}$. Since $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and $\tan(t) = -\frac{3}{4} < 0$, it must be the case that t lies between $-\frac{\pi}{2}$ and 0. As a result, we choose $\sin(t) = -\frac{3}{5}$.

2. (a) If we let $t = \arctan(x)$, then $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and $\tan(t) = x$. We look for a way to express $\tan(2\arctan(x)) = \tan(2t)$ in terms of x. Before we get started using identities, we note that $\tan(2t)$ is undefined when $2t = \frac{\pi}{2} + \pi k$ for integers k, which means we need to exclude any of the values $t = \frac{\pi}{4} + \frac{\pi}{2}k$, where k is an integer, which lie in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We find that we need to discard $t = \pm\frac{\pi}{4}$ from the discussion, so we are now working with t in $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \cup \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Returning to $\arctan(2t)$, we note the double angle identity $\tan(2t) = \frac{2\tan(t)}{1-\tan^2(t)}$, is valid for values of t under consideration, hence we get $\tan(2\arctan(x)) = \tan(2t) = \frac{2\tan(t)}{1-\tan^2(t)} = \frac{2x}{1-x^2}$. To find where this equivalence is valid we first note that the domain of $\arctan(x)$ is all real numbers, so the only exclusions come from the x values which correspond to $t = \pm\frac{\pi}{4}$, the values where $\tan(2t)$ is undefined. Since $x = \tan(t)$, we exclude $x = \tan\left(\pm\frac{\pi}{4}\right) = \pm 1$. Hence, $\tan(2\arctan(x)) = \frac{2x}{1-x^2}$ holds[5] for $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

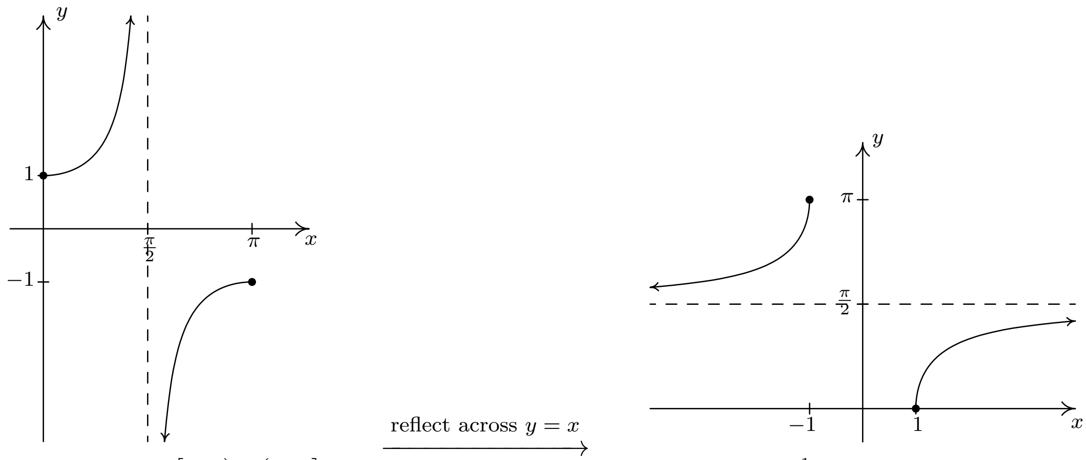
(b) We let $t = \text{arccot}(2x)$ so that $0 < t < \pi$ and $\cot(t) = 2x$. In terms of t, $\cos(\text{arccot}(2x)) = \cos(t)$, and our goal is to express the latter in terms of x. Since $\cos(t)$ is always defined, there are no additional restrictions on t, and we can begin using identities to get expressions for $\cos(t)$ and $\cot(t)$. The identity $\cot(t) = \frac{\cos(t)}{\sin(t)}$ is valid for t in $(0, \pi)$, so if we can get $\sin(t)$ in terms of x, then we can write $\cos(t) = \cot(t)\sin(t)$ and be done. The identity $1 + \cot^2(t) = \csc^2(t)$ holds for all t in $(0, \pi)$ and relates $\cot(t)$ and $\csc(t) = \frac{1}{\sin(t)}$, so we substitute $\cot(t) = 2x$ and get $1 + (2x)^2 = \csc^2(t)$. Thus, $\csc(t) = \pm\sqrt{4x^2 + 1}$ and since t is between 0 and π, we know $\csc(t) > 0$, so we choose $\csc(t) = \sqrt{4x^2 + 1}$. This gives $\sin(t) = \frac{1}{\sqrt{4x^2+1}}$, so that $\cos(t) = \cot(t)\sin(t) = \frac{2x}{\sqrt{4x^2+1}}$. Since $\text{arccot}(2x)$ is defined for all real numbers x and we encountered no additional restrictions on t, we have the equivalence $\cos\left(\text{arccot}(2x)\right) = \frac{2x}{\sqrt{4x^2+1}}$ for all real numbers x. □

[5] Why not just start with $\frac{2x}{1-x^2}$ and find its domain? After all, it gives the correct answer - in this case. There are lots of incorrect ways to arrive at the correct answer. It pays to be careful.

The last two functions to invert are secant and cosecant. There are two generally acceptable ways to restrict the domains of these functions so that they are one-to-one. One approach simplifies the Trigonometry associated with the inverse functions, but complicates the Calculus; the other makes the Calculus easier, but the Trigonometry less so. We present both points of view.

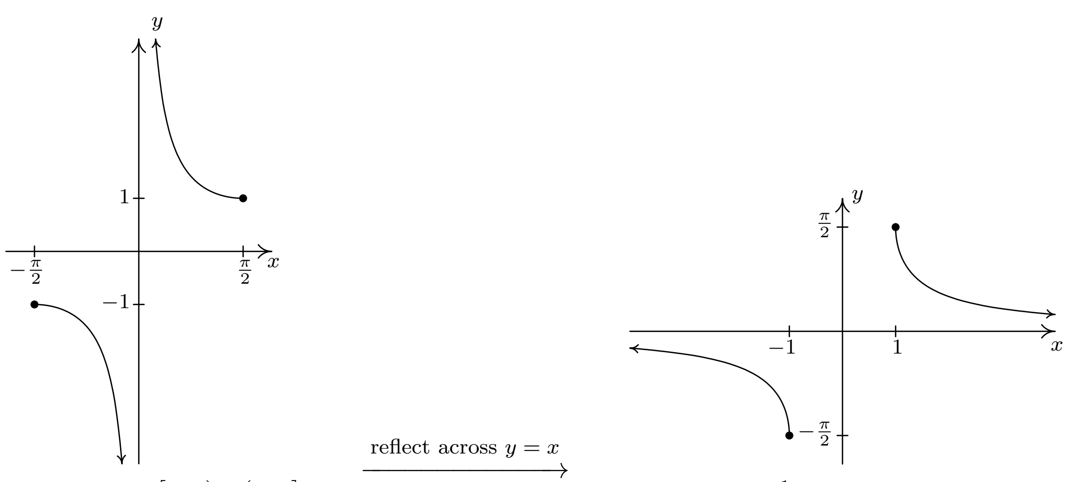
10.6.1 Inverses of Secant and Cosecant: Trigonometry Friendly Approach

In this subsection, we restrict the secant and cosecant functions to coincide with the restrictions on cosine and sine, respectively. For $f(x) = \sec(x)$, we restrict the domain to $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



$f(x) = \sec(x)$ on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ $\qquad$ $f^{-1}(x) = \operatorname{arcsec}(x)$

and we restrict $g(x) = \csc(x)$ to $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.



$f(x) = \csc(x)$ on $\left[0, \frac{\pi}{2}\right) \cup \left(0, \frac{\pi}{2}\right]$ $\qquad$ $f^{-1}(x) = \operatorname{arccsc}(x)$

Note that for both arcsecant and arccosecant, the domain is $(-\infty, -1] \cup [1, \infty)$. Taking a page from Section 2.2, we can rewrite this as $\{x : |x| \geq 1\}$. This is often done in Calculus textbooks, so we include it here for completeness. Using these definitions, we get the following properties of the arcsecant and arccosecant functions.

Theorem 10.28. **Properties of the Arcsecant and Arccosecant Functions**[a]

- Properties of $F(x) = \operatorname{arcsec}(x)$
  - Domain: $\{x : |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
  - as $x \rightarrow -\infty$, $\operatorname{arcsec}(x) \rightarrow \frac{\pi}{2}^{+}$; as $x \rightarrow \infty$, $\operatorname{arcsec}(x) \rightarrow \frac{\pi}{2}^{-}$
  - $\operatorname{arcsec}(x) = t$ if and only if $0 \leq t < \frac{\pi}{2}$ or $\frac{\pi}{2} < t \leq \pi$ and $\sec(t) = x$
  - $\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ provided $|x| \geq 1$
  - $\sec\left(\operatorname{arcsec}(x)\right) = x$ provided $|x| \geq 1$
  - $\operatorname{arcsec}(\sec(x)) = x$ provided $0 \leq x < \frac{\pi}{2}$ or $\frac{\pi}{2} < x \leq \pi$
- Properties of $G(x) = \operatorname{arccsc}(x)$
  - Domain: $\{x : |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
  - as $x \rightarrow -\infty$, $\operatorname{arccsc}(x) \rightarrow 0^{-}$; as $x \rightarrow \infty$, $\operatorname{arccsc}(x) \rightarrow 0^{+}$
  - $\operatorname{arccsc}(x) = t$ if and only if $-\frac{\pi}{2} \leq t < 0$ or $0 < t \leq \frac{\pi}{2}$ and $\csc(t) = x$
  - $\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$ provided $|x| \geq 1$
  - $\csc\left(\operatorname{arccsc}(x)\right) = x$ provided $|x| \geq 1$
  - $\operatorname{arccsc}(\csc(x)) = x$ provided $-\frac{\pi}{2} \leq x < 0$ or $0 < x \leq \frac{\pi}{2}$
  - additionally, arccosecant is odd

[a] . . . assuming the "Trigonometry Friendly" ranges are used.

Example 10.6.3.

1. Find the exact values of the following.

   (a) $\operatorname{arcsec}(2)$

   (b) $\operatorname{arccsc}(-2)$

   (c) $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right)$

   (d) $\cot\left(\operatorname{arccsc}(-3)\right)$

2. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.

   (a) $\tan(\operatorname{arcsec}(x))$

   (b) $\cos(\operatorname{arccsc}(4x))$

Solution.

1. (a) Using Theorem 10.28, we have $\operatorname{arcsec}(2) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

(b) Once again, Theorem 10.28 comes to our aid giving $\text{arccsc}(-2) = \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

(c) Since $\frac{5\pi}{4}$ doesn't fall between 0 and $\frac{\pi}{2}$ or $\frac{\pi}{2}$ and π, we cannot use the inverse property stated in Theorem 10.28. We can, nevertheless, begin by working 'inside out' which yields $\text{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right) = \text{arcsec}(-\sqrt{2}) = \arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

(d) One way to begin to simplify $\cot\left(\text{arccsc}\left(-3\right)\right)$ is to let $t = \text{arccsc}(-3)$. Then, $\csc(t) = -3$ and, since this is negative, we have that t lies in the interval $\left[-\frac{\pi}{2}, 0\right)$. We are after $\cot\left(\text{arccsc}\left(-3\right)\right) = \cot(t)$, so we use the Pythagorean Identity $1 + \cot^2(t) = \csc^2(t)$. Substituting, we have $1 + \cot^2(t) = (-3)^2$, or $\cot(t) = \pm\sqrt{8} = \pm 2\sqrt{2}$. Since $-\frac{\pi}{2} \leq t < 0$, $\cot(t) < 0$, so we get $\cot\left(\text{arccsc}\left(-3\right)\right) = -2\sqrt{2}$.

2. (a) We begin simplifying $\tan(\text{arcsec}(x))$ by letting $t = \text{arcsec}(x)$. Then, $\sec(t) = x$ for t in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, and we seek a formula for $\tan(t)$. Since $\tan(t)$ is defined for all the t values in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, we have no additional restrictions on t. The identity $1 + \tan^2(t) = \sec^2(t)$ is valid for all values t under consideration, and we get substitute $\sec(t) = x$ to get $1 + \tan^2(t) = x^2$. Hence, $\tan(t) = \pm\sqrt{x^2 - 1}$. If t belongs to $\left[0, \frac{\pi}{2}\right)$ then $\tan(t) \geq 0$; if, on the the other hand, t belongs to $\left(\frac{\pi}{2}, \pi\right]$ then $\tan(t) \leq 0$. As a result, we get a piecewise defined function for $\tan(t)$

$$\tan(t) = \begin{cases} \sqrt{x^2 - 1}, & \text{if } 0 \leq t < \frac{\pi}{2} \\ -\sqrt{x^2 - 1}, & \text{if } \frac{\pi}{2} < t \leq \pi \end{cases}$$

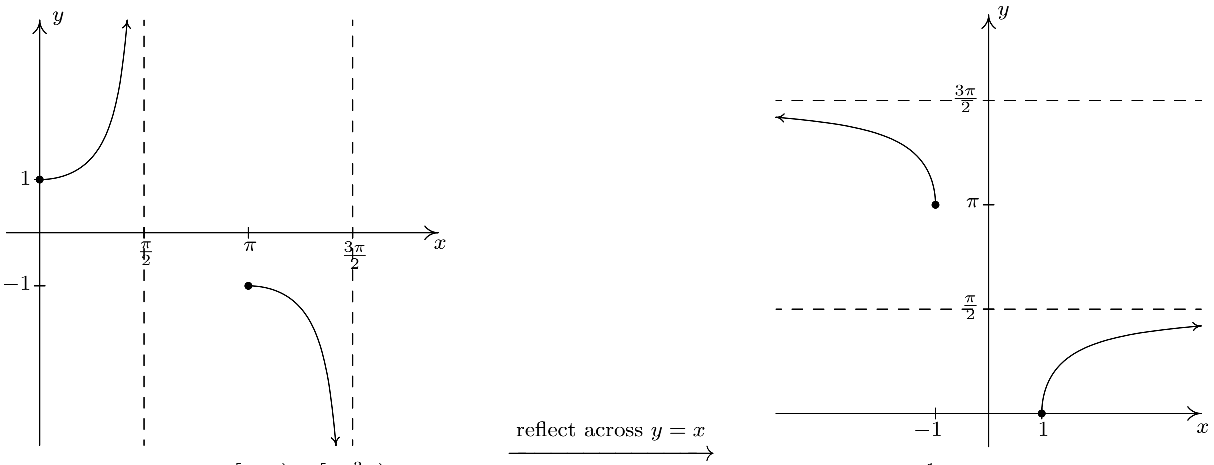
Now we need to determine what these conditions on t mean for x. We know that the domain of $\text{arcsec}(x)$ is $(-\infty, -1] \cup [1, \infty)$, and since $x = \sec(t)$, $x \geq 1$ corresponds to $0 \leq t < \frac{\pi}{2}$, and $x \leq -1$ corresponds to $\frac{\pi}{2} < t \leq \pi$. Since we encountered no further restrictions on t, the equivalence below holds for all x in $(-\infty, -1] \cup [1, \infty)$.

$$\tan(\text{arcsec}(x)) = \begin{cases} \sqrt{x^2 - 1}, & \text{if } x \geq 1 \\ -\sqrt{x^2 - 1}, & \text{if } x \leq -1 \end{cases}$$

(b) To simplify $\cos(\text{arccsc}(4x))$, we start by letting $t = \text{arccsc}(4x)$. Then $\csc(t) = 4x$ for t in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$. Our objective is to write $\cos(\text{arccsc}(4x)) = \cos(t)$ in terms of x. Since $\cos(t)$ is defined for all t, we do not encounter any additional restrictions on t. From $\csc(t) = 4x$, we get $\sin(t) = \frac{1}{4x}$. The identity $\cos^2(t) + \sin^2(t) = 1$ holds for all values of t and substituting for $\sin(t)$ yields $\cos^2(t) + \left(\frac{1}{4x}\right)^2 = 1$. Solving, we get $\cos(t) = \pm\sqrt{\frac{16x^2-1}{16x^2}} = \pm\frac{\sqrt{16x^2-1}}{4|x|}$. Since t belongs to $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, we know $\cos(t) \geq 0$, so we choose $\cos(t) = \frac{\sqrt{16-x^2}}{4|x|}$. (The absolute values here are necessary, since x could be negative.) Since the domain of $\text{arccsc}(x)$ requires $|x| \geq 1$, the domain of $\text{arccsc}(4x)$ requires $|4x| \geq 1$. Using Theorem 2.3, we can rewrite this as the compound inequality $4x \leq -1$ or $4x \geq 1$. Solving, we get $x \leq -\frac{1}{4}$ or $x \geq \frac{1}{4}$. Since we had no additional restrictions on t, the equivalence $\cos(\text{arccsc}(4x)) = \frac{\sqrt{16x^2-1}}{4|x|}$ holds for all x in $\left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, \infty\right)$. □

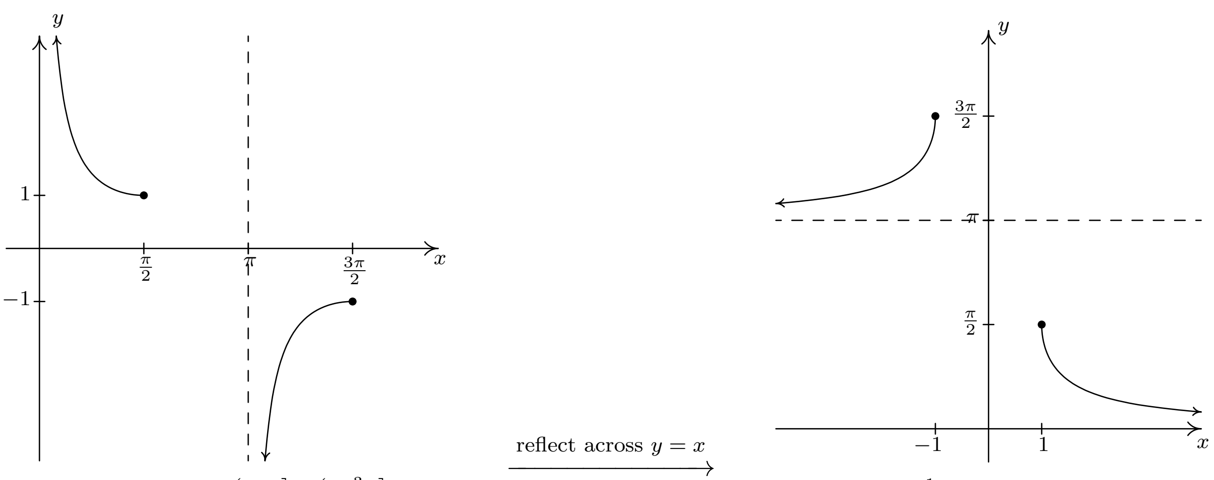
10.6.2 Inverses of Secant and Cosecant: Calculus Friendly Approach

In this subsection, we restrict $f(x) = \sec(x)$ to $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$



$f(x) = \sec(x)$ on $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ $\qquad$ $f^{-1}(x) = \operatorname{arcsec}(x)$

and we restrict $g(x) = \csc(x)$ to $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$.



$g(x) = \csc(x)$ on $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ $\qquad$ $g^{-1}(x) = \operatorname{arccsc}(x)$

Using these definitions, we get the following result.

Theorem 10.29. **Properties of the Arcsecant and Arccosecant Functions**[a]

- Properties of $F(x) = \operatorname{arcsec}(x)$
  - Domain: $\{x : |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$
  - as $x \to -\infty$, $\operatorname{arcsec}(x) \to \frac{3\pi}{2}^{-}$; as $x \to \infty$, $\operatorname{arcsec}(x) \to \frac{\pi}{2}^{-}$
  - $\operatorname{arcsec}(x) = t$ if and only if $0 \leq t < \frac{\pi}{2}$ or $\pi \leq t < \frac{3\pi}{2}$ and $\sec(t) = x$
  - $\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ for $x \geq 1$ only[b]
  - $\sec(\operatorname{arcsec}(x)) = x$ provided $|x| \geq 1$
  - $\operatorname{arcsec}(\sec(x)) = x$ provided $0 \leq x < \frac{\pi}{2}$ or $\pi \leq x < \frac{3\pi}{2}$
- Properties of $G(x) = \operatorname{arccsc}(x)$
  - Domain: $\{x : |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$
  - as $x \to -\infty$, $\operatorname{arccsc}(x) \to \pi^{+}$; as $x \to \infty$, $\operatorname{arccsc}(x) \to 0^{+}$
  - $\operatorname{arccsc}(x) = t$ if and only if $0 < t \leq \frac{\pi}{2}$ or $\pi < t \leq \frac{3\pi}{2}$ and $\csc(t) = x$
  - $\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$ for $x \geq 1$ only[c]
  - $\csc(\operatorname{arccsc}(x)) = x$ provided $|x| \geq 1$
  - $\operatorname{arccsc}(\csc(x)) = x$ provided $0 < x \leq \frac{\pi}{2}$ or $\pi < x \leq \frac{3\pi}{2}$

[a] . . . assuming the "Calculus Friendly" ranges are used.
[b] Compare this with the similar result in Theorem 10.28.
[c] Compare this with the similar result in Theorem 10.28.

Our next example is a duplicate of Example 10.6.3. The interested reader is invited to compare and contrast the solution to each.

Example 10.6.4.

1. Find the exact values of the following.
   - (a) $\operatorname{arcsec}(2)$
   - (b) $\operatorname{arccsc}(-2)$
   - (c) $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right)$
   - (d) $\cot(\operatorname{arccsc}(-3))$
2. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.
   - (a) $\tan(\operatorname{arcsec}(x))$
   - (b) $\cos(\operatorname{arccsc}(4x))$

Solution.

1. (a) Since $2 \geq 1$, we may invoke Theorem 10.29 to get $\operatorname{arcsec}(2) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

   (b) Unfortunately, -2 is not greater to or equal to 1, so we cannot apply Theorem 10.29 to $\operatorname{arccsc}(-2)$ and convert this into an arcsine problem. Instead, we appeal to the definition. The real number $t = \operatorname{arccsc}(-2)$ lies between 0 and $\frac{\pi}{2}$ or between π and $\frac{3\pi}{2}$ and satisfies $\csc(t) = -2$. We have $t = \frac{7\pi}{6}$, so $\operatorname{arccsc}(-2) = \frac{7\pi}{6}$.

   (c) Since $\frac{5\pi}{4}$ lies between π and $\frac{3\pi}{2}$, we may apply Theorem 10.29 directly to simplify $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right) = \frac{5\pi}{4}$. We encourage the reader to work this through using the definition as we have done in the previous examples to see how it goes.

   (d) To simplify $\cot(\operatorname{arccsc}(-3))$ we let $t = \operatorname{arccsc}(-3)$ so that $\cot(\operatorname{arccsc}(-3)) = \cot(t)$. We know $\csc(t) = -3$, and since this is negative, t lies between π and $\frac{3\pi}{2}$. Using the Pythagorean Identity $1 + \cot^2(t) = \csc^2(t)$, we find $1 + \cot^2(t) = (-3)^2$ so that $\cot(t) = \pm\sqrt{8} = \pm 2\sqrt{2}$. Since t is in the interval $\left(0, \frac{3\pi}{2}\right]$, we know $\cot(t) > 0$. Our answer is $\cot(\operatorname{arccsc}(-3)) = 2\sqrt{2}$.

2. (a) To simplify $\tan(\operatorname{arcsec}(x))$, we let $t = \operatorname{arcsec}(x)$ so $\sec(t) = x$ for t in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$. Our goal is to express $\tan(\operatorname{arcsec}(x)) = \tan(t)$ in terms of x. Since $\tan(t)$ is defined for all t under consideration, we have no additional restrictions on t. The identity $1 + \tan^2(t) = \sec^2(t)$ is valid for all t under discussion, so we substitute $\sec(t) = x$ to get $1 + \tan^2(t) = x^2$. We get $\tan(t) = \pm\sqrt{x^2 - 1}$, but since t lies in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$, $\tan(t) \geq 0$, so we choose $\tan(t) = \sqrt{x^2 - 1}$. Since we found no additional restrictions on t, the equivalence $\tan(\operatorname{arcsec}(x)) = \sqrt{x^2 - 1}$ holds on the domain of $\operatorname{arcsec}(x)$, $(-\infty, -1] \cup [1, \infty)$.

   (b) If we let $t = \operatorname{arccsc}(4x)$, then $\csc(t) = 4x$ for t in $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$. Then $\cos(\operatorname{arccsc}(4x)) = \cos(t)$ and our objective is to express the latter in terms of x. Since $\cos(t)$ is defined everywhere, we have no additional restrictions on t. From $\csc(t) = 4x$, we have $\sin(t) = \frac{1}{\csc(t)} = \frac{1}{4x}$ which allows us to use the Pythagorean Identity, $\cos^2(t) + \sin^2(t) = 1$, which holds for all values of t. We get $\cos^2(t) + \left(\frac{1}{4x}\right)^2 = 1$, or $\cos(t) = \pm\sqrt{\frac{16x^2-1}{16x^2}} = \pm\frac{\sqrt{16x^2-1}}{4|x|}$. If t lies in $\left(0, \frac{\pi}{2}\right]$, then $\cos(t) \geq 0$. Otherwise, t belongs to $\left(\pi, \frac{3\pi}{2}\right]$ in which case $\cos(t) \leq 0$. Assuming $0 < t \leq \frac{\pi}{2}$, we choose $\cos(t) = \frac{\sqrt{16x^2-1}}{4|x|}$. Since $\csc(t) \geq 1$ in this case and $\csc(t) = 4x$, we have $4x \geq 1$ or $x \geq \frac{1}{4}$. Hence, in this case, $|x| = x$ so $\cos(t) = \frac{\sqrt{16x^2-1}}{4|x|} = \frac{\sqrt{16x^2-1}}{4x}$. For $\pi < t \leq \frac{3\pi}{2}$, we choose $\cos(t) = -\frac{\sqrt{16x^2-1}}{4|x|}$ and since $\csc(t) \leq -1$ here, we get $x \leq -\frac{1}{4} < 0$ so $|x| = -x$. This leads to $\cos(t) = -\frac{\sqrt{16x^2-1}}{4|x|} = -\frac{\sqrt{16x^2-1}}{4(-x)} = \frac{\sqrt{16x^2-1}}{4x}$ in this case, too. Hence, we have established that, in all cases: $\cos(\operatorname{arccsc}(4x)) = \frac{\sqrt{16x^2-1}}{4x}$. Since the domain of $\operatorname{arccsc}(x)$ requires $|x| \geq 1$, $\operatorname{arccsc}(4x)$ requires $|4x| \geq 1$ or, using Theorem 2.3, for x to lie in $\left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, \infty\right)$. Since we found no additional restrictions on t, $\cos(\operatorname{arccsc}(4x)) = \frac{\sqrt{16x^2-1}}{4x}$ for all x in $\left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, \infty\right)$. □

10.6.3 Using a Calculator to Approximate Inverse Function Values.

In the sections to come, we will have need to approximate the values of the inverse circular functions. On most calculators, only the arcsine, arccosine and arctangent functions are available and they are usually labeled as $\sin^{-1}, \cos^{-1}$ and $\tan^{-1}$, respectively. If we are asked to approximate these values, it is a simple matter to punch up the appropriate decimal on the calculator. If we are asked for an arccotangent, arcsecant or arccosecant, however, we often need to employ some ingenuity, as the next example illustrates.

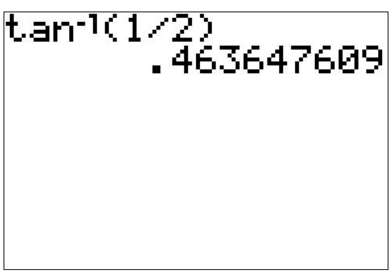
Example 10.6.5. Use a calculator to approximate the following values to four decimal places.

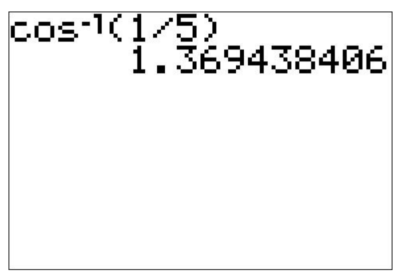
1. $\operatorname{arccot}(2)$
2. $\operatorname{arcsec}(5)$
3. $\operatorname{arccot}(-2)$
4. $\operatorname{arccsc}(-5)$

Solution.

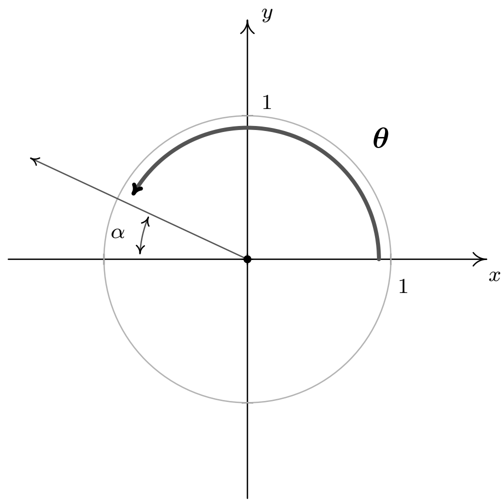
1. Since $2 > 0$, we can use a property listed in Theorem 10.27 to write $\operatorname{arccot}(2) = \arctan\left(\frac{1}{2}\right)$. In 'radian' mode, we find $\operatorname{arccot}(2) = \arctan\left(\frac{1}{2}\right) \approx 0.4636$.

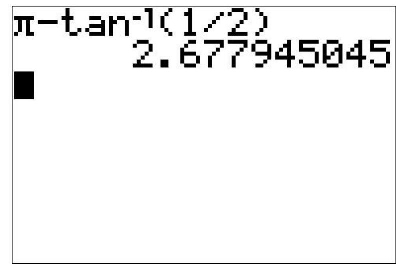
2. Since $5 \geq 1$, we can invoke either Theorem 10.28 or Theorem 10.29 to write $\operatorname{arcsec}(5) = \arccos\left(\frac{1}{5}\right) \approx 1.3694$.



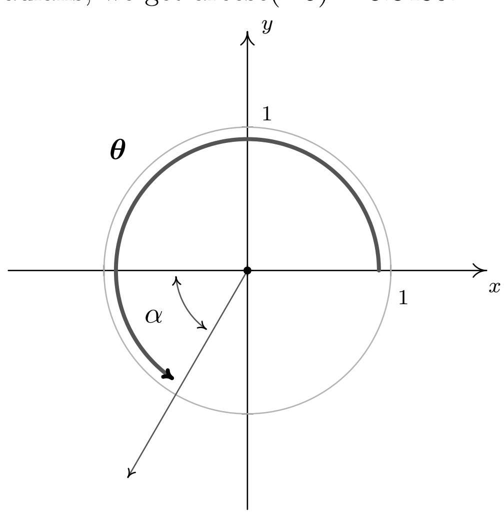


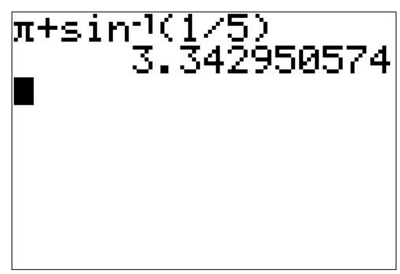
3. Since the argument, -2, is negative we cannot directly apply Theorem 10.27 to help us find $\operatorname{arccot}(-2)$. Let $t = \operatorname{arccot}(-2)$. Then t is a real number between 0 and π with $\cot(t) = -2$. Let $\theta = t$ radians. Then θ is an angle between 0 and π with $\cot(\theta) = -2$. Since $\cot(\theta) < 0$, we know θ must be a Quadrant II angle. Consider the reference angle for θ, α, as pictured below. By definition, $0 < \alpha < \frac{\pi}{2}$ and by the Reference Angle Theorem, Theorem 10.2, it follows that $\cot(\alpha) = 2$. By definition, then, $\alpha = \operatorname{arccot}(2)$ radians which we can rewrite using Theorem 10.27 as $\arctan\left(\frac{1}{2}\right)$. Since $\theta + \alpha = \pi$, we have $\theta = \pi - \alpha = \pi - \arctan\left(\frac{1}{2}\right) \approx 2.6779$ radians. Since $\theta = t$ radians, we have $\operatorname{arccot}(-2) \approx 2.6779$.





4. If the range of $\operatorname{arccsc}(x)$ is taken to be $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, we can use Theorem 10.28 to get $\operatorname{arccsc}(-5) = \arcsin\left(-\frac{1}{5}\right) \approx -0.2014$. If, on the other hand, the range of $\operatorname{arccsc}(x)$ is taken to be $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$, then we proceed as in the previous problem. Let $t = \operatorname{arccsc}(-5)$ and let $\theta = t$ radians. Then $\csc(\theta) = -5$ which means $\pi \leq \theta < \frac{3\pi}{2}$. Let α be the reference angle for θ. Then $0 < \alpha < \frac{\pi}{2}$ and $\csc(\alpha) = 5$. Hence, $\alpha = \operatorname{arccsc}(5) = \arcsin\left(\frac{1}{5}\right)$ radians, where the last equality comes from Theorem 10.29. Since, in this case, $\theta = \pi + \alpha = \pi + \arcsin\left(\frac{1}{5}\right) \approx 3.3430$ radians, we get $\operatorname{arccsc}(-5) \approx 3.3430$.





□

The inverse trigonometric functions are typically found in applications whenever the measure of an angle is required. One such scenario is presented in the following example.

Example 10.6.6. [6] The roof on the house below has a '6/12 pitch.' This means that when viewed from the side, the roof line has a rise of 6 feet over a run of 12 feet. Find the angle of inclination

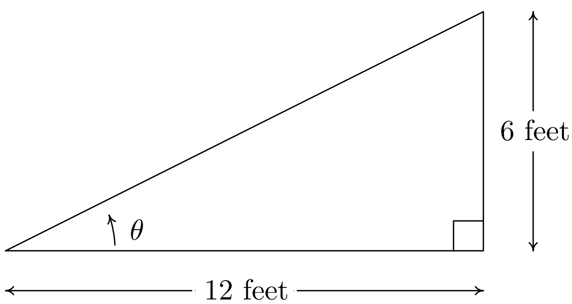
[6]The authors would like to thank Dan Stitz for this problem and associated graphics.

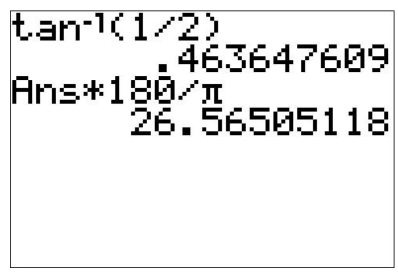
from the bottom of the roof to the top of the roof. Express your answer in decimal degrees, rounded to the nearest hundredth of a degree.

Front View

Side View

SOLUTION. If we divide the side view of the house down the middle, we find that the roof line forms the hypotenuse of a right triangle with legs of length 6 feet and 12 feet. Using Theorem 10.10, we find the angle of inclination, labeled θ below, satisfies $\tan(\theta) = \frac{6}{12} = \frac{1}{2}$. Since θ is an acute angle, we can use the arctangent function and we find $\theta = \arctan\left(\frac{1}{2}\right)$ radians. Converting degrees to radians,[7] we find $\theta = \left(\arctan\left(\frac{1}{2}\right) \text{ radians}\right)\left(\frac{180 \text{ degrees}}{\pi \text{ radians}}\right) \approx 26.56^\circ$.





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10.6.4 Solving Equations Using the Inverse Trigonometric Functions.

In Sections 10.2 and 10.3, we learned how to solve equations like $\sin(\theta) = \frac{1}{2}$ for angles θ and $\tan(t) = -1$ for real numbers t. In each case, we ultimately appealed to the Unit Circle and relied on the fact that the answers corresponded to a set of 'common angles' listed on page 619. If, on the other hand, we had been asked to find all angles with $\sin(\theta) = \frac{1}{3}$ or solve $\tan(t) = -2$ for real numbers t, we would have been hard-pressed to do so. With the introduction of the inverse trigonometric functions, however, we are now in a position to solve these equations. A good parallel to keep in mind is how the square root function can be used to solve certain quadratic equations. The equation $x^2 = 4$ is a lot like $\sin(\theta) = \frac{1}{2}$ in that it has friendly, 'common value' answers $x = \pm 2$. The equation $x^2 = 7$, on the other hand, is a lot like $\sin(\theta) = \frac{1}{3}$. We know[8] there are answers, but we can't express them using 'friendly' numbers.[9] To solve $x^2 = 7$, we make use of the square root

[7] Or, alternatively, setting the calculator to 'degree' mode.
[8] How do we know this again?
[9] This is all, of course, a matter of opinion. For the record, the authors find $\pm\sqrt{7}$ just as 'nice' as ± 2.

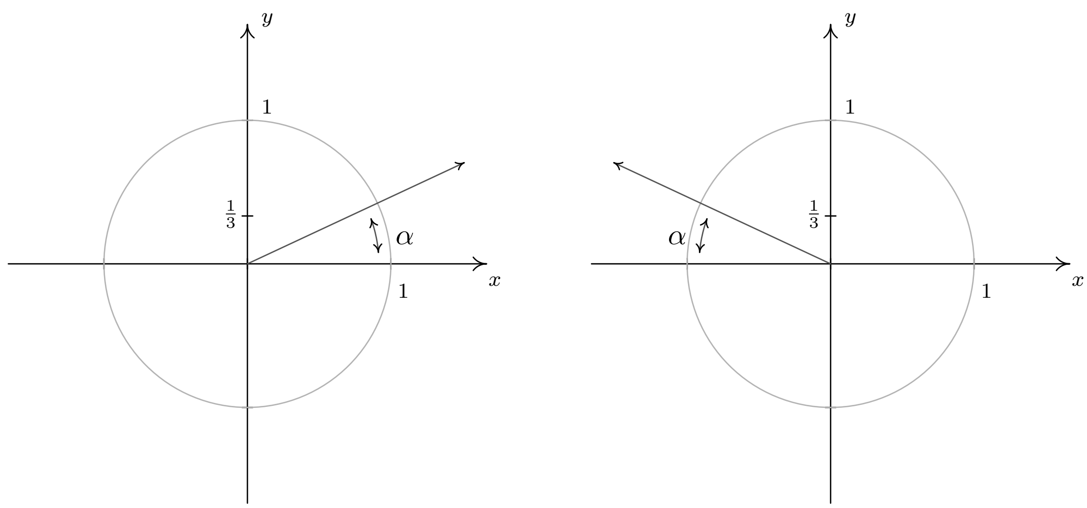
function and write $x = \pm\sqrt{7}$. We can certainly *approximate* these answers using a calculator, but as far as exact answers go, we leave them as $x = \pm\sqrt{7}$.[10] In the same way, we will use the arcsine function to solve $\sin(\theta) = \frac{1}{3}$, as seen in the following example.

Example 10.6.7. Solve the following equations.

1. Find all angles θ for which $\sin(\theta) = \frac{1}{3}$.
2. Find all real numbers t for which $\tan(t) = -2$
3. Solve $\sec(x) = -\frac{5}{3}$ for x.

Solution.

1. If $\sin(\theta) = \frac{1}{3}$, then the terminal side of θ, when plotted in standard position, intersects the Unit Circle at $y = \frac{1}{3}$. Geometrically, we see that this happens at two places: in Quadrant I and Quadrant II.



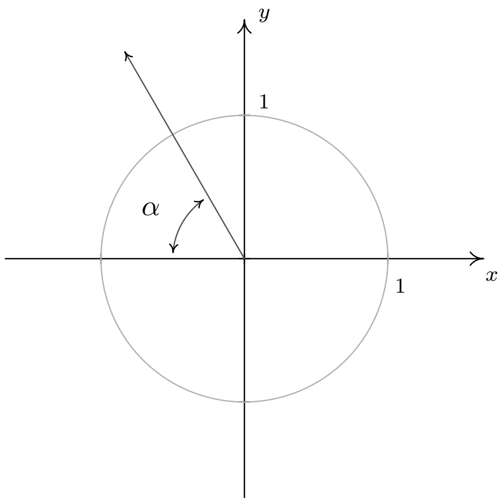
The quest now is to find the measures of these angles. Since $\frac{1}{3}$ isn't the sine of any of the 'common angles' discussed earlier, we are forced to use the inverse trigonometric functions, in this case the arcsine function, to express our answers. By definition, the real number $t = \arcsin\left(\frac{1}{3}\right)$ satisfies $0 < t < \frac{\pi}{2}$ with $\sin(t) = \frac{1}{3}$, so we know our solutions have a reference angle of $\alpha = \arcsin\left(\frac{1}{3}\right)$ radians. The solutions in Quadrant I are all coterminal with α and so our solution here is $\theta = \alpha + 2\pi k = \arcsin\left(\frac{1}{3}\right) + 2\pi k$ for integers k. Turning our attention to Quadrant II, one angle with a reference angle of α is $\pi - \alpha$. Hence, all solutions here are of the form $\theta = \pi - \alpha + 2\pi k = \pi - \arcsin\left(\frac{1}{3}\right) + 2\pi k$, for integers k.

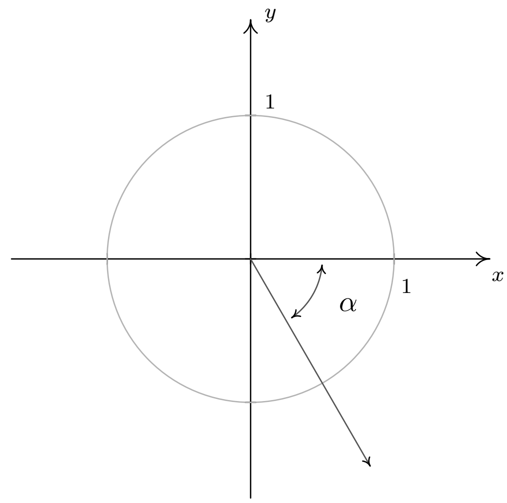
2. Even though we are told t represents a real number, it we can visualize this problem in terms of angles on the Unit Circle, so at least mentally,[11] we cosmetically change the equation to

[10] We could solve $x^2 = 4$ using square roots as well to get $x = \pm\sqrt{4}$, but, we would simplify the answers to $x = \pm 2$.

[11] In practice, this is done mentally, or in a classroom setting, verbally. Carl's penchant for pedantry wins out here.

$\tan(\theta) = -2$. Tangent is negative in two places: in Quadrant II and Quadrant IV. If we proceed as above using a reference angle approach, then the reference angle α must satisfy $0 < \alpha < \frac{\pi}{2}$ and $\tan(\alpha) = 2$. Such an angle is $\alpha = \arctan(2)$ radians. A Quadrant II angle with reference angle α is $\pi - \alpha$. Hence, the Quadrant II solutions to the equation are $\theta = \pi - \alpha + 2\pi k = \pi - \arctan(2) + 2\pi k$ for integers k. A Quadrant IV angle with reference angle α is $2\pi - \alpha$, so the Quadrant IV solutions are $\theta = 2\pi - \alpha + 2\pi k = 2\pi - \arctan(2) + 2\pi k$ for integers k. As we saw in Section 10.3, these solutions can be combined.[12] One way to describe all the solutions is $\theta = -\arctan(2) + \pi k$ for integers k. Remembering that we are solving for real numbers t and not angles θ measured in radians, we write our final answer as $t = -\arctan(2) + \pi k$ for integers k.



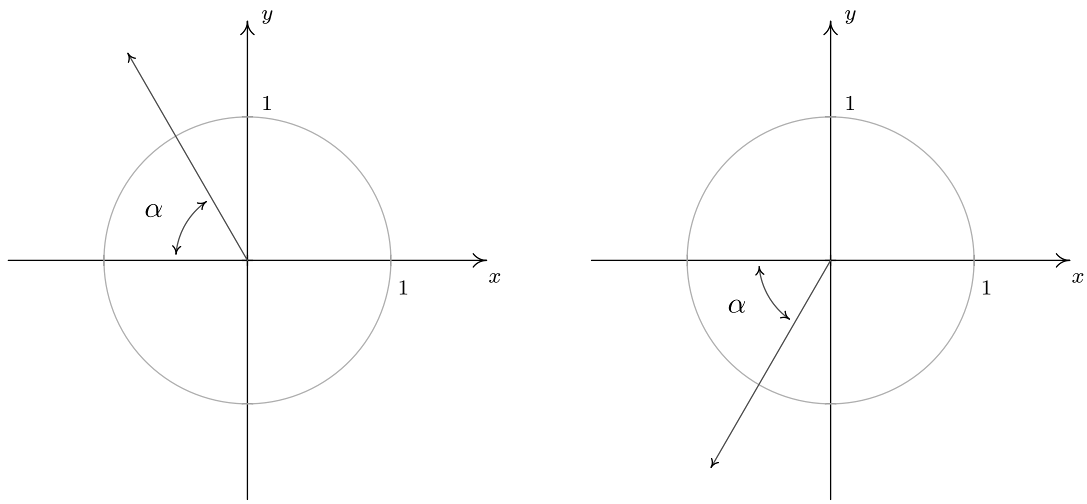


Alternatively, we can forgo the 'angle' approach altogether and we note that $\tan(t) = -2$ only once on its fundamental period $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. By definition, this happens at the value $t = \arctan(-2)$. Since the period of tangent is π, we can capture all solutions by adding integer multiples of π and get our solution $t = \arctan(-2) + \pi k$ for integers k.

3. The last equation we are asked to solve, $\sec(x) = -\frac{5}{3}$, poses two immediate problems. First, we are not told whether or not x represents an angle or a real number. We assume the latter, but note that, once again, we will use angles and the Unit Circle to solve the equation regardless. Second, as we have mentioned, there is no universally accepted range of the arcsecant function. For that reason, we adopt the advice given in Section 10.3 and convert this to the cosine problem $\cos(x) = -\frac{3}{5}$. Adopting an angle approach, we consider the equation $\cos(\theta) = -\frac{3}{5}$ and note our solutions lie in Quadrants II and III. The reference angle α satisfies $0 < \alpha < \frac{\pi}{2}$ with $\cos(\alpha) = \frac{3}{5}$. We look to the arccosine function for help. The real number $t = \arccos\left(\frac{3}{5}\right)$ satisfies $0 < t < \frac{\pi}{2}$ and $\cos(t) = \frac{3}{5}$, so our reference angle is $\alpha = \arccos\left(\frac{3}{5}\right)$ radians. Proceeding as before, we find the Quadrant II solutions to be $\theta = \pi - \alpha + 2\pi k = \pi - \arccos\left(\frac{3}{5}\right) + 2\pi k$ for integers k. In Quadrant III, one angle with reference angle α is $\pi + \alpha$, so our solutions here are $\theta = \pi + \alpha + 2\pi k = \pi + \arccos\left(\frac{3}{5}\right) + 2\pi k$ for

[12] When in doubt, write them out!

integers k. Passing back to real numbers, we state our solutions as $t = \pi - \arccos\left(\frac{3}{5}\right) + 2\pi k$ or $t = \pi + \arccos\left(\frac{3}{5}\right) + 2\pi k$ for integers k.



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It is natural to wonder if it is possible to skip the 'angle' argument in number 3 as we did in number 2 in Example 10.6.7 above. It is true that one solution to $\cos(x) = -\frac{3}{5}$ is $x = \arccos\left(-\frac{3}{5}\right)$ and since the period of the cosine function is 2π, we can readily express one family of solutions as $x = \arccos\left(-\frac{3}{5}\right) + 2\pi k$ for integers k. The problem with this is that there is another family of solutions. While expressing this family of solutions in terms of $\arccos\left(-\frac{3}{5}\right)$ isn't impossible, it certainly isn't as intuitive as using a reference angle.[13] In general, equations involving cosine and sine (and hence secant or cosecant) are usually best handled using the reference angle idea thinking geometrically to get the solutions which lie in the fundamental period $[0, 2\pi)$ and then add integer multiples of the period 2π to generate all of the coterminal answers and capture all of the solutions. With tangent and cotangent, we can ignore the angular roots of trigonometry altogether, invoke the appropriate inverse function, and then add integer multiples of the period, which in these cases is π. The reader is encouraged to check the answers found in Example 10.6.7 - both analytically and with the calculator (see Section 10.6.3). With practice, the inverse trigonometric functions will become as familiar to you as the square root function. Speaking of practice . . .

[13]In our humble opinion, of course!