63. Representation of functions as power series

Consider a power series

$$1 - x^2 + x^4 - x^6 + x^8 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

It is a geometric series with $q = -x^2$ and therefore it converges for all $|q| = x^2 < 1$ or $x \in (-1, 1)$. Using the formula for the sum of a geometric series, one infers that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^{2n} \quad \text{for all} \quad -1 < x < 1$$

This shows that the function $1/(1+x^2)$ can be represented as a power series in the open interval $(-1, 1)$. Note well that the found representation is valid *only* in the interval of convergence of the power series despite that the function $1/(1+x^2)$ is defined on the entire real line.

In general, one can construct a representation of a function by a power series in $(x - a)$ for some a. The interval of validity of this representation depends on the choice of a.

Example 102. *Find a representation of* $1/x$ *as a power series in* $(x - a)$, $a > 0$, *and determine the interval of its validity.*

Solution: Put $y = x - a$. The function can rewritten in the form that resembles the sum of a geometric series:

$$\frac{1}{x} = \frac{1}{a(1 + y/a)} = \frac{1}{a}\sum_{n=0}^{\infty}\Big(-\frac{y}{a}\Big)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{a^{n+1}}(x-a)^n\,, \quad x \in (0, 2a)$$

The geometric series converges if $|q| = |-y/a| = |y|/a < 1$ and, hence, this representation is valid only if $-a < y < a$ or $-a < x - a < a$ or $0 < x < 2a$. □

63.1. Differentiation and integration of power series. A formula for the sum of a power series $\sum c_n x^n$ is often complicated and, in most cases, cannot even be found explicitly. How can functions defined by a power series be differentiated and integrated? If a function is a finite sum $f(x) = u_1(x) + \cdots + u_n(x)$, then the derivative is the sum of derivatives $f' = u_1' + \cdots + u_n'$ and, similarly, the integral is the sum of integrals $\int f dx = \int u_1 dx + \cdots + \int u_n dx$. This is *not* generally true for infinite sums. As an example, consider a function defined by the series

$$f(x) = \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty}\frac{\sin(nx)}{n^2}$$

By comparison with a $p-$series: $|u_n(x)| = |\sin(nx)|/n^2 \le 1/n^2$, this series converges for all x because $\sum 1/n^2$ converges. If the series is differentiated just like a finite sum, i.e. **term-by-term**, $u_n'(x) = \cos(nx)/n$, then the series $\sum u_n'(x)$ diverges for $x = 2\pi k$ for any integer k as the harmonic series $\sum 1/n$. So, $f'(2\pi k)$ does not exist. Thus, *although the terms $u_n(x)$ are differentiable functions in the interval of convergence of the series $\sum u_n$, the series of derivatives $\sum u_n'$ may not converge and, hence, $f = \sum u_n$ may not be differentiable everywhere in its domain.*

It appears that if $u_n(x) = c_n(x-a)^n$, that is, $\sum u_n(x)$ is a power series, then the term-by-term differentiation or integration is justified. A proof of this assertion goes beyond the scope of this course.

THEOREM 46. (Differentiation and integration of power series) *If the power series $\sum c_n(x-a)^n$ has a non-zero radius of convergence $R > 0$, then the function f defined by*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + \cdots = C + \sum_{n=0}^{\infty} c_n\,\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of these power series are both R.

Thus, for *power series* the differentiation or integration and the summation can be carried out in any order:

$$\frac{d}{dx}\sum c_n(x-a)^n = \sum \frac{d}{dx}[c_n(x-a)^n]$$

$$\int \Big(\sum c_n(x-a)^n\Big)dx = \sum \int [c_n(x-a)^n]dx$$

Remark. Theorem 46 states the radius of convergence of a power series does not change after differentiation or integration of the series. This does not mean that the *interval of convergence* does not change. It may happen that the original series converges at an end-point, whereas the differentiated series diverges there.

EXAMPLE 103. *Find the intervals of convergence for f, f', and f'' if $f(x) = \sum_{n=1}^{\infty} x^n/n^2$*

Solution: Here $c_n = 1/n^2$ and, hence, $\sqrt[n]{|c_n|} = 1/\sqrt[n]{n^2} = (1/\sqrt[n]{n})^2 \to 1 = \alpha$. So the radius of convergence is $R = 1/\alpha = 1$. For $x = \pm 1$, the series is a $p-$series $\sum 1/n^2$ which converges ($p = 2 > 1$). Thus, $f(x)$ is defined on the closed interval $x \in [-1, 1]$. By Theorem 1.27, the derivatives $f'(x) = \sum_{n=1}^{\infty} x^{n-1}/n$ and $f''(x) = \sum_{n=2}^{\infty}(n - 1)x^{n-2}/n$ have the same radius of convergence $R = 1$. For $x = -1$, the series $f'(-1) = \sum(-1)^{n-1}/n$ is the alternating harmonic series which converges, whereas the series $f''(-1) = \sum(-1)^n(n-1)/n$ diverges because the sequence of its terms does not converge to zero: $|(-1)^n(n-1)/n| = 1 - 1/n \to 1 \neq 0$. For $x = 1$, the series $f'(1) = \sum 1/n$ is the harmonic series and, hence, diverges. The series $f''(1) = \sum(n-1)/n$ also diverges ($(n-1)/n$ does not converge to zero). Thus, the intervals of convergence for f, f', and f'' are, respectively, $[-1, 1]$, $[-1, 1)$, and $(-1, 1)$. □

The term-by-term integration of a power series can be used to obtain a power series representation of antiderivatives.

EXAMPLE 104. *Find a power series representation for* $\tan^{-1}(x)$

Solution:

$$\tan^{-1}(x) = \int \frac{dx}{1+x^2} = \int \Big(\sum_{n=0}^{\infty}(-x^2)^n\Big)dx = C + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{2n+1}$$

Since $\tan^{-1}(0) = 0$, the integration constant C satisfies the condition $0 = C + 0$ or $C = 0$. The geometric series with $q = -x^2$ converges if $|q| < 1$. Hence, the radius of convergence of the series for $\tan^{-1}(x)$ is $R = 1$ (the power series representation is valid for $x \in (-1, 1)$). □ In particular, the number $1/\sqrt{3}$ is less than the radius of convergence of the power series for $\tan^{-1}(x)$. So, the number $\tan^{-1}(1/\sqrt{3}) = \pi/6$ can be written as the numerical series by substituting $x = 1/\sqrt{3}$ into the power series for $\tan^{-1}(x)$. This leads to the following representation of the number π:

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n}$$

63.2. Power series and differential equations. A power series representation is often used to solve **differential equations**. A relation between a function $f(x)$, its argument x, and its derivatives $f'(x)$, $f''(x)$ etc. is called a differential equation. A function $f(x)$ that satisfies a differential equation is generally difficult to find in a closed form. A power series representation turns out to be helpful. Since in this representation a function is defined by a sequence $\{c_n\}$, $f(x) = \sum c_n x^n$, and so are its derivatives $f^{(k)}(x)$, a differential equation imposes conditions on c_n which are solved recursively.

EXAMPLE 105. *Find a power series representation of the solution of the equation $f'(x) = f(x)$ and determine its radius of convergence.*

Solution: Put $f(x) = \sum c_n x^n$ and, hence, $f'(x) = \sum n c_n x^{n-1}$. Then the equation $f' = f$ gives:

$$c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

By matching the coefficients at the monomial terms 1, x, x^2, x^3, etc., one finds:

$$c_0 = c_1 \quad c_2 = \frac{c_1}{2}, \quad c_3 = \frac{c_2}{3}, \ldots, \quad c_n = \frac{c_{n-1}}{n}$$

Using the latter relation recursively:

$$c_n = \frac{1}{n} c_{n-1} = \frac{1}{n(n-1)} c_{n-2} = \frac{1}{n(n-1)(n-2)} c_{n-3} = \cdots = \frac{c_0}{n!}$$

So, $f(x) = c_0 \sum_{n=1} x^n/n!$ where c_0 is a constant (the equation is satisfied for any choice of c_0). By the ratio test, the series converges for all x (so $R = \infty$). Indeed, $c_n = 1/n!$ and $c_{n+1}/c_n = 1/(n+1) \to 0 = \alpha$ and, hence, $R = 1/\alpha = \infty$. □

For this simple differential equation, it is not difficult to find $f(x) = c_0 e^x$ by recalling the properties of the exponential function: $(e^x)' = e^x$. The condition $f(0) = e^0 = 1$ determines the constant $c_0 = 1$. Thus, the exponential function has the following power series representation

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{61}$$

The series converges on the entire real line. In particular, the number e has the following series representation

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$$

63.3. Approximation of definite integrals. If an indefinite integral of $f(x)$ is difficult to obtain, then the evaluation of the integral $\int_a^b f(x)dx$ poses a problem. A power series representation offers a simple way to approximate the value of the integral. Suppose that $f(x) = \sum c_n x^n$ for $-R < x < R$. By Theorem 46, for any

$-R < a < b < R$,

$$\begin{aligned}\int_a^b f(x)dx &= \sum c_n \int_a^b x^n dx = \sum c_n \frac{b^{n+1}}{n+1} - \sum c_n \frac{a^{n+1}}{n+1} \\ &\approx \sum_{k=0}^{n} c_k \frac{b^{k+1}}{k+1} - \sum_{k=0}^{n} c_k \frac{a^{k+1}}{k+1}\end{aligned}$$

Errors of the approximation of the series sum by finite sums have been discussed earlier.

EXAMPLE 106. *How many terms does one need in the power series approximation of the integral of $f(x) = e^{-x^2}$ over the inteval $[0, 1]$ to make the absolute error smaller than 10^{-5}?*

Solution: Note first that the indefinite integral $\int e^{-x^2} dx$ cannot be expressed in elementary functions! So, a direct use of the fundamental theorem of calculus becomes problematic. However, $\int e^{-x^2} dx$ can be represented as a power series that converges on the entire real line by replacing x in Eq. (61) by $(-x^2)$. One has

$$\int_0^1 e^{-x^2} dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 x^{2k} dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} \approx \sum_{k=0}^{n} \frac{(-1)^k}{k!(2k+1)}$$

To determine n in the finite sum approximation of the series, recall the alternating series estimation theorem (Theorem 37) where $b_n = 1/(n!(2n+1))$:

$$\Big| \int_0^1 e^{-x^2} dx - \sum_{k=0}^{n} \frac{(-1)^k}{k!(2k+1)} \Big| \le b_{n+1} = \frac{1}{(n+1)!(2n+3)} < 10^{-5}$$

A direct calculation shows that $b_7 \approx 1.32 \cdot 10^{-5}$ and $b_8 \approx 1.46 \cdot 10^{-6}$. So $n = 7$ is sufficient to approximate the integral with the required accuracy. □